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NON-HOMOGENEOUS FRACTIONAL BOUNDARY VALUE PROBLEM IN SPACES OF GENERALIZED FUNCTIONS

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We prove the existence and uniqueness theorem and get the representation, by means of the Green vector-function, of the solution of the problem

$$u_t^{(\beta)}(x,t) - a^2\Delta u(x,t) = F(x,t), \quad (x,t) \in \Omega \times (0,T], \ \ a = const$$
$$u(x,t) = F_1(x,t), \quad (x,t) \in \partial\Omega \times (0,T], \qquad u(x,0) = F_2(x), \quad x \in \Omega$$

with Riemann-Liouville fractional derivative $u_t^{(\beta)}$ of the order $\beta \in (0,1)$ and F, F_1, F_2 from spaces of generalized functions D'.

Key words: fractional derivative, generalized function, boundary value problem, Green vector-function.

1. Introduction. The conditions of classical solvability of the first boundary value problem to equation

$$D_t^{\beta} u(x,t) - a^2\Delta u(x,t) = F(x,t), \quad a = const$$

in bounded domain $\Omega \times (0,T]$, with regulating derivative

$$D_t^{\beta} u(x,t) = \frac{1}{\Gamma(1-\beta)}\Big(\frac{\partial}{\partial t}\int\limits_0^t \frac{u(x,\tau)}{(t-\tau)^{\beta}} d\tau - \frac{u(x,0)}{t^{\beta}}\Big)$$

of the order $\beta \in (0;1)$, under homogeneous boundary conditions, were obtained by Luchko Yu. [1], Meerschaert M.M., Nane Erkan and Vallaisamy P. [2]. The solution was constructed by means of Fourier rows on eigen functions of corresponding Sturm-Lioville problem.

There were proved in [3], [4] the existence and uniqueness theorem and the representation, by means of Green function, of classical solution of fractional Cauchy problem

$$D_t^{\beta} u(x,t) = A(x,D)u(x,t), \quad (x,t) \in R^N \times [0,T],$$
$$u(x,0) = g_1(x), \quad x \in \mathbb{R}^N$$

with continuous function g_1 having some growth at infinity, elliptic differential second order operator $A(x,D)$ with smooth coefficients depending on space variables $x \in \mathbb{R}^n$. Such regulating fractional derivative was used by the authors of [5]-[10] and the others. The classical solution of corresponding fractional Cauchy problem in the case $\beta > 1$ and $A(x,D) = \Delta$ was constructed in [9]. The representation, by means of Green function, of the solution was obtained. In [11] and [12] the unique solvability of the fractional Cauchy problem with given data – slowly increasing generalized functions and from weight spaces of generalized functions, respectively, was established.

We prove the unique solvability of the first boundary value problem to equation

$$u_t^{(\beta)}(x,t) - a^2\Delta u(x,t) = F(x,t), \quad (x,t) \in \Omega \times (0,T], \quad a = const,$$

with Riemann-Liouville fractional derivative $u_t^{(\beta)}$ of the order $\beta \in (0,1)$, in spaces of generalized functions of type D'. This paper is organized as follows. The main significances and terminology are given in section 2. We state the problem in section 3. To prove the main result (in section 5) we study the properties of conjugated Green operators in section 4. We finish this paper with some remarks in section 6 and addition on some requisite properties of the H-function of Fox in section 7.

2. Auxiliary significances. Let Ω be boundary domain in $\mathbb{R}^N$, $N = 2,3,\ldots$, $S = \partial\Omega$ – the boundary of domain Ω (of class C^∞), $Q_T = \Omega \times (0,T]$, $Q_{1T} = S \times (0,T]$,

$D(\bar{Q}_T) = C^{\infty,(0)}(\bar{Q}_T) = \{\varphi \in C^\infty(\bar{Q}_T) : D_t^l\varphi|_{t=T} = 0,\ l = 0,1,2,\ldots\}$,

$D(\bar{Q}_{1T}) = C^{\infty,(0)}(\bar{Q}_{1T}) = \{\varphi \in C^\infty(\bar{Q}_{1T}) : D_t^l\varphi|_{t=T} = 0,\ l = 0,1,2,\ldots\}$,

$D(R^N) = C_0^\infty(R^N)$ – the space of indefinitely differentiable functions with compact supports in R^N,

$D(\bar{\Omega}) = C^\infty(\bar{\Omega})$,

$D'(\mathbb{R}^N)$, $D'(\bar{\Omega})$, $D'(\bar{Q}_T)$ and $D'(\bar{Q}_{1T})$ – the spaces of linear continuous functionals (generalized functions) on $D(\mathbb{R}^N)$, $D(\bar{\Omega})$, $D(\bar{Q}_T)$ and $D(\bar{Q}_{1T})$, respectively,

(f,φ) – the value of $f \in D'(\mathbb{R}^N)$ onto the basic function $\varphi \in D(\mathbb{R}^N)$,

$(f,\varphi)_0$ – the value of $f \in D'(\bar{Q}_T)$ onto the basic function $\varphi \in D(\bar{Q}_T)$,

$(f,\varphi)_1$ – the value of $f \in D'(\bar{Q}_{1T})$ onto the basic function $\varphi \in D(\bar{Q}_{1T})$,

$(f,\varphi)_2$ – the value of $f \in D'(\bar{\Omega})$ onto the basic function $\varphi \in D(\bar{\Omega})$.

We denote by $\hat{*}$ the operation of convolution of generalized function g and basic function φ: $(g\hat{*}\varphi)(x) = (g(\xi),\varphi(x+\xi))$, and by $*$ – the operation of convolution of generalized functions f and g – the generalized function $f * g$: $(f * g,\varphi) = (f, g\hat{*}\varphi)$ for all basic function φ. We use the function $f_\lambda \in D'_+(R) = \{f \in D'(\mathbb{R}) : f = 0 \text{ при } t < 0\}$:

$$f_\lambda(t) = \frac{\theta(t)t^{\lambda-1}}{\Gamma(\lambda)} \text{ for } \lambda > 0 \quad \text{and} \quad f_\lambda(t) = f'_{1+\lambda}(t) \text{ for } \lambda \leqslant 0,$$

where $\theta(t)$– Heaviside function, $\Gamma(\lambda)$– Gamma-function. The function

$$u_t^{(\beta)} = f_{-\beta} * u$$

is called the Riemann-Lioville derivative of the order β of the function $u \in D'_+(\mathbb{R})$. If the regulating derivative $D^\beta u(t)$ exists then

$$D^\beta u(t) = u^{(\beta)}u(t) - f_{1-\beta}(t)u(0).$$

The relations underway

$$f_\lambda * f_\mu = f_{\lambda+\mu}, \qquad f_\lambda \hat{*} f_\mu = f_{\lambda+\mu},$$

and for $(x,t)\in Q_T$, $v\in D(\bar{Q}_T)$, $\beta\in(0;1)$

$$f_{-\beta}(t)\hat{*}v(x,t)=f'_{1-\beta}(t)\hat{*}v(x,t)=-f_{1-\beta}(t)\hat{*}v_t(x,t)=-\frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\int\limits_t^T\frac{v(x,\eta)}{(\eta-t)^\beta}d\eta.$$

We denote by $C_{2,\beta}(Q_T)$ the class of restricted twice continuously differentiable by variables x functions $v(x,t)$, $(x,t)\in\bar{Q}_T$, vanishing at $t\ge T$ and with continuous $D_t^\beta v(x,t)$ in Q_T.

We introduce the operators

$$\hat{L}:(\hat{L}v)(x,t)\equiv f_{-\beta}(t)\hat{*}v(x,t)-a^2\Delta v(x,t),\quad (x,t)\in Q_T,\quad v\in D(\bar{Q}_T),$$

$$L:(Lv)(x,t)\equiv f_{-\beta}(t)*v(x,t)-a^2\Delta v(x,t),\quad (x,t)\in Q_T,\quad v\in D'(\bar{Q}_T),$$

$$L^{reg}:\ (L^{reg}v)(x,t)\equiv D_t^\beta v(x,t)-a^2\Delta v(x,t),\quad (x,t)\in Q_T,\quad v\in C_{2,\beta}(Q_T),$$

the functional space

$$X(\bar{Q}_T)=\{\varphi\in C^{\infty,(0)}(\bar{Q}_T)\ :\ \hat{L}\varphi\in D(\bar{Q}_T),\ \varphi|_{\bar{Q}_{1T}}=0\}$$

which does not empty (see following lemma 3) and $X'(\bar{Q}_T)$ – the space of linear continuous functionals on $X(\bar{Q}_T)$, denote by $(f,\varphi)_0$ the value of $f\in X'(\bar{Q}_T)$ onto $\varphi\in X(\bar{Q}_T)$.

3. Problem's definition.

Supposition (L): Let $\beta\in(0,1)$, $F\in X'(\bar{Q}_T)$, $F_1\in D'(\bar{Q}_{1T})$, $F_2\in D'(\bar{\Omega})$.

Under supposition (L) we study the problem

$$f_{-\beta}(t)*u(x,t)-a^2\Delta u(x,t)=F(x,t),\quad (x,t)\in Q_T, \tag{1}$$

$$u(x,t)=F_1(x,t),\quad (x,t)\in Q_{1T},\qquad u(x,0)=F_2(x),\quad x\in\Omega. \tag{2}$$

Definition 1. *The function $u\in D'(\bar{Q}_T)$ satisfying the equality*

$$(u,\hat{L}\psi)_0=(F,\psi)_0+(F_1,\frac{\partial\psi}{\partial\nu})_1+(F_2,\int\limits_0^T f_{1-\beta}(t)\psi(\cdot,t)dt)_2\qquad\forall\psi\in X(\bar{Q}_T) \tag{3}$$

is called the solution of the problem (1), (2). Here $\nu(x)=(\nu_1(x),\dots,\nu_N(x))$ – the unite vector of the inner normal to the surface S at the point $x\in S$.

Note that for $u\in C_{2,\beta}(Q_T)$, $\psi\in X(\bar{Q}_T)$ the Green formula

$$\int\limits_{Q_T}u(x,t)(\hat{L}\psi)(x,t)dxdt=\int\limits_{Q_T}(L^{reg}u)(x,t)\psi(x,t)dxdt+ \tag{4}$$

$$+a^2\int\limits_0^T dt\int\limits_S u(x,t)\frac{\partial\psi(x,t)}{\partial\nu}dS+\int\limits_\Omega u(x,0)dx\int\limits_0^T f_{1-\beta}(t)\psi(x,t)dt$$

holds. We may prove it as the corresponding formula in [11].

We may consider the problem (1), (2) as the generalization of the problem

$$(L^{reg}u)(x,t)=g_0(x,t),\quad (x,t)\in Q_T \tag{5}$$

$$u(x,t)=g_1(x,t),\quad (x,t)\in Q_{1T},\qquad u(x,0)=g_2(x),\quad x\in\Omega \tag{6}$$

with regular data g_0, g_1, g_2. It obtains from following theorem 1 that under rather regular given functions $F = g_0$, $F_1 = g_1$, $F_2 = g_2$ the solutions of the problems (1), (2) and (5), (6) coincide.

4. Green vector-function.

Definition 2. *The vector-function $(G_0(x,t), G_1(x,t), G_2(x,t))$ such that under rather regular functions g_0, g_1, g_2 the function*

$$u(x,t) = \int_0^t d\tau \int_{\mathbb{R}^n} G_0(x-y,t-\tau)g_0(y,\tau)dy+ \tag{7}$$

$$+\int_{Q_{1t}} G_1(x-y,t-\tau)g_1(y,\tau)dSd\tau + \int_{\Omega} G_2(x-y,t)g_2(y)dy, \quad (x,t)\in Q_T$$

is the classical (of class $C_{2,\beta}(Q_T)$) solution of the problem (5), (6) is called the Green vector-function of the problem (1), (2) (as well as of the problem (5), (6)).

It follows from the definition 2 that

$$(LG_0)(x,t) = \delta(x,t), \quad (x,t)\in Q_T \quad \text{where } \delta - \text{delta-function of Dirac},$$

$$(L^{reg}G_1)(x,t) = 0, \quad (x,t)\in Q_T, \qquad G_1(x,t) = \delta(x,t), \quad (x,t)\in Q_{1T},$$

$$(L^{reg}G_2)(x,t) = 0, \quad (x,t)\in Q_T, \quad G_2(x,0) = \delta(x), \quad x\in\Omega.$$

In [3], [4] the properties of the Green operators on $C_{2,\beta}(\mathbb{R}^N\times(0,T])$ were studied. We study the conjugated Green operators

$$(\hat{\mathfrak{G}}_0\varphi)(y,\tau) = \int_\tau^T dt \int_\Omega G_0(x-y,t-\tau)\varphi(x,t)dx,$$
$$(\hat{\mathfrak{G}}_1\varphi)(y,\tau) = \int_\tau^T dt \int_\Omega G_1(x-y,t-\tau)\varphi(x,t)dx,$$
$$(\hat{\mathfrak{G}}_2\varphi)(y) = \int_0^T dt \int_\Omega G_2(x-y,t)\varphi(x,t)dxdt, \quad \varphi\in D(\bar{Q}_T)$$

by methods of the works [4], [13], [11].

Lemma 1. *For all $\psi \in X(\bar{Q}_T)$*

$$(\hat{\mathfrak{G}}_0(\hat{L}\psi))(y,\tau) = \psi(y,t), \quad (y,t)\in\bar{Q}_T, \tag{8}$$

$$(\hat{\mathfrak{G}}_1(\hat{L}\psi))(y,\tau) = \frac{\partial\psi(y,t)}{\partial\nu}, \quad (y,t)\in\bar{Q}_{1T}, \tag{9}$$

$$(\hat{\mathfrak{G}}_2(\hat{L}\psi))(y) = \int_0^T f_{1-\beta}(t)\psi(y,t)dt, \quad y\in\Omega. \tag{10}$$

Proof. If we substitute the solution (7) of the classical first boundary value problem (5), (6) into the formula (4), then for all $\psi \in X(\bar{Q}_T)$ we obtain

$$\int\limits_{Q_T} \Big(\int\limits_0^t d\tau \int\limits_\Omega G_0(x-y,t-\tau)g_0(y,\tau)dy\Big)(\hat{L}\psi)(x,t)dxdt+$$
$$\int\limits_{Q_T} \Big(\int\limits_0^t d\tau \int\limits_S G_1(x-y,t-\tau)g_1(y,\tau)dS\Big)(\hat{L}\psi)(x,t)dxdt+$$
$$+\int\limits_{Q_T} \Big(\int\limits_\Omega G_2(x-y,t)g_2(y)dy\Big)(\hat{L}\psi)(x,t)dxdt =$$
$$= \int\limits_{Q_T} g_0(x,t)\psi(x,t)dxdt + \int\limits_{Q_{1T}} g_1(x,t)\tfrac{\partial\psi(x,t)}{\partial\nu}dSdt+$$
$$+\int\limits_{Q_T} g_2(x)f_{1-\beta}(t)\psi(x,t)dxdt,$$

that is

$$\int\limits_{Q_T} \Big(\int\limits_\tau^T dt \int\limits_\Omega G_0(x-y,t-\tau)(\hat{L}\psi)(x,t)dx\Big)g_0(y,\tau)dyd\tau+$$
$$+\int\limits_{Q_{1T}} \Big(\int\limits_\tau^T dt \int\limits_\Omega G_1(x-y,t-\tau)(\hat{L}\psi)(x,t)dx\Big)g_1(y,\tau)dSd\tau+$$
$$+\int\limits_\Omega \Big(\int\limits_{Q_T} G_2(x-y,t)(\hat{L}\psi)(x,t)dxdt\Big)g_2(y)dy =$$
$$= \int\limits_{Q_T} g_0(x,t)\psi(x,t)dxdt + \int\limits_{Q_{1T}} g_1(x,t)\tfrac{\partial\psi(x,t)}{\partial\nu}dSdt+$$
$$+\int\limits_{Q_T} g_2(x)f_{1-\beta}(t)\psi(x,t)dxdt.$$

The correctness of the lemma follows from the arbitrariness of g_0, g_1, g_2 .

Lemma 2. *Mapping* $\hat{\mathfrak{G}}_0$ *acts:* $D(\bar{Q}_T) \to C^{\infty,(0)}(\bar{Q}_T)$.

Proof. It follows from the results of [14] that the fundamental function $G(x,t)$ of the operator L is given by

$$G(x,t) = \frac{\pi^{-N/2}t^{\beta-1}}{|x|^N} H^{2,1}_{2,3}\left(\frac{|x|^2}{4a^2t^\beta}\middle|\begin{matrix}(1,1) & (\beta,\beta) & \\ (1,1) & (N/2,1) & (1,1)\end{matrix}\right), \tag{11}$$

where $H^{m,n}_{p,q}\left(z\middle|\begin{matrix}(a_1,\alpha_1) & \dots & (a_p,\alpha_p)\\ (b_1,\beta_1) & \dots & (b_q,\beta_q)\end{matrix}\right) := H(z)$ is the H-function of Fox [15].

By property (B) (in addition) of the H-function we simplify the expression (11) and get the representation

$$G(x,t) = \frac{\pi^{-N/2}t^{\beta-1}}{|x|^N} H^{2,0}_{1,2}\left(\frac{|x|^2}{4a^2t^\beta}\middle|\begin{matrix}(\beta,\beta) & \\ (1,1) & (N/2,1)\end{matrix}\right). \tag{12}$$

We use following significances for $H^{m,n}_{p,q}$:
$\Delta^* = \sum_{i=1}^q \beta_i - \sum_{i=1}^p \alpha_i$,
$a^* = \sum_{i=1}^n \alpha_i - \sum_{i=n+1}^p \alpha_i + \sum_{i=1}^m \beta_i - \sum_{i=m+1}^q \beta_i$.

Since $\Delta^* = 2-\beta$, it follows from theorem 1.1 [15] that function $G(x,t)$ exists for all $x \neq 0$, $t > 0$ ($\Delta^* \neq 0$).

Using property (E) (in addition) of the H-function we find

$$\frac{dG(x,t)}{d|x|}=\frac{\partial G(|x|,t)}{\partial |x|}=-\frac{\pi^{-N/2}t^{\beta-1}}{|x|^{N+1}}H^{3,0}_{2,3}\left(\frac{|x|^2}{4a^2t^\beta}\Big|\begin{matrix}(\beta,\beta) & (N,2) & \\ (N+1,2) & (1,1) & (N/2,1)\end{matrix}\right),$$

whence

$$\frac{\partial G(x,t)}{\partial \nu}=-\frac{\pi^{-N/2}t^{\beta-1}}{|x|^{N+1}}H^{3,0}_{2,3}\left(\frac{|x|^2}{4a^2t^\beta}\begin{matrix}(\beta,\beta) & (N,2) & \\ (N+1,2) & (1,1) & (N/2,1)\end{matrix}\right)\sum_{j=1}^{N}\frac{x_j\nu_j(x)}{|x|},\quad (13)$$

$x\in S$. Since $\Delta^*=2-\beta$, it follows from theorem 1.1 [15] that function $\frac{\partial \tilde G_0(x,t)}{\partial\nu}$ exists for all $x\neq 0$, $t>0$.

As in [3], [4] one may prove by Levi method that the functions G_0 and G (and its derivatives) have equal character of singularities.

Let $\gamma=(\gamma_1,\dots,\gamma_N)$ ($\gamma_j\in\mathbb{Z}_+$, $j=1,\dots,N$) be multi-index, $|\gamma|=\gamma_1+\dots+\gamma_N$,

$D^\gamma_x=D^\gamma=(\frac{\partial}{\partial x})^\gamma=\frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1}\dots\partial x_N^{\gamma_N}}$.

Because that $G(x,t)=G(|x|,t)$ (and therefore, $(\frac{\partial}{\partial x})^\gamma G(x-y,t)=(-\frac{\partial}{\partial y})^\gamma G(x-y,t)$ for all multi-index γ), by property of the convolution of generalized function and basic $\varphi\in D(\bar Q_T)$ we obtain

$$\frac{\partial}{\partial\tau}\int\limits_\tau^T G(x-y,t-\tau)\varphi(x,t)dt=\frac{\partial}{\partial\tau}(G(x-y,t-\tau)\hat{*}\varphi(x,\tau)=$$

$$=G(x-y,t-\tau)\hat{*}\frac{\partial\varphi(x,\tau)}{\partial\tau}=\int\limits_\tau^T G(x-y,t-\tau)\frac{\partial\varphi(x,t)}{\partial t}dt$$

(and therefore, $(\frac{\partial}{\partial\tau})^{\gamma_0}\int\limits_{Q_T}G(x-y,t-\tau)\varphi(x,t)dxdt=\int\limits_{Q_T}G(x-y,t-\tau)(\frac{\partial}{\partial t})^{\gamma_0}\varphi(x,t)dxdt$). In respect continuity of functions $D^\gamma_xD^{\gamma_0}_t\varphi(x,t)$ on $\bar Q_T$ for all $\gamma_0\in\mathbb{Z}_+$ and multi-index γ, it is enough to demonstrate the uniform convergence of the integrals

$$\int\limits_\tau^T dt\int\limits_\Omega|G(x-y,t-\tau)|dx,\quad \int\limits_\tau^T dt\int\limits_S|G(x-y,t-\tau)|dS_x,\quad (14)$$

$$\int\limits_\tau^T dt\int\limits_S\Big|\frac{\partial G(x-y,t-\tau)}{\partial\nu_x}\Big|dS_x.\quad (15)$$

In [15] the asymptotic of the H-functions is constructed. Using the realization of the conditions (1.1.6) and (1.3.2) from this work for obtaining of the estimates of the H-functions we adapt the corollary 1.10.2 [15]:

$$\left|H^{q,0}_{p,q}\left(z\Big|\begin{matrix}(a_1,\alpha_1) & \dots & (a_p,\alpha_p)\\ (b_1,\beta_1) & \dots & (b_q,\beta_q)\end{matrix}\right)\right|\le C|z|^{\frac{\mu+\frac12}{\Delta^*}}e^{-h|z|^{\frac{1}{\Delta^*}}}\quad\text{for real}\quad z\to\infty,\quad (16)$$

where $\mu=\sum\limits_{j=1}^q b_j-\sum\limits_{i=1}^p a_i+\frac{p-q}{2}$, C and $h=h(a,\beta)$ are positive constants. Further we denote by $P,\hat C,c,C_i,C_i^*,C_i',c_i,c_i',\hat c_i,c_i^*$ $(i=0,1,2,3)$ some positive constants.

Using (12) and (13) we find $\mu + \frac{1}{2} = \frac{N}{2} + 1 - \beta$ for G, $\mu + \frac{1}{2} = \frac{N}{2} + 2 - \beta$ for $\frac{\partial G}{\partial \nu}$, $\Delta^* = 2 - \beta$ for both cases. Then by (16) we get the following estimates:

$$|G(x,t)| \leq \frac{\hat{C} t^{\beta-1}}{|x|^N} \cdot \Big(\frac{|x|^2}{t^\beta}\Big)^{\frac{N+2-2\beta}{2(2-\beta)}} e^{-h\left(\frac{|x|^2}{t^\beta}\right)^{\frac{1}{2-\beta}}} \leq \frac{C_0}{t^{1-\beta}|x|^N} \quad \text{for} \quad |x|^2 > t^\beta, \qquad (17)$$

$$\Big|\frac{\partial G(x,t)}{\partial \nu}\Big| \leq \frac{\hat{C} t^{\beta-1}}{|x|^{N+1}} \cdot \Big(\frac{|x|^2}{t^\beta}\Big)^{1+\frac{N}{2(2-\beta)}} e^{-h\left(\frac{|x|^2}{t^\beta}\right)^{\frac{1}{2-\beta}}} \leq \frac{C_1}{t^{1-\beta}|x|^{N+1}} \quad \text{for} \quad |x|^2 > t^\beta. \qquad (18)$$

By corollary from theorem 1.12 [15] the following estimates

$$\left| H^{m,n}_{p,q}\left(z\middle| \begin{matrix} (a_1,\alpha_1) & \dots & (a_p,\alpha_p) \\ (b_1,\beta_1) & \dots & (b_q,\beta_q) \end{matrix}\right)\right| \leq C|z|^\varrho |logz|^{N^*-1}, \quad z \to 0 \qquad (19)$$

hold where $\varrho = \min\limits_{1\leq j\leq m} \frac{Reb_j}{\beta_j}$, N^* is the greatest order of the poles $b_{jl} = \frac{-b_j - l}{\beta_j}$, $1 \leq j \leq m$, $l = 0, 1, \dots$ of the function $\Gamma(b_j + \beta_j s)$.

We find $\varrho = \min\{1, \frac{N}{2}\} = 1$ for $N \geq 2$ ($\varrho = 1/2$ for $N = 1$) and get

$$|G(x,t)| \leq \frac{C_0^*}{t|x|^{N-2}}, \quad \Big|\frac{\partial G(x,t)}{\partial \nu}\Big| \leq \frac{C_1^*}{t|x|^{N-1}}, \quad \text{for} \quad |x|^2 < t^\beta \qquad (20)$$

in the case of the simple poles b_{jl} (that is $N \geq 3$),

$$|G(x,t)| \leq \frac{C_0^*}{t} ln\frac{t^{\beta/2}}{|x|}, \quad \Big|\frac{\partial G(x,t)}{\partial \nu}\Big| \leq \frac{C_1^*}{t|x|} ln\frac{t^{\beta/2}}{|x|}, \quad \text{for} \quad |x|^2 < t^\beta, \quad N = 2.$$

Now we may prove the uniform convergence of integrals (14),(15). Let for all $(y,\tau) \in \bar{Q}_T$, $\varepsilon > 0$

$Q_{T,\varepsilon}(y,\tau) = Q_{T,\varepsilon} = \{(x,t) \in \bar{Q}_T \ : \ |x-y| < \varepsilon, \ 0 < t-\tau < \varepsilon^{2/\beta}\}$,

$Q_{1T,\varepsilon}(y,\tau) = Q_{1T,\varepsilon} = \{(x,t) \in \bar{Q}_{1T} \ : \ |x-y| < \varepsilon, \ 0 < t-\tau < \varepsilon^{2/\beta}\}$.

Then in the case $N \geq 3$, for all $(y,\tau) \in \bar{Q}_T$, $\varepsilon > 0$ the formulas (17), (18), (20) imply following requisite estimates

$$\int\limits_{Q_{T,\varepsilon}(y,\tau)} |G(x-y,t-\tau)| dxdt \leq \int\limits_{(x,t)\in Q_{T,\varepsilon}:|x-y|^2<(t-\tau)^\beta} |G(x-y,t-\tau)| dxdt +$$

$$+ \int\limits_{(x,t)\in Q_{T,\varepsilon}:|x-y|^2>(t-\tau)^\beta} |G(x-y,t-\tau)| dxdt \leq$$

$$\leq C_0^* \int\limits_0^{\varepsilon^{2/\beta}} \frac{dt}{t} \int\limits_{x\in\Omega:|x-y|^2<t^\beta} |x-y|^{2-N} dx + C_0 \int\limits_0^{\varepsilon^{2/\beta}} \frac{dt}{t^{1-\beta}} \int\limits_{x\in\Omega:t^\beta<|x-y|^2<\varepsilon^2} |x-y|^{-N} dx \leq$$

$$\leq c_0 \int\limits_0^{\varepsilon^{2/\beta}} [t^{-1} \int\limits_0^{t^{\beta/2}} r dr + t^{\beta-1} \int\limits_{t^{\beta/2}}^{\varepsilon} r^{-1} dr] dt \leq c_0^* \varepsilon^2,$$

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$$\int\limits_{Q_{1T,\varepsilon}(y,\tau)} |G(x-y,t-\tau)|dS_x dt \le c_0 \int\limits_0^{\varepsilon^{2/\beta}} [t^{-1+\frac{\beta}{2}} + t^{\beta-1}(t^{-\beta/2} - \varepsilon^{-1})]dt \le c_0^* \varepsilon,$$

$$\int\limits_{Q_{1T,\varepsilon}(y,\tau)} \Big|\frac{\partial G(x-y,t-\tau)}{\partial \nu_x}\Big| dS dt \le C_1^* \int\limits_0^{\varepsilon^{2/\beta}} \frac{dt}{t} \int\limits_{x\in S: |x-y|^2<t^\beta} |x-y|^{1-N+s} dS +$$

$$+C_1 \int\limits_0^{\varepsilon^{2/\beta}} \frac{dt}{t^{1-\beta}} \int\limits_{x\in S: t^\beta<|x-y|^2<\varepsilon^2} |x-y|^{-N-1+s} dS \le$$

$$\le c_1 \int\limits_0^{\varepsilon^{2/\beta}} [t^{-1} \int\limits_0^{t^{\beta/2}} r^{s-1} dr + t^{\beta-1} \int\limits_{t^{\beta/2}}^{\varepsilon} r^{s-2} dr] dt \le$$

$$\le c_1' \int\limits_0^{\varepsilon^{2/\beta}} [t^{-1+\frac{s\beta}{2}} + t^{\beta-1}(t^{(s-2)\beta/2} - \varepsilon^{s-1})]dt \le c_1^* \varepsilon^s.$$

Here $s \in (0,1)$ – smoothness of surface S is taken into consideration. We get similar results in the case $N=2$ and for G_0 instead of G.

Lemma 3. *For arbitrary $\varphi \in D(\bar{Q}_T)$ there exists $\psi \in X(\bar{Q}_T)$ such that*

$$(\hat{L}\psi)(x,t) = \varphi(x,t), \qquad (x,t) \in Q_T.$$

Proof. As in [11] we show that the function

$$\psi(y,\tau) = \int\limits_\tau^T dt \int\limits_\Omega G_0(x-y,t-\tau)\varphi(x,t)dx$$

is desired.

Lemma 4. $G_1(x-y,t) = \frac{\partial G_0(x-y,t)}{\partial \nu_y}, \quad (x,t) \in Q_T,\ (y,\tau) \in Q_{1T},$

$$G_2(x,t) = f_{1-\beta}(t) * G_0(x,t), \quad (x,t) \in Q_T.$$

Proof. It follows from (8) and (9) that for all $\psi \in X(\bar{Q}_T)$, $(y,\tau) \in Q_{1T}$

$$\int\limits_\tau^T \int\limits_\Omega \frac{\partial G_0(x-y,t-\tau)}{\partial \nu_y} (\hat{L}\psi)(x,t)dxdt = \int\limits_\tau^T \int\limits_\Omega G_1(x-y,t-\tau)(\hat{L}\psi)(x,t)dxdt = \frac{\partial \psi(y,\tau)}{\partial \nu_y},$$

whence

$$\int\limits_\tau^T \int\limits_\Omega \Big[\frac{\partial G_0(x-y,t-\tau)}{\partial \nu_y} - G_1(x-y,t-\tau)\Big](\hat{L}\psi)(x,t)dxdt = 0.$$

By lemma 3, for every $\varphi \in D(\bar{Q}_T)$, there exists the function $\psi \in X(\bar{Q}_T)$ such that $\hat{L}\psi = \varphi$ in Q_T. Then

$$\int_{\tau}^{T}\int_{\Omega}\big[\frac{\partial G_0(x-y,t-\tau)}{\partial \nu_y} - G_1(x-y,t-\tau)\big]\varphi(x,t)dxdt = 0 \quad \forall\varphi \in D(\bar{Q}_T).$$

By known lemma of Du Bois-Reymond we get the first formula in assertion of lemma. We obtain the second formula in assertion of lemma reciprocally, from formulas (8) and (10).

Note, that by means of Fourier rows on eigen ortonormed functions $\omega_m(y)$ ($m = 1, 2, \dots$) of the Sturm-Lioville problem

$$\Delta\omega_m + \lambda_m\omega_m = 0, \quad y \in \Omega, \quad \omega_m(y) = 0, \quad y \in \partial\Omega$$

we find the main Green function

$$G_0(x-y,t-\tau) = (t-\tau)^{\beta-1}\sum_{m=0}^{\infty} E_\beta(-\lambda_m a^2(t-\tau)^\beta)\omega_m(x)\omega_m(y)$$

where $E_\beta(z) = E_{\beta^{-1}}(z,\beta) = \sum\limits_{p=0}^{\infty}\frac{z^p}{\Gamma(p\beta+\beta)}$ – Mittag-Leffler function [7] having the estimate $E_\beta(-a^2\lambda_m(t-\tau)^\beta) \le \frac{C}{1+a^2\lambda_m(t-\tau)^\beta}$.

Lemma 5. *Following mappings*

$\hat{\mathfrak{G}}_0 : D(\bar{Q}_T) \to X(\bar{Q}_T), \quad \hat{\mathfrak{G}}_1 : D(\bar{Q}_T) \to C^{\infty,(0)}(\bar{Q}_{1T}), \quad \hat{\mathfrak{G}}_2 : D(\bar{Q}_T) \to C^\infty(\bar{\Omega})$ *hold.*

Proof. . It follows from lemmas 2 and 3 that $\hat{\mathfrak{G}}_0 : D(\bar{Q}_T) \to X(\bar{Q}_T)$. By lemma 4 $G_1(x-y,t) = \frac{\partial G_0(x-y,t)}{\partial\nu_y}$, $(x,t) \in Q_T$, $(y,\tau) \in Q_{1T}$. Let $\tilde{G}_1(x-y,t) = \frac{\partial G(x-y,t)}{\partial \nu_y}$, $(x,t) \in Q_T$, $(y,\tau) \in Q_{1T}$. It follows from (18) and (20) that

$$|\tilde{G}_1(x-y,t)| \le \frac{C_1}{t^{1-\beta}|x-y|^{N+1}} \quad \text{for} \quad |x-y|^2 > t^\beta,$$

$$|\tilde{G}_1(x-y,t)| \le \frac{C_1^*}{t|x-y|^{N-1}}, \quad \text{for} \quad |x-y|^2 < t^\beta, \quad N \ge 3,$$

$$|\tilde{G}_1(x-y,t)| \le \frac{C_1^*}{t|x-y|}ln\frac{t^{\beta/2}}{|x-y|}, \quad \text{for} \quad |x-y|^2 < t^\beta, \quad N = 2$$

and the similar estimates for $G_1(x-y,t)$ hold.

We find function $G_2(x,t)$ using lemma 4 and properties of the H-functions. Being that $G_2(x,t) = f_{1-\beta}(t) * G_0(x,t)$, $\tilde{G}_2(x,t) = f_{1-\beta}(t) * G(x,t)$, we use the property (F) (in addition) about fractional differentiation of the H-function reforming the expression (12) before it.

Because in our case ($n = 0$) the condition (1.1.6) hold automatically, the property (D) (in addition) for $\lambda = \frac{|x|^2}{4a^2}$ implies

$$G(x,t) = \frac{\pi^{-N/2}t^{\beta-1}}{|x|^N}\frac{|x|^2}{4a^2}\sum_{k=0}^{\infty}\frac{1}{k!}(1-\frac{|x|^2}{4a^2})^k H^{2,0}_{1,2}\left(\frac{1}{t^\beta}\Big|\begin{matrix}(\beta,\beta) \\ (1+k,1) \quad (N/2,1)\end{matrix}\right).$$

Using property (C) (in addition) of the H-functions we get

$$G(x,t)=\tfrac{\pi^{-N/2}t^{\beta-1}}{4a^2|x|^{N-2}}\sum_{k=0}^{\infty}\tfrac{1}{k!}(1-\tfrac{|x|^2}{4a^2})^k H^{0,2}_{2,1}\left(t^\beta\Big|\begin{matrix}(-k,1) & (1-N/2,1)\\ (1-\beta,\beta) & \end{matrix}\right).$$

At last, by property (F), we find that

$$\tilde G_2(x,t)=\tfrac{\pi^{-N/2}}{4a^2|x|^{N-2}}\sum_{k=0}^{\infty}\tfrac{1}{k!}(1-\tfrac{|x|^2}{4a^2})^k H^{0,3}_{3,2}\left(t^\beta\Big|\begin{matrix}(1-\beta,\beta) & (-k,1) & (1-N/2,1)\\ (1-\beta,\beta) & (0,\beta) & \end{matrix}\right).$$

Using properties (B) and (C) of the H-function, we reform this expression toward following

$$\tilde G_2(x,t)=\tfrac{\pi^{-N/2}}{4a^2|x|^{N-2}}\sum_{k=0}^{\infty}\tfrac{1}{k!}(1-\tfrac{|x|^2}{4a^2})^k H^{0,2}_{2,1}\left(t^\beta\Big|\begin{matrix}(-k,1) & (1-N/2,1)\\ (0,\beta) & \end{matrix}\right)=$$
$$=\tfrac{\pi^{-N/2}}{4a^2|x|^{N-2}}\sum_{k=0}^{\infty}\tfrac{1}{k!}(1-\tfrac{|x|^2}{4a^2})^k H^{2,0}_{1,2}\left(\tfrac{1}{t^\beta}\Big|\begin{matrix}(1,\beta) & \\ (1+k,1) & (N/2,1)\end{matrix}\right),$$

and lastly, by property (D), we obtain that $\tilde G_2(x,t)$ is given by

$$\tilde G_2(x,t)=\frac{\pi^{-N/2}}{|x|^N}H^{2,0}_{1,2}\left(\frac{|x|^2}{4a^2t^\beta}\Big|\begin{matrix}(1,\beta) & \\ (1,1) & (N/2,1)\end{matrix}\right). \tag{21}$$

Since $\Delta^*=2-\beta$, it follows from the theorem 1.1 [15] that the functions $\tilde G_2(x,t)$ and $G_2(x,t)$ exist for all $x\neq 0$, $t>0$.

Note that function (21) is such as constructed in [3] and also [16] Green function of the Cauchy problem for the equation with regulating derivative.

As in proof of lemma 2, in respect continuity of functions $D_x^\gamma D_t^{\gamma_0}\varphi(x,t)$, $(x,t)\in\bar Q_T$ for all $\gamma_0\in\mathbb{Z}_+$ and multi-index γ, it remains to demonstrate the uniform convergence of integrals

$$\int\limits_{Q_T}|\tilde G_i(x-y,t)|dxdt,\qquad \int\limits_{Q_{1T}}|\tilde G_i(x-y,t)|dSdt,\quad i=1,2,\qquad \int\limits_{Q_{1T}}\Big|\frac{\partial\tilde G_2(x-y,t)}{\partial\nu_x}\Big|dSdt.$$

By property (E) of the differentiation of the H-function we get

$$\frac{\partial\tilde G_2(x,t)}{\partial\nu_x}=-\frac{\pi^{-N/2}}{|x|^{N+1}}H^{3,0}_{2,3}\left(\frac{|x|^2}{4a^2t^\beta}\Big|\begin{matrix}(1,\beta) & (N,2) & \\ (N+1,2) & (1,1) & (N/2,1)\end{matrix}\right)\sum_{j=1}^N\frac{x_j\nu_j(x)}{|x|}. \tag{22}$$

We find $\mu+\frac12=\frac N2$ for the H-function in (21), $\mu+\frac12=\frac N2+1$ for the H-function in (22). Then by estimate (16) we get

$$|\tilde G_2(x,t)|\le\frac{C}{|x|^N}\cdot\Big(\frac{|x|^2}{t^\beta}\Big)^{\frac{N}{2(2-\beta)}}e^{-h\left(\frac{|x|^2}{t^\beta}\right)^{\frac{1}{2-\beta}}}\le\frac{C_2}{|x|^N}\quad\text{for}\quad |x|^2>t^\beta,$$

$$\Big|\frac{\partial\tilde G_2(x,t)}{\partial\nu_x}\Big|\le\frac{C}{|x|^{N+1}}\cdot\Big(\frac{|x|^2}{t^\beta}\Big)^{\frac{N+2}{2(2-\beta)}}e^{-h\left(\frac{|x|^2}{t^\beta}\right)^{\frac{1}{2-\beta}}}\le\frac{C_3}{|x|^{N+1}}\quad\text{for}\quad |x|^2>t^\beta,$$

and by estimate (19) for $|x|^2<t^\beta$

$$\Big|\frac{\partial\tilde G_1(x,t)}{\partial\nu_x}\Big|\le\frac{C_1'}{t|x|^N},\quad |\tilde G_2(x,t)|\le\frac{C_2^*}{t^\beta|x|^{N-2}},\quad \Big|\frac{\partial G_2(x,t)}{\partial\nu_x}\Big|\le\frac{C_3^*}{t^\beta|x|^{N-1}}\quad\text{if}\quad N\ge3,$$

$$|\frac{\partial \tilde{G}_1(x,t)}{\partial \nu_x}| \leq \frac{C_1'}{t|x|^2} ln \frac{t^{\beta/2}}{|x|}, \quad |\tilde{G}_2(x,t)| \leq \frac{C_2^*}{t^\beta} ln \frac{t^{\beta/2}}{|x|},$$

$$|\frac{\partial \tilde{G}_2(x,t)}{\partial \nu_x}| \leq \frac{C_3^*}{t^\beta |x|} ln \frac{t^{\beta/2}}{|x|} \quad \text{if} \quad N=2.$$

Now, as in proof of lemma 2, in the case $N \geq 3$ we find following estimates:

$$\int\limits_{Q_{T,\varepsilon}(y,\tau)} |\tilde{G}_1(x-y,t-\tau)| dxdt \leq \int\limits_{(x,t)\in Q_{T,\varepsilon}:|x-y|^2<(t-\tau)^\beta} |\tilde{G}_1(x-y,t-\tau)| dxdt +$$

$$+ \int\limits_{(x,t)\in Q_{T,\varepsilon}:|x-y|^2>(t-\tau)^\beta} |\tilde{G}_1(x-y,t-\tau)| dxdt \leq$$

$$\leq C_1^* \int\limits_0^{\varepsilon^{2/\beta}} \frac{dt}{t} \int\limits_{x\in\Omega:|x-y|^2<t^\beta} |x-y|^{1-N} dx + C_1 \int\limits_0^{\varepsilon^{2/\beta}} \frac{dt}{t^{1-\beta}} \int\limits_{x\in\Omega:t^\beta<|x-y|^2<\varepsilon^2} |x-y|^{-N-1} dx \leq$$

$$\leq c_1 \int\limits_0^{\varepsilon^{2/\beta}} [t^{-1} \int\limits_0^{t^{\beta/2}} dr + t^{\beta-1} \int\limits_{t^{\beta/2}}^{\varepsilon} r^{-2} dr] dt \leq c_1^* \varepsilon \quad \forall (y,\tau) \in Q_{1T},$$

$$\int\limits_{Q_{T,\varepsilon}(y,0)} |\tilde{G}_2(x-y,t)| dxdt \leq$$

$$\leq C_2^* \int\limits_0^{\varepsilon^{2/\beta}} t^{-\beta} dt \int\limits_{x\in\Omega:|x-y|^2<t^\beta} |x-y|^{2-N} dx + C_2 \int\limits_0^{\varepsilon^{2/\beta}} dt \int\limits_{x\in\Omega:t^\beta<|x-y|^2<\varepsilon^2} |x-y|^{-N} dx \leq$$

$$\leq c_2 \int\limits_0^{\varepsilon^{2/\beta}} [t^{-\beta} \int\limits_0^{t^{\beta/2}} r dr + \int\limits_{t^{\beta/2}}^{\varepsilon} r^{-1} dr] dt \leq c_2' \int\limits_0^{\varepsilon^{2/\beta}} [1 + ln \frac{\varepsilon}{t^{\beta/2}}] dt \leq c_2^* \varepsilon^{2/\beta} \quad \forall y \in \bar{\Omega},$$

$$\int\limits_{Q_{1T,\varepsilon}(y,\tau)} |\tilde{G}_1(x-y,t-\tau)| dS_x dt = \int\limits_{Q_{1T,\varepsilon}(y,\tau)} |\frac{\partial G(x-y,t-\tau)}{\partial \nu}| dS_x dt \leq c_1^* \varepsilon^s$$

$$\forall (y,\tau) \in \bar{Q}_T, \quad s \in (0,1)$$

(see proof of lemma 2),

$$\int\limits_{Q_{1T,\varepsilon}(y,0)} |\tilde{G}_2(x-y,t)| dS_x dt = \int\limits_0^{\varepsilon^{2/\beta}} dt \int\limits_S |\tilde{G}_2(x-y,t)| dS_x \leq$$

$$\leq c \int\limits_0^{\varepsilon^{2/\beta}} [\int\limits_{x\in S:|x-y|^2<t^\beta} |\tilde{G}_2(x-y,t)| dS + \int\limits_{x\in S:t^\beta<|x-y|^2<\varepsilon^2} |\tilde{G}_2(x-y,t)| dS] dt \leq$$

$$\leq c \int_0^{\varepsilon^{2/\beta}} [C_2^* t^{-\beta} \int_{x\in S:|x-y|<t^\beta} |x-y|^{2-N} dS + C_2 \int_{x\in S:t^\beta<|x-y|^2<\varepsilon^2} |x-y|^{-N} dS]dt \leq$$

$$\leq \hat{c}_2 \int_0^{\varepsilon^{2/\beta}} [1+t^{-\beta/2} - \varepsilon^{-1}]dt \leq c_2^* \varepsilon^{\frac{1}{\beta}-1} \quad \forall y \in \bar{\Omega},$$

$$\int_{Q_{1T,\varepsilon}(y,0)} |\frac{\partial \tilde{G}_2(x-y,t)}{\partial \nu_x}|dS \leq$$

$$\leq C_3^* \int_0^{\varepsilon^{2/\beta}} [t^{-\beta} \int_{x\in S:|x-y|^2<t^\beta} |x-y|^{1-N} dx + C_3 \int_{x\in S:t^\beta<|x-y|^2<\varepsilon^2} |x-y|^{-1-N} dx]dt \leq$$

$$\leq c_3 \int_0^{\varepsilon^{2/\beta}} [t^{-\beta} \int_0^{t^{\beta/2}} r^{s-1} dr + \int_{t^{\beta/2}}^{\varepsilon} r^{s-3} dr]dt \leq c_3^* \varepsilon^{\frac{2(1-\beta)}{\beta}+s}, \quad \forall y \in \bar{\Omega}, \quad s \in (0,1), \quad \varepsilon > 0.$$

We get similar results in the case $N = 2$.

5. The existence and uniqueness theorem.

Theorem 1. *Under supposition (L) the unique solution $u \in D'(\bar{Q}_T)$ of the problem (1), (2) exists. It is given by the formula*

$$(u, \varphi)_{Q_T} = (F, \hat{\mathfrak{G}}_0 \varphi)_0 + (F_1, \hat{\mathfrak{G}}_1 \varphi)_1 + (F_2, \hat{\mathfrak{G}}_2 \varphi)_2 \quad \forall \varphi \in D(\bar{Q}_T). \tag{23}$$

Proof. It follows from lemma 5 that

$$\hat{\mathfrak{G}}_0 \varphi \in X(\bar{Q}_T), \quad \hat{\mathfrak{G}}_1 \varphi \in C^{\infty,(0)}(Q_{1T}), \quad \hat{\mathfrak{G}}_2 \varphi \in D(\bar{\Omega}) \quad \forall \varphi \in D(\bar{Q}_T).$$

So, the right-hand side in formula (23) makes sense and the function $u \in D'(\bar{Q}_T)$ is defined by (23).

Substituting the function (23) into identity (3), using lemma 1, we show that the function (23) satisfies the problem (1), (2):

$$(u, \hat{L}\psi)_{Q_T} = (F, \hat{\mathfrak{G}}_0(\hat{L}\psi))_0 + (F_1, \hat{\mathfrak{G}}_1(\hat{L}\psi))_1 + (F_2, \hat{\mathfrak{G}}_2(\hat{L}\psi))_2 =$$

$$= (F, \psi)_0 + (F_1, \tfrac{\partial \psi}{\partial \nu})_1 + (F_2(x), \int_0^T f_{1-\beta}(t)\psi(x,t)dt)_2 \qquad \forall \psi \in X(\bar{Q}_T).$$

Let u_1, u_2 be solutions of the problem (1), (2). It follows from the solution's definition that function $u = u_1 - u_2$ satisfies the condition

$$(u, \hat{L}\psi)_{Q_T} = 0 \qquad \forall \psi \in X(\bar{Q}_T).$$

By lemma 3, for every $\varphi \in D(\bar{Q}_T)$, there exists function $\psi \in X(\bar{Q}_T)$ such that $\hat{L}\psi = \varphi$ in $\bar{Q}_T$. Then, from the previous identity, $(u, \varphi)_{Q_T} = 0$ for every $\varphi \in D(\bar{Q}_T)$, that is $u = 0$ in $D'(\bar{Q}_T)$. The theorem is proved.

6. Final remarks. The result of the theorem 1 may by improved: by learning new properties of conjugated Green operators as in [12] we may find the character of the solution's singularities at the boundary of the domain subject to the singularities of the right-hand side of the equation and subject to the orders of the singularities of given generalized functions in initial and boundary conditions.

The obtained results have expansion to the equation

$$u_t^{(\beta)} - \sum_{j=1}^{n} b_j u_{x_j}^{(\alpha)} = F(x,t), \quad (x,t) \in \Omega \times (0,T]$$

with partial Riemann-Liouville fractional derivatives $u_{x_j}^{(\alpha)}$ and constant coefficients b_j, $j=\overline{1,n}$ under the condition $\sum\limits_{j=1}^{n} b_j p_j^{\alpha} \geq C_0$ for all $p \in \mathbb{R}^n, |p|=1$ (the estimates of the fundamental solutions were given in [17]) and also to the equation

$$u_t^{(\beta)} + a^2(-\Delta)^{\alpha/2} u = F(x,t), \quad (x,t) \in \Omega \times (0,T]$$

where the fractional n-dimensional Laplace operator $(-\Delta)^{\alpha/2}$ is defined by its Fourier transform: $\mathfrak{F}[(-\Delta)^{\alpha/2}\psi(x)] = |\lambda|^{\alpha}\mathfrak{F}[\psi(x)]$.

We may also study the generalized boundary value meaning of corresponding problems for semi-linear equations

$$u_t^{(\beta)}(x,t) - a^2\Delta u(x,t) = f(x,t,u(x,t)), \quad (x,t) \in \Omega \times (0,T]$$

by the methods of [18].

7. Addition. Here we adduce some properties of the H-function of Fox

$$H_{p,q}^{m,n}\left(z\middle|\begin{matrix}(a_1,\alpha_1) & \dots & (a_p,\alpha_p)\\ (b_1,\beta_1) & \dots & (b_q,\beta_q)\end{matrix}\right) := H(z)$$

from [15] which were applied in previous sections. We have

$$H(z) = \int\limits_{\mathbb{C}} \mathfrak{H}(s) z^{-s} ds,$$

where

$$\mathfrak{H}(s) = \frac{\Pi_{j=1}^{m}\Gamma(b_j+\beta_j s)\Pi_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\Pi_{i=n+1}^{q}\Gamma(a_i+\alpha_i s)\Pi_{j=m+1}^{p}\Gamma(1-b_j-\beta_j s)},$$

$$z^{-s} = \exp[-s(log|z| + iargz)], \quad z \neq 0, \quad i^2 = -1,$$

$\mathbb{C}$ is infinite contour which separates all the poles $b_{jl} = \frac{-b_j - l}{\beta_j}$, $1 \leq j \leq m$, $l = 0, 1, \dots$ of the functions $\Gamma(b_j + \beta_j s)$ to the left and all the poles $a_{ik} = \frac{1-a_i-k}{\alpha_i}$, $1 \leq i \leq n$, $k = 0, 1, \dots$ of the functions $\Gamma(1-a_i-\alpha_i s)$ to the right (under supposition that these poles do not coincide).

(A) – *property 2.1* [15]. The H-function is symmetric in the set of pares (a_1,α_1), $\dots$, (a_n,α_n); in $(a_{n+1},\alpha_{n+1}),\dots,(a_p,\alpha_p)$; in $(b_1,\beta_1),\dots,(b_m,\beta_m)$ and in $(b_{m+1},\beta_{m+1}),\dots,(b_q,\beta_q)$.

(B) – *property 2.2* [15]. For $n \geq 1$, $q \geq m$

$$H^{m,n}_{p,q}\left(z\middle|\begin{matrix}(a_1,\alpha_1) & \dots & (a_p,\alpha_p) & \\ (b_1,\beta_1) & \dots & (b_{q-1},\beta_{q-1}) & (a_1,\alpha_1)\end{matrix}\right)=$$

$$=H^{m,n-1}_{p-1,q-1}\left(z\middle|\begin{matrix}(a_2,\alpha_2) & \dots & (a_p,\alpha_p)\\ (b_1,\beta_1) & \dots & (b_{q-1},\beta_{q-1})\end{matrix}\right).$$

(C) – *property 2.3* [15].

$$H^{m,n}_{p,q}\left(\tfrac{1}{z}\middle|\begin{matrix}(a_1,\alpha_1) & \dots & (a_p,\alpha_p)\\ (b_1,\beta_1) & \dots & (b_q,\beta_q)\end{matrix}\right)=H^{n,m}_{q,p}\left(z\middle|\begin{matrix}(1-b_1,\beta_1) & \dots & (1-b_q,\beta_q)\\ (1-a_1,\alpha_1) & \dots & (1-a_p,\alpha_p)\end{matrix}\right).$$

(D) – *theorem 2.1* [15]. For all $\lambda\in C$, $m>0$ and under condition (1.1.6) of this work,

$$H^{m,n}_{p,q}\left(\lambda z\middle|\begin{matrix}(a_1,\alpha_1) & \dots & (a_p,\alpha_p)\\ (b_1,\beta_1) & \dots & (b_q,\beta_q)\end{matrix}\right)=$$

$$=\lambda^{\frac{b_1}{\beta_1}}\sum_{k=0}^{\infty}\frac{1}{k!}(1-\lambda^{\frac{1}{\beta_1}})^k H^{m,n}_{p,q}\left(z\middle|\begin{matrix}(a_1,\alpha_1) & \dots & (a_p,\alpha_p)\\ (b_1+k\beta_1,\beta_1) & \dots & (b_q,\beta_q)\end{matrix}\right).$$

(E) – *property 2.8* [15] *about the differentiation*. For $\omega,c\in\mathbb{C}$, $\sigma>0$, $k=0,1,\dots$

$$\left(\frac{d}{dz}\right)^k\left[z^{\omega}H^{m,n}_{p,q}\left(cz^{\sigma}\middle|\begin{matrix}(a_1,\alpha_1) & \dots & (a_p,\alpha_p)\\ (b_1,\beta_1) & \dots & (b_q,\beta_q)\end{matrix}\right)\right]=$$

$$=z^{\omega-k}H^{m,n+1}_{p+1,q+1}\left(cz^{\sigma}\middle|\begin{matrix}(-\omega,\sigma) & (a_1,\alpha_1) & \dots & (a_p,\alpha_p)\\ (b_1,\beta_1) & \dots & (b_q,\beta_q) & (k-\omega,\sigma)\end{matrix}\right)=$$

$$=(-1)^k z^{\omega-k}H^{m+1,n}_{p+1,q+1}\left(cz^{\sigma}\middle|\begin{matrix}(a_1,\alpha_1) & \dots & (a_p,\alpha_p) & (-\omega,\sigma)\\ (k-\omega,\sigma) & (b_1,\beta_1) & \dots & (b_q,\beta_q)\end{matrix}\right).$$

(F) – *theorem 2.7* [15] *about fractional differentiation*. For $a^*>0$, $\sigma\min\limits_{1\le j\le m}[\frac{Reb_j}{\beta_j}]+Re\omega>-1$, $\varrho>0$

$$f_{\varrho}(z)*\left[z^{\omega}H^{m,n}_{p,q}\left(z^{\sigma}\middle|\begin{matrix}(a_1,\alpha_1) & \dots & (a_p,\alpha_p)\\ (b_1,\beta_1) & \dots & (b_q,\beta_q)\end{matrix}\right)\right]=$$

$$=z^{\omega+\varrho}H^{m,n+1}_{p+1,q+1}\left(z^{\sigma}\middle|\begin{matrix}(-\omega,\sigma) & (a_1,\alpha_1) & \dots & (a_p,\alpha_p)\\ (b_1,\beta_1) & \dots & (b_q,\beta_q) & (-\omega-\varrho,\sigma)\end{matrix}\right).$$

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НЕОДНОРІДНІ КРАЙОВІ ЗАДАЧІ ДЛЯ РІВНЯНЬ З ДРОБОВОЮ ПОХІДНОЮ В ПРОСТОРАХ УЗАГАЛЬНЕНИХ ФУНКЦІЙ

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Доведено теорему існування та єдиності, одержано зображення за допомогою вектор-функції Гріна розв'язку задачі

$$u_t^{(\beta)}(x,t) - a^2\Delta u(x,t) = F(x,t), \quad (x,t) \in \Omega \times (0,T], \ \ a = const$$

$$u(x,t) = F_1(x,t), \quad (x,t) \in \partial\Omega \times (0,T], \qquad u(x,0) = F_2(x), \quad x \in \Omega$$

з дробовою похідною Рімана-Ліувілля $u_t^{(\beta)}$ порядку $\beta \in (0,1)$ та F, F_1, F_2 із просторів узагальнених функцій D'.

Ключові слова: похідна дробового порядку, узагальнена функція, крайова задача, вектор-функція Гріна.

НЕОДНОРОДНЫЕ КРАЕВЫЕ ЗАДАЧИ ДЛЯ УРАВНЕНИЙ С ДРОБНОЙ ПРОИЗВОДНОЙ В ПРОСТРАНСТВАХ ОБОБЩЕННЫХ ФУНКЦИЙ

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Доказана теорема существования и единственности, получено представление с помощью вектор-функции Грина решения задачи

$$u_t^{(\beta)}(x,t) - a^2\Delta u(x,t) = F(x,t), \quad (x,t) \in \Omega \times (0,T], \ \ a = const$$

$$u(x,t) = F_1(x,t), \quad (x,t) \in \partial\Omega \times (0,T], \qquad u(x,0) = F_2(x), \quad x \in \Omega$$

с дробной производной Римана-Лиувилля $u_t^{(\beta)}$ порядка $\beta \in (0,1)$ и F, F_1, F_2 из пространств обобщенных функций D'.

Ключевые слова: производная дробного порядка, обобщенная функция, краевая задача, вектор-функция Грина.