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A NONLOCAL INVERSE PROBLEM FOR THE DIFFUSION EQUATION

Mykola IVANCHOV

Ivan Franko National University of Lviv,
Universitetska Str., 1, Lviv, 79000
e-mail: ivanchov@franko.lviv.ua.

An inverse problem for nonlocal diffusion equation is considered. The existence and uniqueness conditions of the solution to the problem are obtained.

Key words: inverse problem, nonlocal diffusion equation.

1. The statement of the problem and the main results. We consider a nonlocal inverse problem for the diffusion equation with an unknown coefficient $a(s) > 0$

$$u_t = a\left(\int_0^h u(x,t)dx\right) u_{xx} + f(x,t), \quad (x,t) \in Q_T \equiv \{(x,t): \ 0 < x < h, \ 0 < t < T\}, \quad (1)$$

subject to the initial condition

$$u(x,0) = \varphi(x), \quad x \in [0,h], \quad (2)$$

the boundary conditions

$$u(0,t) = \mu_1(t), \quad u(h,t) = \mu_2(t), \quad t \in [0,T], \quad (3)$$

and the additional condition

$$a\left(\int_0^h u(x,t)dx\right) u_x(0,t) = \mu_3(t), \quad t \in [0,T]. \quad (4)$$

In the case where the function $a = a(s)$ is known, problem (1)-(3) may be considered as a mathematical model of the migration of a population when the velocity of migration depends on the total population in the considered domain [1], [2]. The inverse problem (1)-(4) corresponds to the situation when the velocity of migration is unknown.

Conditions for existence and uniqueness of solution for problem (1)-(4) are given in the following theorems.

Theorem 1. *Suppose that the following assumptions hold:*

1) $\varphi \in C^2[0,h]$, $\mu_i \in C^1[0,T]$, $i=1,2$, $\mu_3 \in C[0,T]$, $f \in C^{1,0}(\overline{Q}_T)$;

2) $\varphi'(x) > 0$, $x \in [0,h]$; $\mu_1'(t) - f(0,t) \le 0$, $\mu_2'(t) - f(h,t) \ge 0$, $\mu_3(t) > 0$, $\int_0^h f(x,t)dx - \mu_3(t) \ge 0$, $t \in [0,T]$; $f_x(x,t) \ge 0$, $(x,t) \in \overline{Q}_T$;

3) $\varphi(0) = \mu_1(0)$, $\varphi(h) = \mu_2(0)$.

Then there exists a solution (a,u) *for problem (1)-(4) from the space* $C[0,S] \times C^{2,1}(\overline{Q}_T)$ *such that* $a(s) > 0, s \in [0,S]$, *where the number* $S > 0$ *is determined by the problem data.*

Theorem 2. *Under the condition*

$$\mu_3(t) \neq 0, \quad t \in [0,T],$$

problem (1)-(4) cannot have more than one solution in the space $C[0,S] \times C^{2,1}(\overline{Q}_T)$, *where the number* $S > 0$ *is determined by the problem data.*

2. Existence of solution. In order to reduce problem (1)-(4) to an equation with respect to $a(\cdot)$, we suppose, for instant, that a solution $(a(s), u(x,t)) \in C(\mathbb{R}_+) \times C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q}_T), a(s) > 0, s > 0$ is known. Denote

$$b(t) := a\left(\int_0^h u(x,t)dx\right), \quad \theta(t) := \int_0^t b(\tau)d\tau, \quad t \in [0,T].$$

Using the Green function $G_1(x,t,\xi,\tau;u)$ we find the solution of problem (1)-(3) [3]:

$$u(x,t) = \int_0^h G_1(x,t,\xi,0;u)\varphi(\xi)d\xi + \int_0^t G_{1\xi}(x,t,0,\tau;u)b(\tau)\mu_1(\tau)d\tau -$$
$$- \int_0^t G_{1\xi}(x,t,h,\tau;u)b(\tau)\mu_2(\tau)d\tau + \int_0^h\int_0^t G_1(x,t,\xi,\tau;u)f(\xi,\tau)d\xi d\tau, \quad (x,t) \in \overline{Q}_T, \quad (5)$$

where the Green functions are defined as follows:

$$G_k(x,t,\xi,\tau;u) = \frac{1}{2\sqrt{\pi(\theta(t)-\theta(\tau))}} \sum_{n=-\infty}^{\infty}\left(\exp\left(-\frac{(x-\xi+2nh)^2}{4(\theta(t)-\theta(\tau))}\right) +\right.$$
$$\left. + (-1)^k \exp\left(-\frac{(x-\xi+2nh)^2}{4(\theta(t)-\theta(\tau))}\right)\right), \quad k \in \{1,2\}.$$

It is easy to verify the equality

$$G_{1x}(x,t,\xi,\tau;u) = -G_{2\xi}(x,t,\xi,\tau;u).$$

Taking it into account, we find from (5) the derivative u_x :

$$u_x(x,t)=\int_0^h G_2(x,t,\xi,0;u)\varphi'(\xi)d\xi-\int_0^t G_2(x,t,0,\tau;u)(\mu_1'(\tau)-f(0,\tau))d\tau+$$
$$+\int_0^t G_2(x,t,h,\tau;u)(\mu_2'(\tau)-f(h,\tau))d\tau+\int_0^h\int_0^t G_2(x,t,\xi,\tau;u)f_\xi(\xi,\tau)d\xi d\tau,$$
$$(x,t)\in\overline{Q}_T. \qquad (6)$$

Since

$$\int_0^h G_2(x,t,\xi,0;u)\varphi'(\xi)d\xi\geq\min_{[0,h]}\varphi'(x)>0,$$

under the assumptions 2) of the Th.1 we have

$$u_x(x,t)>0,\quad (x,t)\in\overline{Q}_T.$$

Hence, we obtain from (4) the following equation with respect to $b(t)$:

$$b(t)=\mu_3(t)\Bigg(\int_0^h G_2(0,t,\xi,0;u)\varphi'(\xi)d\xi-\int_0^t G_2(0,t,0,\tau;u)(\mu_1'(\tau)-f(0,\tau))d\tau+$$
$$+\int_0^t G_2(0,t,h,\tau;u)(\mu_2'(\tau)-f(h,\tau))d\tau+\int_0^h\int_0^t G_2(0,t,\xi,\tau;u)f_\xi(\xi,\tau)d\xi d\tau\Bigg)^{-1},$$
$$t\in[0,T]. \qquad (7)$$

In order to prove the existence of solution of the equation (7) we apply the Schauder fixed-point theorem. Firstly we establish a priori estimates of solutions of equation (7).

It is evident that from the assumptions 2) of Th.1 we have

$$u_x(0,t)\geq\int_0^h G_2(0,t,\xi,0;u)\varphi'(\xi)d\xi\geq\min_{[0,h]}\varphi'(x)>0,\quad t\in[0,T].$$

Thus we obtain

$$b(t)\leq B_1,\quad t\in[0,T], \qquad (8)$$

where the constant $B_1>0$ is determined by the problem data.

In order to evaluate solutions of equation (7) from above, we use the inequalities [3]

$$G_2(0,t,0,\tau;u)\leq C_1+\frac{C_2}{\sqrt{\theta(t)-\theta(\tau)}},\quad G_2(0,t,h,\tau;u)\leq C_3,\quad t\in[0,T],$$

where the constants $C_k, k\in\{1,2,3\}$, do not depend on $b(t)$. Taking into account these inequalities we find

$$u_x(0,t)\leq C_4+C_5\int_0^t\frac{d\tau}{\sqrt{\theta(t)-\theta(\tau)}},\quad t\in[0,T].$$

Thus, we obtain from (7) the following inequality:

$$b(t) \geq \frac{C_6}{C_4 + C_5 \int\limits_0^t \frac{d\tau}{\sqrt{\theta(t)-\theta(\tau)}}}, \quad t \in [0,T]. \tag{9}$$

Denote $b_{\min} := \min\limits_{[0,T]} b(t)$. It is easy to deduce from (9) the inequality

$$b_{\min} \geq \frac{C_6}{C_4 + \frac{C_7}{\sqrt{b_{\min}}}}.$$

Solving it, we obtain

$$b_{\min} \geq B_0 > 0,$$

where the constant B_0 is determined by the given constants. It means that we have the estimation of $b(t)$ from below:

$$b(t) \geq B_0, \quad t \in [0,T]. \tag{10}$$

Rewrite equation (7) as follows

$$b(t) = Pb(t), \quad t \in [0,T], \tag{11}$$

where the operator P is determined by the right-hand side of equation (7). Denote $\mathcal{N} := \{b \in C([0,T]) : B_0 \leq b(t) \leq B_1\}$. In view of estimates (8) and (10), it is evident that the operator P maps $\mathcal{N}$ into itself. It is shown in [3] that the operator P is compact on the set $\mathcal{N}$. Applying the Schauder fixed-point theorem, we deduce the existence of a solution $b = b(t)$ of equation (7) such that $b \in C([0,T])$ and $b(t) > 0,\ t \in [0,T]$. Substituting into (5) the function $b(t)$, found from (11), we obtain the known function $u = u(x,t)$.

To finish the proof of the existence of solution for problem (1)-(4), we have to determine the function $a = a(s)$. Denote

$$q(t) = \int\limits_0^h u(x,t)dx.$$

Note that the function $q = q(t)$ is known. From the definition of $b(t)$ we have

$$b(t) = a(q(t)), \quad t \in [0,T].$$

Find the derivative

$$q'(t) = \int\limits_0^h u_t(x,t)dx.$$

Using equation (1) and taking into account the assumptions of Th.1, we find

$$q'(t) = b(t)u_x(1,t) - \mu_3(t) + \int\limits_0^h f(x,t)dx > 0, \quad t \in [0,T].$$

It means that there exists a continuous function $q^{-1}(s)$ defined on the interval $[0, S]$ such that

$$q(q^{-1}(s)) \equiv s, \quad s \in [0, S],$$

where $S = \max\limits_{[0,T]} q(t)$. From this we find the function $a(s)$:

$$a(s) = b(q^{-1}(s)), \quad s \in [0, S]. \tag{12}$$

Thus, the proof of Theorem 1 is complete.

3. Uniqueness of solution. Suppose that there exist two solutions (a_k, u_k), $k \in \{1, 2\}$, of problem (1)-(4). Denote

$$b_k(t) := a_k \left(\int\limits_0^h u_k(x,t)dx \right), \ k \in \{1,2\}, \ b(t) := b_1(t) - b_2(t), \ u(x,t) := u_1(x,t) - u_2(x,t).$$

The functions $(b(t), u(x, t))$ verify the following system

$$u_t = b_1(t)u_{xx} + b(t)u_{2xx}(x,t), \quad (x,t) \in Q_T, \tag{13}$$

$$u(x, 0) = 0, \quad x \in [0, h], \tag{14}$$

$$u(0,t) = 0, \quad u(h,t) = 0, \quad t \in [0,T], \tag{15}$$

$$b_1(t)u_x(0,t) = -b(t)u_{2x}(0,t), \quad t \in [0,T]. \tag{16}$$

Find the solution of problem (13)-(15):

$$u(x,t) = \int\limits_0^t \int\limits_0^h G_1^{(1)}(x,t,\xi,\tau)b(\tau)u_{2\xi\xi}(\xi,\tau)d\xi d\tau, \quad (x,t) \in \overline{Q}_T, \tag{17}$$

where $G_1^{(1)}(x, t, \xi, \tau)$ is the Green function for the equation

$$u_t = b_1(t)u_{xx}$$

with boundary conditions (15). After substituting (17) into (16) we obtain the following equation with respect to unknown $b(t)$:

$$b(t) = -\frac{b_1(t)b_2(t)}{\mu_3(t)} \int\limits_0^t \int\limits_0^h G_{1x}^{(1)}(0,t,\xi,\tau)b(\tau)u_{2\xi\xi}(\xi,\tau)d\xi d\tau, \quad (x,t) \in \overline{Q}_T. \tag{18}$$

This is a second kind homogeneous Volterra integral equation with integrable kernel. Consequently, $b(t) \equiv 0, \ t \in [0, T]$. It follows from (17) that $u(x, t) \equiv 0, \ (x, t) \in \overline{Q}_T$. The proof of Theorem 2 is complete.

References

1. *Chipot M.* On the asymptotic behaviour of some nonlocal problems / *Chipot M., Lovat B.* // Positivity. – 1999. – Vol. 3. – P. 65-81.
2. *Chipot M.* Asymptotic behaviour of some nonlocal diffusion problems / *Chipot M., Molinet L.* // Applicable Analysis. – 2001. – Vol. 80. – P. 279-315.
3. *Ivanchov M.* Inverse Problems for Equations of Parabolic Type / *Ivanchov M.* – Lviv: VNTL Publishers, 2003.

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ПРО НЕЛОКАЛЬНУ ОБЕРНЕНУ ЗАДАЧУ ДЛЯ РІВНЯННЯ ДИФУЗІЇ

Микола ІВАНЧОВ

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, Львів, 79000
e-mail: ivanchov@franko.lviv.ua.

Розглянуто обернену задачу для нелокального рівняння дифузії. Отримано умови існування та єдиності її розв'язку.

Ключові слова: обернена задача, нелокальне рівняння дифузії.

О НЕЛОКАЛЬНОЙ ОБРАТНОЙ ЗАДАЧЕ ДЛЯ УРАВНЕНИЯ ДИФФУЗИИ

Николай ИВАНЧОВ

Львовский национальный университет имени Ивана Франко,
ул. Университетская, 1, Львов, 79000
e-mail: ivanchov@franko.lviv.ua.

Рассмотрена обратная задача для нелокального уравнения диффузии. Получены условия существования и единственности её решения.

Ключевые слова: обратная задача, нелокальное уравнение диффузии.