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FILTERS AND THEIR TRIVIALITY

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Filters of modules are studied. Necessary and sufficient conditions for all preradical filters to be trivial are given.

Key words: ring, module, preradical.

All rings are considered to be associative with unit $1\neq 0$ and all modules are left unitary.

Let R be a ring. The category of left R-modules will be denoted by $R-Mod$. We shall write $N \leq M$ if N is a submodule of M. The set of all R-endomorphisms of M will be denoted by $End(M)$. Let $J(M)$denote the Jacobson radical of M. Let $N \leq M$ and $f \in End(M)$. Put

$$(N:f)_M = \{x \in M \mid f(x) \in N\}, \quad End(M)_N = \{f \in End(M) \mid f(M) \subseteq N\}.$$

Let E be some non-empty collection of submodules of a left R-module M.

Consider the following conditions:

$$L \in E, L \leq N \leq M \Rightarrow N \in E; \tag{1}$$

$$L \in E, f \in End(M) \Rightarrow (L:f)_M \in E; \tag{2}$$

$$N, L \in E \Rightarrow N \bigcap L \in E; \tag{3}$$

$$N \in E, N \in Gen(M), L \leq N \leq M \wedge \forall g \in End(M)_N : (L:g)_M \in E \Rightarrow L \in E; \tag{4}$$

Definition 1. *A non-empty collection E of submodules of a left R-module M satisfying* (1)*,* (2)*,* (3) *is called a preradical filter of M (see* [2]*).*

Definition 2. *A non-empty collection E of submodules of a left R-module M satisfying* (1)*,* (2)*,* (4) *is called a radical filter of M(see* [2]*).*

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Definition 3. *A preradical (radical) filter E of a left R-module M is said to be trivial if either $E = \{L \,|L \leq M\}$ or $E = \{M\}$.*

Theorem 1. *Let M be a left R-module such that $M = M_1 \oplus M_2 \oplus ... \oplus M_n$, where $M_i = Tr_M(M_i)$ for each $i \in \{1, 2, ..., n\}$ and $\forall S : S \leq M \Rightarrow S \in Gen(M)$. If E_i is a radical [preradical] filter of M_i for each $i \in \{1, 2, ..., n\}$, then $E = \{J_1 + J_2 + ... + J_n|$ $J_i \in E_i(i \in \{1, 2, ..., n\})\}$ is a radical [preradical] filter of M.*

Proof. Let E_i is a radical [preradical] filter of M_i for each $i \in \{1, 2, ..., n\}$. Put

$$E = \{J_1 + J_2 + ... + J_n \,|J_i \in E_i(i \in \{1, 2, ..., n\})\}.$$

(1) Let $J_1 + J_2 + ... + J_n \leq S \leq M$, where $J_1 \in E_1, J_2 \in E_2, ..., J_n \in E_n$. As in the above, by a similar argument, $S = S_1 + S_2 + ... + S_n$, where $S_i = Tr_S(M_i) = f_i(S)$. Hence $J_i \leq S_i$ for any $i \in \{1, 2, ..., n\}$. By (1) for radical [preradical] filters, $S_i \in E_i$ for any $i \in \{1, 2, ..., n\}$. Hence $S \in E$.

(2) Let $J_1 \in E_1, J_2 \in E_2, ..., J_n \in E_n, F \in End(M)$. Since M_i is fully invariant for any $i \in \{1, 2, ..., n\}$, $F(M_i) \subseteq M_i$ for any $i \in \{1, 2, ..., n\}$. Consider

$$F_i : M_i \to M_i, F_i(m) = F(m), (m \in M_i).$$

Hence $F_i \in End(M_i)$. We claim that

$$(J_1 : F_1)_{M_1} + (J_2 : F_2)_{M_2} + ... + (J_n : F_n)_{M_n} = ((J_1 + J_2 + ... + J_n) : F)_M.$$

Indeed, let $x \in M, x = x_1 + x_2 + ... + x_n, \forall i \in \{1, 2, ..., n\} : x_i \in M_i$. Hence

$$x \in (J_1 : F_1)_{M_1} + (J_2 : F_2)_{M_2} + ... + (J_n : F_n)_{M_n} \Leftrightarrow \forall i \in \{1, 2, ..., n\} : x_i \in (J_i : F_i)_{M_i} \Leftrightarrow$$

$$\Leftrightarrow \forall i \in \{1, 2, ..., n\} : F_i(x_i) \in J_i \Leftrightarrow \forall i \in \{1, 2, ..., n\} : F(x_i) \in J_i \Leftrightarrow$$

$$\Leftrightarrow F(x) \in J_1 + J_2 + ... + J_n \Leftrightarrow x \in ((J_1 + J_2 + ... + J_n) : F)_M.$$

By (2) for radical [preradical] filters, $(J_i : F_i)_{M_i} \in E_i$ for any $i \in \{1, 2, ..., n\}$. Therefore $((J_1 + J_2 + ... + J_n) : F)_M \in E$.

(3) $J_1 \in E_1, J_2 \in E_2, ..., J_n \in E_n, T_1 \in E_1, T_2 \in E_2, ..., T_n \in E_n$. By (3) for preradical filters, $J_1 \bigcap T_1 \in E_1, J_2 \bigcap T_2 \in E_2, ..., J_n \bigcap T_n \in E_n$. Hence

$$(J_1 + J_2 + ... + J_n) \bigcap (T_1 + T_2 + ... + T_n) = J_1 \bigcap T_1 + J_2 \bigcap T_2 + ... + J_n \bigcap T_n \in E.$$

(4) Let $N_1 \in E_1, N_2 \in E_2, ..., N_n \in E_n$, $L \leq N \leq M$, $\forall g \in End(M)_N : (L : g)_M \in E$, where $N = N_1 + N_2 + ... + N_n$. As in the above consideration, we obtain $L = L_1 \oplus L_2 \oplus ... \oplus L_n$, where $L_i = Tr_L(M_i) = f_i(L)$. Since $L \leq N$, it is easily seen that $L_i \leq N_i$ for any $i \in \{1, 2, ..., n\}$. Let g_i be an arbitrary element of $End(M_i)_{N_i}$. Consider

$$g : M \to M, x_1 + x_2 + ... + x_i + ... + x_n \mapsto g_i(x_i), (x_1 \in M_1, x_2 \in M_2, ..., x_n \in M_n).$$

Hence $g \in End(M)_N$. It is obvious that $f_i((L : g)_M) = (L_i : g_i)_{M_i}$. Since $(L : g)_M \in E$, $(L_i : g_i)_{M_i} = f_i((L : g)_M) \in E_i$.

Claim that

$$\forall s \in \{1, 2, ..., n\} \;\; \forall K \leq M_s : \;\; K \in Gen(M_s).$$

Indeed, let $K \leq M_s$. Since $K \in Gen(M)$, $Tr_K(M) = K$. By Proposition 8.20 [1], $K = Tr_K(M) = \sum_{i=1}^{n} Tr_K(M_i)$. But $Tr_K(M_i) \leq K \bigcap Tr_M(M_i) = K \bigcap M_i \leq M_s \bigcap M_i = 0$ for any $s \neq i$. Hence $K = Tr_K(M) = Tr_K(M_s)$. Therefore $K \in Gen(M_s)$.

Whence $N_i \in Gen(M_i)$ for any $i \in \{1, 2, ..., n\}$. Now we obtain $N_i \in E_i, N_i \in Gen(M_i), L_i \leq N_i \leq M_i \wedge \forall g_i \in End(M_i)_{N_i} : (L_i : g_i)_{M_i} \in E_i$. By (4) for radical [preradical] filter E_i of M_i, $L_i \in E_i$. Therefore $L = L_1 + L_2 + ... + L_n \in E$.

Corollary 1. *Let $R = R_1 \oplus R_2 \oplus ... \oplus R_n$, where R_i is a non-zero two-sided ideal for each $i \in \{1, 2, ..., n\}$. If E_i is a radical [preradical] filter of R_i for each $i \in \{1, 2, ..., n\}$, then $E = \{J_1 + J_2 + ... + J_n \,|J_i \in E_i(i \in \{1, 2, ..., n\})\}$ is a radical [preradical] filter of R.*

Proof. It is easy to see that $R_i = Tr_R(R_i)$ for each $i \in \{1, 2, ..., n\}$ and $\forall S : S \leq R \Rightarrow S \in Gen(R)$.

Theorem 2. *If M is a left R-module with $J(M) \neq M$, then every preradical filter of M is trivial if and only if M is a finitely generated semisimple module and all minimal submodules of M are isomorphic.*

Proof. ($\Rightarrow$) Assume that every preradical filter of M is trivial. Let Ssbe the class of all semisimple modules of M. Consider

$$F := \{L \leq M \quad | \quad M/L \in Ss\}.$$

Since $J(M) \neq M$, $F \neq \{M\}$.

(1) Let $L \leq K, L \in F$. Then there exists an exact sequence $M/L \to M/K \to 0$. Hence $K \in F$.

(2) Let $L \in F, f \in End(M)$. Since there exists an exact sequence $0 \to M/(L : f)_M \to M/L$, $(L : f)_M \in F$.

(3) Let $L, N \in F$. Since there exists an exact sequence $0 \to M/(L \bigcap N) \to M/L \times M/N$, $L \bigcap N \in F$.

Therefore F is a preradical filter.

Since F is a preradical filter and $F \neq \{M\}$, $0 \in F$. Hence M is semisimple.

We shall show that all minimal submodules of M are isomorphic. Suppose that L, N are non-isomorphic minimal submodules of M. Hence $Tr_M(L), Tr_M(N)$ are fully invariant submodules of M. Since L, N are non-isomorphic, $Tr_M(L), Tr_M(N)$ are independent. Hence $Tr_M(L) \bigcap Tr_M(N) = 0$. Since $0 \neq L \subseteq Tr_M(L) \& 0 \neq N \subseteq Tr_M(N) \& Tr_M(L) \bigcap Tr_M(N) = 0$, $0 \neq Tr_M(L) \neq M$. Taking into account that $Tr_M(L)$ is a fully invariant submodule of M, it is easily seen that $\{B \leq M \,|Tr_M(L) \leq B\}$ is a non-trivial preradical filter of M, contrary to the fact that every preradical filter of M is trivial.

Since all minimal submodules of M are isomorphic, M has exactly one homogeneous component.

Suppose that M is not a finitely generated module. Hence $M = \underset{i \in A}{\oplus} P_i$, where $P_i \cong P$ for some minimal submodule $P \leq M$, $Card(A) = \infty$.

Put

$$E := \{T \mid T \leq M, \ M/T \quad \text{is finitely generated} \}.$$

Let $a \in A$. Put

$$K := \sum_{i \in A \setminus \{a\}} P_i.$$

It is obvious that $M/K = (K \oplus P_a)/K \cong P_a$ is finitely generated. Therefore $K \in E$. Since $M/0 \cong M$ is not finitely generated, $0 \notin E$.

Hence $E \neq \{T | T \leq M\}$ and $E \neq \{M\}$.

(1) Let $L \in E, L \leq N \leq M$. There exists an exact sequence $M/L \to M/N \to 0$. Since M/L is finitely generated and M/N is an epimorphic image of M/L, M/N is finitely generated. Hence $N \in E$.

(2) Let $L \in E, f \in End(M)$. By Lemma 1 [3], $M/(L : f)_M \cong f(M)/(f(M) \bigcap L)$. By Corollary 3.7 (3) [1, p. 46], $f(M)/(f(M) \bigcap L) \cong (f(M) + L)/L$. Since $(f(M) + L)/L$ is a submodule of a finitely generated semisimple module M/L, $M/(L : f)_M$ is finitely generated. Hence $(L : f)_M \in E$.

(3) Let $N, L \in E$. Hence $M/N, M/L$ are finitely generated semisimple modules. It follows from this that $M/N \times M/L$is finitely generated semisimple. Since there exists an exact sequence $0 \to M/(N \bigcap L) \to M/N \times M/L$ and $M/N \times M/L$ is finitely generated semisimple, $M/(N \bigcap L)$ is finitely generated. Hence $N \bigcap L \in E$.

Now we obtain that E is a non-trivial filter, contrary to the fact that every preradical filter of M is trivial. It means that M is finitely generated.

($\Leftarrow$) Assume that M is a finitely generated semisimple module and all minimal submodules of M are isomorphic. Hence M is a finitely generated semisimple module with a unique homogeneous component. Arguing as in the proof of Theorem 4 [3] we can show that all preradical filters of M are trivial.

Corollary 2. *All preradical filters of R are trivial if and only if $R \cong M_n(T)$ for some division ring T and $n \in \mathbb{N}$.*

Proof. By Theorems 2 and 13.4 [1].

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ФІЛЬТРИ ТА ЇХНЯ ТРИВІАЛЬНІСТЬ

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Вивчено фільтри модулів. Наведено необхідні та достатні умови для тривіальності всіх напередрадикальних фільтрів.

Ключові слова: кільце, модуль, напередрадикал.

ФИЛЬТРЫ И ИХ ТРИВИАЛЬНОСТЬ

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Изучено фильтры модулей. Указано необходимые и достаточные условия для тривиальности всех предрадикальных фильтров.

Ключевые слова: кольцо, модуль, предрадикал.