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ONE-TO-ONE CORRESPONDENCES BETWEEN CLASSES OF OBJECTS OF THE CATEGORY $\bigcup \mathcal{R} - \mathcal{M}od$ AND RADICAL FUNCTORS

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In the paper we study some properties of pretorsion and torsion classes in the category of all modules over different rings and state the relations between these classes and idempotent preradical, radical, pretorsion and torsion functors.

Key words: category, functor, preradical functor, radical functor, pretorsion functor, torsion functor, pretorsion class, torsion class.

1. Introduction. An intensive studying of hereditary torsions began due to the papers of P. Gabriel [1] and J.-M. Maranda [2], where there was stated a bijective correspondence between hereditary torsions and radical filters of rings. In 1966 S.E. Dickson established the correspondence between the classes of objects of an abelian category and the idempotent preradicals [3]. Also, the torsion theory was used in the theory of rings of quotients. All these results were summarized and unified in the monographs of A.I. Kashu [4] and B. Stenström [5].

In 1967, (on the International conference in Riga) L.A. Skorniakov posed the problem of constructing the radical in the category of all modules over different rings. In 1971, O.L. Horbachuk solved this problem and established connection between this radical and radicals in the concrete categories of modules. These investigations were continued by O.L. Horbachuk and N.Yu. Burban in 2008 [6].

In this paper we carry some theorems proved by A.I. Kashu in [4] (for a module category) over the category of all modules over different rings (which is not abelian).

2. Main result. Throughout the whole text, all rings are considered to be associative with $1 \neq 0$ and all modules are left unitary [7, 8]. Let R be a ring. The category of left R–modules will be denoted by $R-$Mod. All necessary definitions and theorems of the Torsion theory and Category theory can be found in [4, 5, 9, 10].

A pair of mappings $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$, where $\varphi\colon R_1 \to R_2$ is an onto ring homomorphism, and $\psi\colon M_1 \to M_2$ is a homomorphism of abelian groups, is called a *semilinear transformation* if $\forall r \in R_1, \forall m_1 \in M_1$

$$\psi(r_1 m_1) = \varphi(r_1)\psi(m_1).$$

Let $\bigcup \mathcal{R} - \mathcal{M}od$ be a category of all left modules over all rings. More precisely, the objects of the category $\bigcup \mathcal{R} - \mathcal{M}od$ are the pairs (R, M), where R is a ring, M is a left module; the set of morphisms $Hom((R_1, M_1), (R_2, M_2))$ is defined as a quotient set of a collection of all semilinear transformations $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$ by the equivalence relation $\sim$ such that $(\varphi, \psi) \sim (\varphi', \psi')$, if $\psi = \psi'$, and the product of morphisms is defined naturally. The class, determined by the semilinear transformation (φ, ψ) will be denoted by $\widetilde{(\varphi, \psi)}$, or, more frequently, (φ, ψ). It is easy to verify that $\bigcup \mathcal{R} - \mathcal{M}od$ is a category.

A class $\widetilde{(\varphi, \psi)}$ is a monomorphism (resp., an epimorphism) in the category $\bigcup \mathcal{R} - \mathcal{M}od$ if ψ is a monomorphism (resp., an epimorphism) in the category of abelian groups. The objects $(R, 0)$ and the morphisms $\widetilde{(\varphi, 0)}$ are zero objects and zero morphisms in the category $\bigcup \mathcal{R} - \mathcal{M}od$, respectively (see [6]).

Now recall some definitions [5, 9, 10].

Definition 1. *Let $\mathcal{A}$ be a category with zero object and zero morphisms, and let $\alpha\colon A \to B$. We will call a morphism $u\colon K \to A$ the kernel of α if $\alpha u = 0$, and if for every morphism $u'\colon K' \to A$ such that $\alpha u' = 0$ we have a unique morphism $\gamma\colon K' \to K$ such that $u\gamma = u'$. The object K is denoted by $Ker\,\alpha$ and the morphism u is denoted by $ker\,\alpha$.*

A morphism $v\colon B \to E$ is called the cokernel of α if $v\alpha = 0$, and if for every morphism $v'\colon B \to E'$ such that $v'\alpha = 0$ we have a unique morphism $\delta\colon E \to E'$ such that $\delta v = v'$. The object E is denoted by $Coker\,\alpha$ and the morphism v is denoted by $coker\,\alpha$.

Definition 2. *If $A' \to A$ is the kernel of some morphism then we call A' a normal subobject of A (or an ideal). Dually, if $A \to A''$ is the cokernel of some morphism, then we call A'' a conormal quotient object of A.*

Definition 3. *A category $\mathcal{A}$ is abelian if*

A0. *$\mathcal{A}$ has the zero object.*
A1. *For every pair of objects there is a direct product and*
A1.* *a direct sum.*
A2. *Every map has the kernel and*
A2.* *the cokernel.*
A3. *Every monomorphism is the kernel of a map.*
A3.* *Every epimorphism is the cokernel of a map.*

In the paper [6] some properties of the category $\bigcup \mathcal{R} - \mathcal{M}od$ were be stated:

1. For arbitrary objects (R_i, M_i), $i \in I$, of the category $\bigcup \mathcal{R} - \mathcal{M}od$ there exists the direct product belonging to $\bigcup \mathcal{R} - \mathcal{M}od$. In particular, the object (R, M), where $R = \prod_{i \in I} R_i$, is a direct product of the rings R_i and $M = \prod_{i \in I} M_i$ is a direct product of the abelian groups M_i, and the morphisms $(s_i, \pi_i)\colon (\prod_{i \in I} R_i, \prod_{i \in I} M_i) \to (R_i, M_i)$, where s_i is the projection of $\prod_{i \in I} R_i$ onto R_i and π_i is the projection of $\prod_{i \in I} M_i$ onto M_i determine a direct product of the objects (R_i, M_i), $i \in I$, in the category $\bigcup \mathcal{R} - \mathcal{M}od$.

2. Every morphism of $\bigcup \mathcal{R}-\mathcal{M}od$ has the kernel. In particular, let $(\varphi,\psi)\colon (R_1,M_1) \to (R_2,M_2)$ be a morphism of the category $\bigcup \mathcal{R}-\mathcal{M}od$. Then the object $(R_1, Ker\psi)$ with a monomorphism $(1_{R_1}, i)\colon (R_1, Ker\psi) \to (R_1,M_1)$, where i is a canonical monomorphism, is the kernel of the morphism (φ,ψ).

3. Every morphism of $\bigcup \mathcal{R}-\mathcal{M}od$ has the cokernel. In particular, let $(\varphi,\psi)\colon (R_1,M_1) \to (R_2,M_2)$ be a morphism of the category $\bigcup \mathcal{R}-\mathcal{M}od$. Then the object $(R_2, M_2/\psi(M_1))$ with an epimorphism $(1_{R_2},\pi)\colon (R_2,M_2) \to (R_2, M_2/\psi(M_1))$, where π is a canonical epimorphism of R_2–modules, is the cokernel of the morphism (φ,ψ) in the category $\bigcup \mathcal{R}-\mathcal{M}od$.

We want to verify whether the category $\bigcup \mathcal{R}-\mathcal{M}od$ is abelian or not, and if it is not abelian, then what axioms are not satisfied.

We know that the category $\bigcup \mathcal{R}-\mathcal{M}od$ satisfies axioms A0, A1, A2 and A2*.

Now, consider the objects $(\mathbb{Z}_2, M_1)$ and $(\mathbb{Z}_3, M_2)$ of the category $\bigcup \mathcal{R}-\mathcal{M}od$ (e. g. $M_1=\mathbb{Z}_2$, $M_2=\mathbb{Z}_3$). Suppose that (R,M) is the direct sum of these objects. In such a case we must have onto ring homomorphisms $\varphi_1\colon \mathbb{Z}_2 \to R$ and $\varphi_2\colon \mathbb{Z}_3 \to R$. Hence $R = Im\varphi_1 = \mathbb{Z}_2$ and $R = Im\varphi_2 = \mathbb{Z}_3$ $(R \neq 0)$. Thus we obtain a contradiction. It means that in the category $\bigcup \mathcal{R}-\mathcal{M}od$ there exist two objects for which the direct sum does not exist. Therefore, axiom A1* is not satisfied.

We want to show that no monomorphism $(\varphi,\psi)\colon (\mathbb{Z},\mathbb{Z}_2) \to (\mathbb{Z}_2,\mathbb{Z}_2)$ is the kernel of any morphism. Let $(\alpha,\beta)\colon (\mathbb{Z}_2,\mathbb{Z}_2) \to (R,M)$ be an arbitrary morphism, then $Ker(\alpha,\beta) = (\mathbb{Z}_2, Ker\beta)$. But all kernels are isomorphic. This provides a contradiction. Hence axiom A3 is not satisfied.

Similarly we can show for example that no epimorphism $(\varphi,\psi)\colon (\mathbb{Z},\mathbb{Z}_4) \to (\mathbb{Z}_6,\mathbb{Z}_2)$ is the cokernel of any morphism. Hence axiom A3* is not satisfied.

In the category $\bigcup \mathcal{R}-\mathcal{M}od$ we shall state the following theorem which holds for abelian categories.

Theorem 1. *Let $(\varphi,\psi)\colon (R,M) \to (R',M')$ be a morphism in $\bigcup \mathcal{R}-\mathcal{M}od$. Then*
1). $ker(coker(ker(\varphi,\psi))) = ker(\varphi,\psi)$;
2). $coker(ker(coker(\varphi,\psi))) = coker(\varphi,\psi)$.

Proof. 1). $ker(coker(ker(\varphi,\psi))) = ker\left(coker\left((R, Ker\psi) \xrightarrow{(1_R,i)} (R,M)\right)\right) =$
$= ker\left((R,M) \xrightarrow{(1_R,\pi)} (R, M/Ker\psi)\right) = (R, Ker\psi) \xrightarrow{(1_R,i)} (R,M) = ker(\varphi,\psi)$.
2). $coker(ker(coker(\varphi,\psi))) = coker\left(ker\left((R',M') \xrightarrow{(1_{R'},\pi)} (R', M'/Im\ \psi)\right)\right) =$
$= coker\left((R', Im\ \psi) \xrightarrow{(1_{R'},i)} (R',M')\right) = (R',M') \xrightarrow{(1_{R'},\pi)} (R', M'/Im\ \psi) =$
$= coker(\varphi,\psi)$. □

Let $\mathcal{A}$ be an arbitrary concrete category with zero objects and zero morphisms. Recall that a category is called concrete if all objects are (structured) sets, morphisms from A to B are (structure preserving) mappings from A to B, the composition of morphisms is the composition of mappings, and the identities are the identity mappings [12].

Definition 4. *A preradical functor (or simply a preradical) on $\mathcal{A}$ is a subfunctor of the identity functor on $\mathcal{A}$. In other words, a preradical functor T assigns to each object A a* *subobject $T(A)$ in such a way that the diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] & B \\ T(A) \arrow[u, "i_1"] \arrow[r, "T(\alpha)"] & T(B) \arrow[u, "i_2"'] \end{tikzcd},$$

where i_1, i_2 are monomorphisms, is commutative.

Definition 5. *A preradical functor T is called idempotent if $T(T(A)) = T(A)$ for every $A \in Ob(\mathcal{A})$.*

To a preradical functor T one can associate two classes of objects of $\mathcal{A}$, namely

$$\mathcal{T}_T = \{A \mid T(A) = A\},$$

$$\mathcal{F}_T = \{A \mid T(A) = 0\}.$$

Remark 1. Throughout the whole text, all preradical functors on the category $\bigcup \mathcal{R} - \mathcal{M}od$ are considered to be such that theirs restrictions on every category $R-$Mod are preradical functors, i. e. $T(R, M) = (R, T_R(M))$, where T_R is the restriction of the functor T on the category $R-$Mod.

Theorem 2. *Let T be a preradical functor on the category $\bigcup \mathcal{R} - \mathcal{M}od$, then*

1) The class $\mathcal{T}_T$ is closed under quotient objects and direct sums (if they exist).

2) The class $\mathcal{F}_T$ is closed under subobjects and direct products.

3) $\mathcal{T}_T \bigcap \mathcal{F}_T = \{(R, 0)\}$.

4) $\mathrm{Hom}\,((R_1, M_1), (R_2, M_2)) = (\varphi, 0)$ *for every $(R_1, M_1) \in \mathcal{T}_T$, $(R_2, M_2) \in \mathcal{F}_T$.*

Proof. 1) Let $(R_1, M_1) \in \mathcal{T}_T$, $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$ be an epimorphism ((R_2, M_2) be a quotient object of (R_1, M_1)). Consider the commutative diagram

$$\begin{tikzcd} (R_1, M_1) \arrow[r, "(\varphi,\psi)"] & (R_2, M_2) \\ T(R_1, M_1) \arrow[u] \arrow[r, "T(\varphi,\psi)"] & T(R_2, M_2) \arrow[u] \end{tikzcd}.$$

Since (φ, ψ) is an epimorphism in the category $\bigcup \mathcal{R} - \mathcal{M}od$, it means that φ is a surjective ring homomorphism and ψ is a surjective homomorphism of abelian groups. Hence $(\varphi, \psi)(R_1, M_1) = (R_2, M_2)$. Since $(R_1, M_1) \in \mathcal{T}_T$, it follows that $T(R_1, M_1) = (R_1, M_1)$. By the definition of a preradical functor we obtain $T(R_2, M_2) = (R_2, M_2)$, i. e. $(R_2, M_2) \in \mathcal{T}_T$.

Let $(R_i, M_i), i \in I$, be an arbitrary family of objects in $\mathcal{T}_T$, for which the direct sum exists. Consider the canonical monomorphisms $(\varphi_i, \psi_i)\colon (R_i, M_i) \to \bigoplus\limits_{i\in I}(R_i, M_i)$ and the commutative diagram

$$\begin{tikzcd}
(R_i, M_i) \arrow[r, "(\varphi_i,\psi_i)"] & \bigoplus\limits_{i\in I}(R_i, M_i) \\
T(R_i, M_i) \arrow[u] \arrow[r, "T(\varphi_i,\psi_i)"] & T\left(\bigoplus\limits_{i\in I}(R_i, M_i)\right) \arrow[u]
\end{tikzcd}.$$

We obtain that $(\varphi_i,\psi_i)(R_i,M_i) \subseteq T\left(\bigoplus\limits_{i\in I}(R_i,M_i)\right)$ for every $i\in I$, and by the definition of the direct sum $\bigoplus\limits_{i\in I}(R_i,M_i) = T\left(\bigoplus\limits_{i\in I}(R_i,M_i)\right)$.

2) Let $(\varphi,\psi)\colon (R_1,M_1)\to(R_2,M_2)$ be a monomorphism, $(R_2,M_2)\in\mathcal{F}_T$. Consider the commutative diagram

$$\begin{tikzcd}
(R_1, M_1) \arrow[r, "(\varphi,\psi)"] & (R_2, M_2) \\
T(R_1, M_1) \arrow[u] \arrow[r, "T(\varphi,\psi)"] & T(R_2, M_2) \arrow[u]
\end{tikzcd}.$$

Since $T(R_2,M_2)=(R_2,0)$, and by the definition of a preradical functor

$$(\varphi,\psi)\left(T(R_1,M_1)\right)\subseteq T(R_2,M_2),$$

it follows that $(\varphi,\psi)\left(T(R_1,M_1)\right)\subseteq(R_2,0)$. Furthermore, (φ,ψ) is a monomorphism in $\bigcup\mathcal{R}-\mathcal{M}od$ (φ is a surjective ring homomorphism and ψ is an injective homomorphism of abelian groups), thereby $T(R_1,M_1)=(R_1,0)$, i. e. $(R_1,M_1)\in\mathcal{F}_T$.

Let $(R_i,M_i), i\in I$, be an arbitrary family of objects in $\mathcal{F}_T$, i. e. $T(R_i,M_i)=(R_i,0)$ $\forall i\in I$. Consider the canonical epimorphisms $(\varphi_i,\psi_i)\colon \prod\limits_{i\in I}(R_i,M_i)\to(R_i,M_i)$ and the commutative diagram

$$\begin{tikzcd}
\prod\limits_{i\in I}(R_i, M_i) \arrow[r, "(\varphi_i,\psi_i)"] & (R_i, M_i) & \\
T\left(\prod\limits_{i\in I}(R_i, M_i)\right) \arrow[u] \arrow[r, "T(\varphi_i,\psi_i)"] & T(R_i, M_i) \arrow[u] \arrow[r, equal] & (R_i, 0)
\end{tikzcd}.$$

Hence $(\varphi_i,\psi_i)\left(T\left(\prod\limits_{i\in I}(R_i,M_i)\right)\right)\subseteq T(R_i,M_i)$ for every $i\in I$. Thus $T\left(\prod\limits_{i\in I}(R_i,M_i)\right)$ $\subseteq\prod\limits_{i\in I}\left(T(R_i,M_i)\right)=\prod\limits_{i\in I}(R_i,0)$, i. e. $T\left(\prod\limits_{i\in I}(R_i,M_i)\right)\in\mathcal{F}_T$.

3) Suppose that $(R,M)\in\mathcal{T}_T\bigcap\mathcal{F}_T$. It means that $T(R,M)=(R,M)$ and $T(R,M)=(R,0)$. Therefore $\mathcal{T}_T\bigcap\mathcal{F}_T=(R,0)$.

4) Let $(R_1, M_1) \in \mathcal{T}_T$, $(R_2, M_2) \in \mathcal{F}_T$, $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$. Consider the commutative diagram

$$\begin{array}{ccc} (R_1, M_1) & \xrightarrow{(\varphi,\psi)} & (R_2, M_2) \\ \uparrow & & \uparrow \\ T(R_1, M_1) & \xrightarrow{T(\varphi,\psi)} & T(R_2, M_2) \end{array} .$$

Since $T(R_1, M_1) = (R_1, M_1)$ and $T(R_2, M_2) = (R_2, 0)$, we obtain the commutative diagram

$$\begin{array}{ccc} (R_1, M_1) & \xrightarrow{(\varphi,\psi)} & (R_2, M_2) \\ \uparrow & & \uparrow \\ (R_1, M_1) & \xrightarrow{T(\varphi,\psi)} & (R_2, 0) \end{array} .$$

Thus $(\varphi, \psi) = (\varphi, 0)$. Hence $Hom\,((R_1, M_1), (R_2, M_2)) = (\varphi, 0)$ for every $(R_1, M_1) \in \mathcal{T}_T$, $(R_2, M_2) \in \mathcal{F}_T$. $\square$

Definition 6. *A class $\mathcal{P}$ of objects of the category $\bigcup \mathcal{R} - \mathcal{M}od$ is called a pretorsion class if it is closed under quotient objects and direct sums (if they exist).*

Theorem 3. *There is a bijective correspondence between the idempotent preradical functors of $\bigcup \mathcal{R} - \mathcal{M}od$ and the pretorsion classes of objects of $\bigcup \mathcal{R} - \mathcal{M}od$.*

Proof. ($\Longrightarrow$) Let T be an idempotent preradical functor, $\mathcal{T}_T = \{(R, M) \mid \ T(R, M) = (R, M)\}$. By Theorem 2, the class $\mathcal{T}_T$ is closed under quotient objects and direct sums (if they exist), hence $\mathcal{T}_T$ is a pretorsion class for the preradical functor T.

($\Longleftarrow$) Let $\mathcal{P}$ be a pretorsion class, (R, M) be an arbitrary object of $\mathcal{P}$. Consider $t_R(R, M) = \sum\limits_{i \in I} \{(R, M_i)\}$, where (R, M_i) are normal subobjects of (R, M), $(R, M_i) \in \mathcal{P}$. We know that t_R is a preradical functor on every category $R-$Mod (see e. g. [4]). Since the pretorsion class $\mathcal{P}$ is closed under quotient objects it follows that the preradical functors t_R generate a preradical functor T on $\bigcup \mathcal{R} - \mathcal{M}od$ (see [6]). $\square$

Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary concrete categories with zero objects and zero morphisms.

Definition 7. *Let T_1 and T_2 be functors from a category $\mathcal{A}$ to a category $\mathcal{B}$. The functor T_1 is called a subfunctor of the functor T_2 (denote $T_1 \leqslant T_2$) if $T_1(A)$ is a subobject of $T_2(A)$ (denote $T_1(A) \subseteq T_2(A)$) for every $A \in Ob(\mathcal{A})$ and the following diagram*

$$\begin{array}{ccc} T_1(A_1) & \xrightarrow{T_1(\varphi)} & T_1(A_2) \\ {\scriptstyle i_1}\downarrow & & {\scriptstyle i_2}\downarrow \\ T_2(A_1) & \xrightarrow{T_2(\varphi)} & T_2(A_2) \end{array}$$

is commutative for every morphism $\varphi\colon A_1 \to A_2$, $A_1, A_2 \in Ob(\mathcal{A})$.

Definition 8. *A functor T_1 is called a normal subfunctor of the functor T_2 if $T_1(A)$ is a normal subobject of $T_2(A)$ for every $A \in Ob(\mathcal{A})$.*

As a rule we will consider the cases, when the categories $\mathcal{A}$ and $\mathcal{B}$ coincide.

Definition 9. *Let $\mathcal{A}$ be a category, T_1 and T_2 be functors on $\mathcal{A}$ such that T_1 is a normal subfunctor of T_2. A factor-functor T_2/T_1 is a functor such that $(T_2/T_1)(A) = T_2(A)/T_1(A)$ $\forall A \in Ob(\mathcal{A})$ and the following diagram is commutative*

$$\begin{array}{ccc} T_1(A_1) & \xrightarrow{T_1(\varphi)} & T_1(A_2) \\ {\scriptstyle i_1}\downarrow & & {\scriptstyle i_2}\downarrow \\ T_2(A_1) & \xrightarrow{T_2(\varphi)} & T_2(A_2) \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ T_2(A_1)/T_1(A_1) & \longrightarrow & T_2(A_2)/T_1(A_2), \end{array}$$

where i_1, i_2 are normal monomorphisms, π_1, π_2 are conormal epimorphisms.

Definition 10. *A preradical functor T on a category $\mathcal{A}$ is called a radical functor if $T(I/T) = 0$, where I is an identity functor.*

Definition 11. *A class $\mathcal{T}_T$ is said to be closed under extensions if for every exact sequence $(R', 0) \to (R', M') \to (R, M) \to (R'', M'') \to (R'', 0)$ with (R', M') and (R'', M'') in $\mathcal{T}_T$, also $(R, M) \in \mathcal{T}_T$. A class $\mathcal{T}_T$ is said to be closed under normal extensions if it is closed under extensions and $R' = R = R''$.*

Theorem 4. *Let T be an idempotent radical functor on the category $\bigcup \mathcal{R} - \mathcal{M}od$, then*

1) The class $\mathcal{T}_T$ is closed under quotient objects, direct sums (if they exist) and normal extensions.

2) The class $\mathcal{F}_T$ is closed under subobjects, direct products and normal extensions.

3) $\mathcal{T}_T = \{(R, M) \in Ob(\bigcup \mathcal{R} - \mathcal{M}od) \mid Hom\,((R, M), (R', M')) = (\varphi, 0)$ $\forall\ (R', M') \in \mathcal{F}_T\}$

4) $\mathcal{F}_T = \{(R', M') \in Ob(\bigcup \mathcal{R} - \mathcal{M}od) \mid Hom\,((R, M), (R', M')) = (\varphi, 0)$ $\forall\ (R, M) \in \mathcal{T}_T\}$

Proof. 1) The class $\mathcal{T}_T$ is closed under quotient objects and direct sums (if they exist) (see theorem 2).

Since we want to prove that $\mathcal{T}_T$ is closed under normal extensions, it is sufficient to work in the category $R - M$od. But in every category $R - M$od the class $\mathcal{T}_T$ is closed under extensions (see [4]).

2) Apply Theorem 2 (2), and the proof of the closure under normal extensions is the same as previous.

3) ($\Longrightarrow$) Let $(R, M) \in \mathcal{T}_T$, then by Theorem 2 $Hom\,((R, M), (R', M')) = (\varphi, 0)$ for every $(R', M') \in \mathcal{F}_T$.

($\Longleftarrow$) Now let (R, M) be a such object that $Hom\,((R, M), (R', M')) = (\varphi, 0)$ for every $(R', M') \in \mathcal{F}_T$. By the condition, T is a radical functor, so

$$T((R, M)/T(R, M)) = (R, 0),$$

i. e. $(R, M)/T(R, M) \in \mathcal{F}_T$. Consider the morphism $(\varphi, \psi)\colon (R, M) \to (R, M)/T(R, M)$. Since $(R, M)/T(R, M) \in \mathcal{F}_T$ and (φ, ψ) is an surjective homomorphism it follows that $(R, M)/T(R, M) = (R, 0)$. Hence $T(R, M) = (R, M)$, i. e. $(R, M) \in \mathcal{T}_T$.

4) ($\Longrightarrow$) Let $(R', M') \in \mathcal{F}_T$, then $Hom\,((R, M), (R', M')) = (\varphi, 0)$ for every $(R, M) \in \mathcal{T}_T$ (by Theorem 2).

($\Longleftarrow$) Conversely, let (R', M') be a such object that $Hom\,((R, M), (R', M')) = (\varphi, 0)$ for every $(R, M) \in \mathcal{T}_T$. Assume that $T(R', M') = (R', M^*)$. Consider a morphism $(\varphi, \psi)\colon (R, M) \to (R', M')$, and the commutative diagram

$$\begin{array}{ccc} (R,M) & \xrightarrow{(\varphi,\psi)} & (R',M') \\ \uparrow & & \uparrow \\ T(R,M) & \xrightarrow{T(\varphi,\psi)} & T(R',M') \end{array}.$$

$T(R, M) \in \mathcal{T}_T$ because T is an idempotent, i. e. $T(T(R', M')) = T(R', M')$. Since $(\varphi, \psi)(R, M) = (R, 0)$, then there exists a nonzero monomorphism $T(R', M') \to (R', M')$. This provides a contradiction. $\square$

Definition 12. *A class $\mathcal{P}$ of objects of the category $\bigcup \mathcal{R} - \mathcal{M}od$ is called a torsion class if it is closed under quotient objects, direct sums (if they exist), and normal extensions.*

Theorem 5. *There is a bijective correspondence between idempotent radical functors of $\bigcup \mathcal{R} - \mathcal{M}od$ and torsion classes of objects of $\bigcup \mathcal{R} - \mathcal{M}od$.*

Proof. ($\Longrightarrow$) Let T be an idempotent radical functor,

$$\mathcal{T}_T = \{(R, M) \mid T(R, M) = (R, M)\}.$$

By Theorem 4, the class $\mathcal{T}_T$ is closed under quotient objects, direct sums (if they exist), and normal extensions, hence $\mathcal{T}_T$ is a torsion class for the radical functor T.

($\Longleftarrow$) Let $\mathcal{P}$ be a torsion class. We know that the class $\mathcal{P}$ generates the preradical functor T on the category $\bigcup \mathcal{R} - \mathcal{M}od$(see Theorem 3). Since the torsion class $\mathcal{P}$ is closed under normal extensions it follows that on every category $R-$Mod the restriction of T is a radical functor. But in the paper [6] it was proved that a preradical functor on $\bigcup \mathcal{R} - \mathcal{M}od$ is a radical functor if and only if its restriction in every category $R-$Mod is a radical functor. $\square$

Definition 13. *Let T be an idempotent preradical functor of the category $\bigcup \mathcal{R} - \mathcal{M}od$, $(R, M) \in \mathcal{T}_T$. If every normal subobject of (R, M) belongs to $\mathcal{T}_T$, then T is called a pretorsion functor.*

Definition 14. *A pretorsion functor is called a torsion functor if it is a radical one.*

Theorem 6. *There is a bijective correspondence between pretorsion functors of $\bigcup \mathcal{R} - \mathcal{M}od$ and pretorsion classes of objects of $\bigcup \mathcal{R} - \mathcal{M}od$, closed under normal subobjects.*

Proof. ($\Longrightarrow$) Let T be a pretorsion functor, $\mathcal{T}_T = \{(R, M) \mid T(R, M) = (R, M)\}$. By Theorem 2 the class $\mathcal{T}_T$ is closed under quotient objects and direct sums (if they exist). By the definition of a pretorsion functor the class $\mathcal{T}_T$ is closed under normal subobjects, hence $\mathcal{T}_T$ is a pretorsion class, closed under normal subobjects.

($\Longleftarrow$) Let $\mathcal{P}$ be a pretorsion class, closed under normal subobjects. In every category $R-$Mod the class $\mathcal{P}$ generates a pretorsion functor (see e. g. [4]). By Theorem 3, if a class is closed under quotient objects and direct sums (if they exist), then it generates a preradical functor of the category $\bigcup\mathcal{R}-\mathcal{M}od$. But if a preradical functor of $\bigcup\mathcal{R}-\mathcal{M}od$ is a pretorsion functor on every $R-$Mod, then it is a pretorsion functor of $\bigcup\mathcal{R}-\mathcal{M}od$. $\square$

Theorem 7. *There is a bijective correspondence between the torsion functors of* $\bigcup\mathcal{R}-\mathcal{M}od$ *and the torsion classes of objects of* $\bigcup\mathcal{R}-\mathcal{M}od$, *closed under normal subobjects.*

Proof. ($\Longrightarrow$) See the proofs of Theorems 5 and 6.

($\Longleftarrow$) Let $\mathcal{P}$ be a torsion class, closed under normal subobjects. We know that the class $\mathcal{P}$ generates a radical functor T on the category $\bigcup\mathcal{R}-\mathcal{M}od$ (see the proof of the theorem 5). Since $\mathcal{P}$ is closed under normal subobjects, it follows that $\mathcal{P}$ is closed under subobjects in every category $R-$Mod. Thus, on every $R-$Mod it generates a torsion functor. And we know that if a radical functor of $\bigcup\mathcal{R}-\mathcal{M}od$ is a torsion functor on every $R-$Mod, then it is a torsion functor of $\bigcup\mathcal{R}-\mathcal{M}od$. $\square$

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ВЗАЄМНО ОДНОЗНАЧНІ ВІДПОВІДНОСТІ МІЖ КЛАСАМИ ОБ'ЄКТІВ КАТЕГОРІЇ $\bigcup \mathcal{R} - \mathcal{M}od$ І РАДИКАЛЬНИМИ ФУНКТОРАМИ

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Вивчено деякі властивості прерадикальних і радикальних класів у категорії всіх модулів над різними кільцями, а також є зв'язок між цими класами та ідемпотентними прерадикальними, радикальними, препеpіодичними та періодичними функторами.

Ключові слова: категорія, функтор, прерадикальний функтор, радикальний функтор, препеpіодичний функтор, періодичний функтор, прерадикальний клас, радикальний клас.

ВЗАИМНО ОДНОЗНАЧНЫЕ СООТВЕТСТВИЯ МЕЖДУ КЛАССАМИ ОБЬЕКТОВ КАТЕГОРИИ $\bigcup \mathcal{R} - \mathcal{M}od$ И РАДИКАЛЬНЫМИ ФУНКТОРАМИ

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Изучено некоторые свойства предрадикальных и радикальных классов в категории всех модулей над разными кольцами, а также устанавливается связь между этими классами и предрадикальными, радикальными, предпериодическими и периодическими функторами.

Ключевые слова: категория, функтор, предрадикальный функтор, радикальный функтор, предпериодический функтор, периодический функтор, предрадикальный класс, радикальный класс.