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ON SOLVABILITY OF MODEL NONHOMOGENEOUS PROBLEMS FOR SEMILINEAR PARABOLIC EQUATIONS WITH VARIABLE EXPONENTS OF NONLINEARITY

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The nonhomogeneous initial-boundary value Dirichlet problem for the equation

$$u_t - \Delta u + g(x,t)|u|^{q(x,t)-2}u = f(x,t)$$

in cylinder domain is considered. If the condition $1 < q_0 \le q(x,t) \le q^0 < 2$ is satisfied, then the existence of the mild solution of this problem is proved.

Key words: nonlinear parabolic equation, nonhomogeneous problem, initial-boundary value problem, variable exponent of nonlinearity, generalized Lebesgue and Sobolev spaces, mild solution, Green function.

1. Introduction. In this paper, we continue our study of semilinear parabolic equations from [1]. Let $n \in \mathbb{N}$ and $T > 0$ be fixed numbers, $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$, $Q_{0,T} = \Omega \times (0,T]$.

We seek the mild solution u of the following problem

$$u_t - \Delta u + g(x,t)|u|^{q(x,t)-2}u = f(x,t), \quad (x,t) \in Q_{0,T}, \tag{1}$$

$$u|_{\partial\Omega\times(0,T)} = d(x,t), \tag{2}$$

$$u|_{t=0} = u_0(x). \tag{3}$$

Here $\Delta u = u_{x_1x_1} + u_{x_2x_2} + \ldots + u_{x_1x_1}$ is the Laplace operator, g, q, f are real valued functions on $Q_{0,T}$, d is a real valued function on $\partial\Omega \times (0,T)$, u_0 is a real valued function on Ω.

Under some conditions for data-in of problem (1)-(3), using the Green function technique, we prove a solvability of these problems with variable exponents of nonlinearity.

The existence of the Green function and its various properties it is well known (see for instance [2], [3], [4], and the references given there). The various problems with variable exponents of nonlinearity and homogeneous boundary conditions are investigate in [5], [6], [7], [8], [9], [10]. The Green function technique for some semilinear parabolic problems with variable exponents of nonlinearity and homogeneous boundary conditions are investigated in [1], [11], [12]. The extensive literature is reviewed in [1].

This paper is organized as follows. In Part 2, we give some definitions and main results. Auxiliary facts are given in Part 3. In Part 4, we prove the main theorem.

2. Motivation of the definition and main results.

2.1. Case of the linear problem. First we consider the problem

$$u_t - \Delta u = h(x,t) \quad \text{in} \quad Q_{0,T}, \tag{4}$$

$$u|_{\partial\Omega\times(0,T)} = 0, \quad u|_{t=0} = 0. \tag{5}$$

Recall that a function $\mathsf{G} = \mathsf{G}(x,t,\xi,s)$ $x,\xi \in \Omega$, $t > s \geq 0$, is called the Green function (see [13, p. 1118]) of the Dirichlet mixed problem for parabolic equation (4) if for every $(\xi,s) \in Q_{0,T}$ the function G satisfies homogeneous equation (4), and the boundary condition $\mathsf{G}|_{\partial\Omega\times(0,T)} = 0$ with respect to the variables $x \in \Omega$, $t > s \geq 0$, and for every function $\varphi \in C(\overline{\Omega})$ we have

$$\lim_{t\to s+0} \int\limits_{\Omega} \mathsf{G}(x,t,\xi,s)\varphi(\xi)\, d\xi = \varphi(x).$$

It is well known that the solution of problem (4)-(5) is

$$u(x,t) = \int\limits_0^t \int\limits_{\Omega} \mathsf{G}(x,t,\xi,s)h(\xi,s)\, d\xi ds. \tag{6}$$

Further we consider a problem with a nonhomogeneous initial condition, i.e.

$$u_t - \Delta u = 0 \quad \text{in} \quad Q_{0,T}, \tag{7}$$

$$u|_{\partial\Omega\times(0,T)} = 0, \tag{8}$$

$$u|_{t=0} = u_0(x). \tag{9}$$

Its solution is

$$u(x,t) = \int\limits_{\Omega} \mathsf{G}(x,t,\xi,0)u_0(\xi)\, d\xi, \quad (x,t) \in Q_{0,T}. \tag{10}$$

Now let us consider a problem with a nonhomogeneous boundary condition, i.e.

$$u_t - \Delta u = 0 \quad \text{in} \quad Q_{0,T}, \tag{11}$$

$$u|_{\partial\Omega\times(0,T)} = d(x,t), \tag{12}$$

$$u|_{t=0} = 0. \tag{13}$$

We assume that there exists a function $\widehat{d} = \widehat{d}(x,t)$ such that

$$\widehat{d} \in C^{2,1}_{x,t}(Q_{0,T}) \cap C(\overline{Q_{0,T}}), \quad \widehat{d}|_{\partial\Omega\times(0,T)} = d(x,t). \tag{14}$$

If we replace u by $\widehat{d} + u^*$, we obtain a new problem

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$$u^*_t - \Delta u^* = h^*(x,t) \quad \text{in} \quad Q_{0,T}, \tag{15}$$

$$u^*|_{\partial\Omega\times(0,T)} = 0, \tag{16}$$

$$u^*|_{t=0} = u^*_0(x), \tag{17}$$

where $h^* = -\widehat{d}_t + \Delta\widehat{d}$, $u^*_0 = -\widehat{d}|_{t=0}$. According to (6), (10), we have

$$u^*(x,t) = \int\limits_{\Omega} \mathsf{G}(x,t,\xi,0)u^*_0(\xi)\, d\xi + \int\limits_0^t \int\limits_{\Omega} \mathsf{G}(x,t,\xi,s)h^*(\xi,s)\, d\xi ds. \tag{18}$$

Therefore the solution of problem (11)-(13) is

$$u(x,t) = \widehat{d}(x,t) - \int\limits_{\Omega} \mathsf{G}(x,t,\xi,0)\widehat{d}(\xi,0)\, d\xi -$$

$$- \int\limits_0^t \int\limits_{\Omega} \mathsf{G}(x,t,\xi,s)(\widehat{d}_t(\xi,s) - \Delta\widehat{d}(\xi,s))\, d\xi ds, \quad (x,t) \in Q_{0,T}. \tag{19}$$

Finally let us consider the general Dirichlet mixed problem for the model equation

$$u_t - \Delta u = h(x,t) \quad \text{in} \quad Q_{0,T}, \tag{20}$$

$$u|_{\partial\Omega\times(0,T)} = d(x,t), \tag{21}$$

$$u|_{t=0} = u_0(x). \tag{22}$$

Let again $\mathsf{G} = \mathsf{G}(x,t,\xi,s)$ be the Green function of the homogeneous Diriclet problem (4)-(5); $\widehat{d} = \widehat{d}(x,t)$ be a function such that $\widehat{d}|_{\partial\Omega\times(0,T)} = d(x,t)$;

$$d^*(x,t) = \widehat{d}(x,t) - \int\limits_{\Omega} \mathsf{G}(x,t,\xi,0)\widehat{d}(\xi,0)\, d\xi -$$

$$- \int\limits_0^t \int\limits_{\Omega} \mathsf{G}(x,t,\xi,s)(\widehat{d}_t(\xi,s) - \Delta\widehat{d}(\xi,s))\, d\xi ds. \tag{23}$$

Clearly if we replace (20)-(22) by three problems, i.e. (4)-(5), (7)-(9), and (11)-(13), then, using formulas (6), (10), (19), we get a solution of problems (20)-(22) such that

$$u(x,t) = d^*(x,t) + \int\limits_{\Omega} \mathsf{G}(x,t,\xi,0)u_0(\xi)\, d\xi + \int\limits_0^t \int\limits_{\Omega} \mathsf{G}(x,t,\xi,s)h(\xi,s)\, d\xi ds. \tag{24}$$

Let $p \geq 1$ be a fixed number. We introduce the following notion (see for comparison [14], [15]). A function u is called a mild solution of problem (20)-(22) if $u \in L^p(Q_{0,T})$, u satisfies equality (24) for a.e. $(x,t) \in Q_{0,T}$. In the same manner we define a mild solution of the mixed problem for parabolic equation (1). More precisely, we replace the function h by $f - g|u|^{q(x,t)-2}u$ and consider equality (24) as nonlinear integral equations. A solution of this integral equation is a solution for problem (1)-(3).

2.2. Main results. Suppose that the following conditions hold:
(G): $g \in L^\infty(Q_{0,T})$;
(Q): $q \in L^\infty(Q_{0,T})$, $1 < q_0 \le q(x,t) \le q^0 < +\infty$, where

$$q_0 \equiv \operatorname*{ess\,inf}_{(x,t)\in Q_{0,T}} q(x,t), \qquad q^0 \equiv \operatorname*{ess\,sup}_{(x,t)\in Q_{0,T}} q(x,t);$$

(DFU): for some $p > 1$ we take $u_0 \in L^p(\Omega)$, $f \in L^p(Q_{0,T})$, $\widehat{d} \in W^{1,p}(Q_{0,T}) \cap L^p(0,T;W^{2,p}(\Omega)) \cap C([0,T];L^p(\Omega))$ such that

$$\widehat{d}|_{\partial\Omega\times(0,T)} = d(x,t).$$

Let $\mathsf{G}(x,t,\xi,s)$ be the Green function of the mixed problem (4)-(5) such that the Gaussian estimate

$$|\mathsf{G}(x,t,\xi,s)| \le \frac{M_1\ \chi_{(0,+\infty)}(t-s)}{(t-s)^{\frac{n}{2}}}\ e^{-M_2\frac{|x-\xi|^2}{t-s}} \tag{25}$$

holds. Here $M_1, M_2 > 0$ are constants, $\chi_{(0,+\infty)}(z)$ is the indicator function of the segment $(0,+\infty)$. Let d^* be given by (23), where $\widehat{d}$ is taken from condition **(DFU)**.

Now we provide Definition of the solution and the main theorem.

Definition 1. *A real valued function $u \in L^p(Q_{0,T})$ is called a mild solution of problems (1)-(3) if for a.e. $(x,t) \in Q_{0,T}$ the equality*

$$u(x,t) = d^*(x,t) + \int_\Omega \mathsf{G}(x,t,\xi,0)u_0(\xi)\ d\xi + \int_0^t\int_\Omega \mathsf{G}(x,t,\xi,s)f(\xi,s)\ d\xi ds -$$

$$-\int_0^t\int_\Omega \mathsf{G}(x,t,\xi,s)g(\xi,s)|u(\xi,s)|^{q(\xi,s)-2}u(\xi,s)\ d\xi ds \tag{26}$$

holds.

Theorem 1. *Suppose that conditions **(Q)** with $q^0 < 2$, **(DFU)** with $p \in (1+\frac{n}{2}, +\infty)$, and **(G)** are satisfied. If there exists the Green function of problem (1)-(3) such that the Gaussian estimate (25) is executed, then problem (1)-(3) has a mild solution.*

In particular, the conditions of Theorem 1 are the conditions on $\partial\Omega$. For example, we recall some facts from [4]. First let us consider a nondecreasing bounded half-additive function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\frac{\omega(t)}{t} \le 2\,\frac{\omega(s)}{s}, \qquad \frac{\omega(t)}{t^\gamma} \le \widetilde{C}\,\frac{\omega(s)}{s^\gamma}, \qquad 0 < s < t,$$

where $\gamma \in (\frac{1}{2}, 1)$, $\widetilde{C} > 0$. By definition, put

$$F(t) = \int_0^t \frac{\omega(s)}{s}\ ds, \quad t \ge 0, \qquad \Phi(\tau) = \int_0^\tau \frac{\omega(t)}{t}\ dt, \quad \tau \ge 0.$$

Suppose that there exist constants $\sigma > 0$, $\widehat{C} > 0$ such that

$$F(\sigma) < +\infty, \quad \Phi(\sigma) < +\infty, \qquad \int\limits_t^{\sigma} \frac{\omega(s)}{s^2}\, ds \le \widehat{C}\, \frac{F(t)}{t}, \quad 0 < t < \sigma.$$

Let $m, N \in \mathbb{N}$ be fixed numbers, $\mathcal{O} \subset \mathbb{R}^N$. Let $C^m(\mathcal{O})$ be the space of all functions φ which, together with all their partial derivatives $D^\alpha \varphi$ of orders $|\alpha| \le m$, are continuous on $\mathcal{O}$. The set of all functions $\psi \in C^m(\mathcal{O})$ such that

$$|\psi|_m^\omega := \sum_{|\alpha| \le m} \sup_{y \in \mathcal{O}} |D^\alpha \psi(y)| + \sum_{|\beta| = m} \sup_{y,z \in \mathcal{O}} \frac{|D^\beta \psi(y) - D^\beta \psi(z)|}{\omega(|y-z|)} < +\infty,$$

is called the Dini space and is denoted by $C^{(m,\omega)}(\mathcal{O})$.

Further for our domain $\Omega \subset \mathbb{R}^n$ we put $S = \partial\Omega$. We say that the surface S belongs to the Dini sets $C^{(m,\omega)}$ if $S = \bigcup\limits_{k=1}^{\ell} S_k$, where for every $k \in \{1, \ldots, \ell\}$ the open surface S_k is given by the rule $x_j = \varphi_j^k(x_j')$, $x_j' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \mathcal{O}^k$, and the following conditions hold:

1) $\mathcal{O}^k$ is a bounded domain from $\mathbb{R}^{n-1}$; 2) $\varphi_j^k \in C^{(m,\omega)}(\mathcal{O}^k)$.

Finally we recall Theorem.

Example 1 (see Theorem 2.8 [4, p. 136])**.** If $\partial\Omega \in C^{(2,\omega)}$, then there exists the Green function of problem (1)-(3) such that the Gaussian estimate (25) is executed.

3. Auxiliary facts.

3.1. A fixed point theorem. First we recall some definitions. Let X, Y be normed spaces, $\mathcal{A} : X \to Y$. A subset $\mathcal{M}$ of a normed space X is called a compact set if every sequence of points in $\mathcal{M}$ has a subsequence converging in X to an element of $\mathcal{M}$ (see [16, p. 6]). $\mathcal{M}$ is called a precompact set if its closure $\overline{\mathcal{M}}$ (in the norm topology) is compact (see [16, p. 7]).

An operator $\mathcal{A} : X \to Y$ is called a compact operator if $\mathcal{A}(\mathcal{M})$ is precompact in Y whenever $\mathcal{M}$ is bounded in X (see [16, p. 8]). $\mathcal{A}$ is a completely continuous operator if it is continuous and compact (see [16, p. 9]). Every bounded linear operator is continuous. Hence every compact linear operator is completely continuous.

Further let us consider some examples.

Example 2. 1) Clearly if $A, B : X \to Y$ are completely continuous operators, and $\alpha, \beta \in \mathbb{R}$, then the operator $\alpha A + \beta B$ also is completely continuous.

2) (see Lemma 1 [1, p. 81]) Let X, Y, Z be Banach spaces, $A : Y \to Z$, $B : X \to Y$. If A is completely continuous, and B is bounded continuous, then the composition of the operators $A \circ B : X \to Z$ is a completely continuous operator.

3) Clearly, every constant operator, i.e. an operator $C : X \to X$ such that

$$\exists\, y \in X \quad \forall\, x \in X : \quad Cx = y \tag{27}$$

is a nonlinear (if $y \ne 0$) completely continuous operator.

Suppose that G is a measurable on $Q_{0,T} \times Q_{0,T}$ real valued function such that almost everywhere in $Q_{0,T} \times Q_{0,T}$ we have Gaussian estimate (25). In particular, $\mathsf{G} = 0$ if $t \le s$.

By $L^p(\Omega)$, where $p \geq 1$, we denote the standard Lebesgue space with respect to the norm

$$||u; L^p(\Omega)|| = \Big(\int\limits_{\Omega} |u(x)|^p \, dx\Big)^{1/p}.$$

In the same manner we define the space $L^p(Q_{0,T})$.

Example 3. (Lemma 3 [1, p. 83]). Suppose the measurable real valued function G satisfies Gaussian estimate (25), the integral operator $\mathcal{J}$ is given by equality

$$(\mathcal{J}z)(x,t) = \int\limits_0^t \int\limits_{\Omega} \mathsf{G}(x,t,\xi,s) z(\xi,s) \, d\xi ds, \quad (x,t) \in Q_{0,T}. \tag{28}$$

If $p > 1 + \frac{n}{2}$, then the operator $\mathcal{J} : L^p(Q_{0,T}) \to L^p(Q_{0,T})$ is completely continuous.

The following theorem plays main role in the proof of our results.

Proposition 1 (the Schauder fixed point theorem)**.** *(* [17, p. 229]*). Let X be a Banach space, $\mathcal{A} : X \to X$ be a completely continuous operator, $\mathcal{M} \subset X$ be a nonempty bounded closed convex set. If $\mathcal{A}(\mathcal{M}) \subset \mathcal{M}$, then $\mathcal{A}$ has a fixed point.*

3.2. Some properties of the integral operators. We will need the following Proposition and Lemmas.

Proposition 2. *(Lemma 2* [1, p. 83]*). Suppose that $r \in [1, +\infty)$ is a fixed number, G is a measurable real valued function such that Gaussian estimate (25) holds,*

$$J_r(x,t,s) = \int\limits_{\Omega} |G(x,t,\xi,s)|^r \, d\xi, \qquad \widehat{J}_r(\xi,t,s) = \int\limits_{\Omega} |G(x,t,\xi,s)|^r \, dx \tag{29}$$

for a.e. $x, \xi \in \Omega$, $0 \leq s < t$. Then there exists a constant $C(r) > 0$ such that

$$0 \leq J_r(x,t,s) \leq \frac{C(r)}{(t-s)^{\frac{n}{2}(r-1)}}, \qquad 0 \leq \widehat{J}_r(\xi,t,s) \leq \frac{C(r)}{(t-s)^{\frac{n}{2}(r-1)}}. \tag{30}$$

Remark 1. From (29), (30) it follows that if the measurable real valued function G satisfies Gaussian estimate (25), then there exists a constant $M > 0$ such that

$$\operatorname*{ess\,sup}_{\substack{(x,t) \in Q_{0,T} \\ s \in (0,T)}} \int\limits_{\Omega} |\mathsf{G}(x,t,\xi,s)| \, d\xi \leq M, \quad \operatorname*{ess\,sup}_{\substack{t \in (0,T) \\ (\xi,s) \in Q_{0,T}}} \int\limits_{\Omega} |\mathsf{G}(x,t,\xi,s)| \, dx \leq M. \tag{31}$$

Lemma 1. *Suppose that the measurable real valued function G satisfies Gaussian estimate (25) and the integral operator $\mathcal{J}_0$ is given by the equality*

$$(\mathcal{J}_0 v)(x,t) = \int\limits_{\Omega} G(x,t,\xi,0) v(\xi) \, d\xi, \quad (x,t) \in Q_{0,T}. \tag{32}$$

Then for every $p \in (1, +\infty)$ the linear operator $\mathcal{J}_0 : L^p(\Omega) \to L^p(Q_{0,T})$ is bounded (therefore it is a continuous operator). In addition, there exists a constant $L_0 > 0$ such that for every $v \in L^p(\Omega)$ the estimate

$$||\mathcal{J}_0 v; L^p(Q_{0,T})|| \leq L_0 ||v; L^p(\Omega)|| \tag{33}$$

holds (notice that L_0 depends on p but does not depend on v).

Proof. Clearly the complete proof of this Lemma follows from estimate (33). Let $p, p' \in (1, +\infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$. Using the Hölder inequality, we obtain

$$||\mathcal{J}_0 v; L^p(Q_{0,T})||^p = \int\limits_{Q_{0,T}} \Big| \int\limits_{\Omega} \mathsf{G}(x,t,\xi,0) v(\xi)\, d\xi \Big|^p dxdt \le$$

$$\le \int\limits_{Q_{0,T}} \Big| \int\limits_{\Omega} |\mathsf{G}|_{s=0}|^{\frac{1}{p'}}\, |\mathsf{G}|_{s=0}|^{\frac{1}{p}} |v|\, d\xi \Big|^p dxdt \le \int\limits_{Q_{0,T}} \Big(\int\limits_{\Omega} |\mathsf{G}|_{s=0}|\, d\xi \Big)^{\frac{p}{p'}} \Big(\int\limits_{\Omega} |\mathsf{G}|_{s=0}|\, |v|^p\, d\xi \Big)\, dxdt.$$

Taking into account estimates (31), the equality $\frac{p}{p'} = p-1$, and the Fubini theorem, we get

$$||\mathcal{J}_0 v; L^p(Q_{0,T})||^p \le M^{\frac{p}{p'}} \int\limits_{Q_{0,T}} dxdt \int\limits_{\Omega} |\mathsf{G}|_{s=0}|\, |v|^p\, d\xi = M^{p-1} \times$$

$$\times \int\limits_0^T dt \int\limits_{\Omega} dx \int\limits_{\Omega} |\mathsf{G}(x,t,\xi,0)|\, |v(\xi)|^p\, d\xi = M^{p-1} \int\limits_{\Omega} \Big(\int\limits_0^T dt \int\limits_{\Omega} |\mathsf{G}(x,t,\xi,0)|\, dx \Big) |v(\xi)|^p\, d\xi \le$$

$$\le M^{p-1} \int\limits_{\Omega} \Big(\int\limits_0^T M\, dt \Big) |v(\xi)|^p\, d\xi \le T\, M^p \int\limits_{\Omega} |v(\xi)|^p\, d\xi.$$

Therefore (33) is true, and the Lemma is proved. □

Lemma 2. *Suppose that the measurable real valued function G satisfies Gaussian estimate (25), the integral operator $\mathcal{J}$ is given by equality (28). Then for every $p \in (1, +\infty)$ the linear operator $\mathcal{J} : L^p(Q_{0,T}) \to L^p(Q_{0,T})$ is bounded (therefore it is a continuous operator). In addition, there exists a constant $L > 0$ such that for every $z \in L^p(Q_{0,T})$ the estimate*

$$||\mathcal{J} z; L^p(Q_{0,T})|| \le L||z; L^p(Q_{0,T})|| \tag{34}$$

holds (notice that the constant L depends on p but does not depend on z).

Proof. Again it is enough to show only estimate (34). Let $p, p' \in (1, +\infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$. Using the methods of Lemma 1, we obtain

$$||\mathcal{J} z; L^p(Q_{0,T})||^p = \int\limits_{Q_{0,T}} \Big| \int\limits_{Q_{0,t}} \mathsf{G}(x,t,\xi,s) z(\xi,s)\, d\xi ds \Big|^p dxdt \le$$

$$\le \int\limits_{Q_{0,T}} \Big| \int\limits_{Q_{0,t}} |\mathsf{G}|^{\frac{1}{p'}}\, |\mathsf{G}|^{\frac{1}{p}} |z|\, d\xi ds \Big|^p dxdt \le$$

$$\le \int\limits_{Q_{0,T}} \Big(\int\limits_{Q_{0,t}} |\mathsf{G}|\, d\xi ds \Big)^{\frac{p}{p'}} \Big(\int\limits_{Q_{0,t}} |\mathsf{G}|\, |z|^p\, d\xi ds \Big)\, dxdt.$$

Using estimate (31), equality $\frac{p}{p'} = p-1$, and the Fubini theorem, we get

$$||\mathcal{J} z; L^p(Q_{0,T})||^p \le (T\, M)^{\frac{p}{p'}} \int\limits_{Q_{0,T}} dxdt \int\limits_{Q_{0,t}} |\mathsf{G}|\, |z|^p\, d\xi ds =$$

$$= (T\,M)^{p-1}\int_0^T dt\int_\Omega dx\int_0^t ds\int_\Omega |\mathsf{G}(x,t,\xi,s)|\,|z(\xi,s)|^p\,d\xi =$$

$$= (T\,M)^{p-1}\int_0^T ds\int_\Omega \Big(\int_s^T dt\int_\Omega |\mathsf{G}(x,t,\xi,s)|\,dx\Big)\,|z(\xi,s)|^p\,d\xi \le$$

$$\le (T\,M)^{p-1}\int_0^T ds\int_\Omega \Big(\int_s^T M\,dt\Big)\,|z(\xi,s)|^p\,d\xi \le (T\,M)^p\int_{Q_{0,T}} |z(\xi,s)|^p\,d\xi ds.$$

This inequality yields (34). The Lemma is proved. □

3.3. Generalized Lebesgue spaces and Nemytskii operator with the variable exponent of nonlinearity. First let us introduce some notation and functional spaces. Suppose that $q \in L^\infty(Q_{0,T})$ satisfies condition **(Q)**. Consider a linear subspace $L^{q(x,t)}(Q_{0,T})$ of the space $L^1(Q_{0,T})$ which consists of v such that $\rho_q(v, Q_{0,T}) < \infty$, where

$$\rho_q(v, Q_{0,T}) := \int_{Q_{0,T}} |v(x,t)|^{q(x,t)}\,dxdt.$$

It is the Banach space with respect to the Luxemburg norm

$$||v; L^{q(x,t)}(Q_{0,T})|| := \inf\{\lambda > 0 \mid \rho_q(v/\lambda, Q_{0,T}) \le 1\}$$

(see [19, p. 599]) and it is called a generalized Lebesgue space. This space was first introduced by W. Orlicz in [18]. Note that if $q(x,t) = q_0 = \mathrm{const}$ for a.e. $(x,t) \in Q_{0,T}$, then $||\cdot; L^{q(x,t)}(Q_{0,T})||$ equals to the standard norm $||\cdot; L^{q_0}(Q_{0,T})||$ of the Lebesgue space $L^{q_0}(Q_{0,T})$. According to [19, p. 599], the conjugate space $[L^{q(x,t)}(Q_{0,T})]^*$ equals $L^{q'(x,t)}(Q_{0,T})$, where the function q' is defined by the equality $\frac{1}{q(x,t)} + \frac{1}{q'(x,t)} = 1$ for a.e. $(x,t) \in Q_{0,T}$. Note also that the set $C(\overline{Q}_{0,T})$ is dense in $L^{q(x,t)}(Q_{0,T})$ (see [19, p. 603]). In addition, the continuous embedding $L^{q(x,t)}(Q_{0,T}) \circlearrowleft L^{r(x,t)}(Q_{0,T})$ holds if $q(x,t) \ge r(x,t)$ (see [19, p. 599-600]).

Suppose that the function q satisfies condition **(Q)**,

$$S_q(s) = \begin{cases} s^{q_0}, & s \in [0,1], \\ s^{q^0}, & s > 1, \end{cases} \qquad S_{1/q}(s) = \begin{cases} s^{1/q^0}, & s \in [0,1], \\ s^{1/q0}, & s > 1, \end{cases} \tag{35}$$

where the constant q_0, q^0 are given by **(Q)** (see Lemma 1 [8, p. 168], Remark 3.1 [10, p. 453]). We will need the following Propositions.

Proposition 3. *(Lemma 4* [1, p. 85]*). Suppose that conditions **(Q)** with* $q^0 < 2$, ***(G)**, are satisfied, the Nemytskii operator* $\mathcal{N}$ *is defined by the formula*

$$(\mathcal{N}z)(x,t) = g(x,t)|z(x,t)|^{q(x,t)-2}z(x,t), \quad (x,t) \in Q_{0,T}. \tag{36}$$

Then for every number $p \in [1, +\infty)$ *the operator* $\mathcal{N} : L^p(Q_{0,T}) \to L^p(Q_{0,T})$ *is bounded and continuous. In addition, there exists a constant* $N_p > 0$ *such that for every* $u, v \in L^p(Q_{0,T})$ *we have*

$$||\mathcal{N}u - \mathcal{N}v; L^p(Q_{0,T})|| \le N_p\Big\{S_{1/h}\Big(||u - v; L^p(Q_{0,T})||^p\Big)\Big\}^{1/p}, \tag{37}$$

$$||\mathcal{N}u; L^p(Q_{0,T})|| \leq N_p \Big\{ S_{1/h}\Big(||u; L^p(Q_{0,T})||^p \Big) \Big\}^{\frac{1}{p}}, \tag{38}$$

where $S_{1/h}$ is a continuous function from (35), and $h(x,t) = \frac{1}{q(x,t)-1}$.

4. Proof of Theorem 1. We define the operators $\mathcal{D}$, $\mathcal{K}$, $\mathcal{N}$, $\mathcal{J}$, $\mathcal{A}$ by the following identities. $\mathcal{D}$ is a constant (see (27)) operator such that

$$(\mathcal{D}z)(x,t) = d^*(x,t), \quad (x,t) \in Q_{0,T}, \tag{39}$$

where d^* is given by (23), $\widehat{d}$ is taken from **(DFU)**. $\mathcal{K}$ is a constant operator such that

$$(\mathcal{K}z)(x,t) = \int_\Omega \mathsf{G}(x,t,\xi,0)u_0(\xi)\, d\xi + \int_0^t \int_\Omega \mathsf{G}(x,t,\xi,s)f(\xi,s)\, d\xi ds. \tag{40}$$

The nonlinear Nemytskii operator $\mathcal{N}$ is given by (36). The linear integral operator $\mathcal{J}$ is given by (28). The operator $\mathcal{A}$ is a combination of the operators $\mathcal{D}$, $\mathcal{K}$, $\mathcal{N}$, $\mathcal{J}$, namely

$$\mathcal{A} = \mathcal{D} + \mathcal{K} - \mathcal{J} \circ \mathcal{N}. \tag{41}$$

Taking into account these notation, we rewrite equality (26) as

$$u = \mathcal{A}u. \tag{42}$$

Then the existence of the solution to problems (1)-(3) means the existence of the fixed point of the operator $\mathcal{A}$. We will show that conditions of the Schauder theorem are satisfied.

Step 1. Lemmas 1, 2 mean that $\mathcal{K}(L^p(Q_{0,T})) \subset L^p(Q_{0,T})$, and

$$||\mathcal{K}z; L^p(Q_{0,T})|| \leq L_0||u_0; L^p(\Omega)|| + L||f; L^p(Q_{0,T})||, \tag{43}$$

where $L_0 > 0$ is taken from (33), $L > 0$ is taken from (34). Similarly $\mathcal{D}(L^p(Q_{0,T})) \subset$ $\subset L^p(Q_{0,T})$, and the estimate

$$||\mathcal{D}z; L^p(Q_{0,T})|| \leq$$

$$\leq ||\widehat{d}; L^p(Q_{0,T})|| + L_0||\widehat{d}|_{t=0}; L^p(\Omega)|| + L||\widehat{d}_t - \Delta\,\widehat{d}; L^p(Q_{0,T})|| \tag{44}$$

holds. Recall that the constant operators are completely continuous.

Step 2. From Proposition 3 it follows that $\mathcal{N} : L^p(Q_{0,T}) \to L^p(Q_{0,T})$ is continuous and bounded operator. In addition, we have estimates (37), (38).

From Example 2 it follows that the operator $\mathcal{J} : L^p(Q_{0,T}) \to L^p(Q_{0,T})$ is completely continuous. In addition, estimate (34) holds.

Using Example 1 and the properties of the operators $\mathcal{N}$, $\mathcal{J}$, we see that $\mathcal{J} \circ \mathcal{N}$ is a completely continuous operator as a composition of completely continuous and bounded continuous operators. Consequently, $\mathcal{A}$ is a completely continuous operator as a sum of the completely continuous operators $\mathcal{D}$, $\mathcal{K}$, and $\mathcal{J} \circ \mathcal{N}$ (see Example 1).

Step 3. Take a sufficiently small $\varepsilon \in (0, \min\{\frac{1}{2}, 2 - q^0\})$, where $q^0 \in (1,2)$ is taken from condition **(Q)**. Then $\varepsilon < \frac{1}{2}$, that is $1 - 2\varepsilon > 0$. In addition, $\varepsilon < 2 - q^0$, i.e. $2 - q^0 - \varepsilon > 0$.

Let $R > 0$ be a sufficiently large number such that

$$\begin{cases} R^{\varepsilon} \geq \max\{\sqrt{||\widehat{d}; L^p(Q_{0,T})||},\ ||\widehat{d}|_{t=0}; L^p(\Omega)||,\ ||\widehat{d}_t - \Delta\,\widehat{d}; L^p(Q_{0,T})||, \\ \qquad ||u_0; L^p(\Omega)||,\ ||f; L^p(Q_{0,T})||,\ L_0,\ L,\ 1\}, \\ \dfrac{5}{R^{1-2\varepsilon}} + \dfrac{N_p}{R^{2-q^0-\varepsilon}} \leq 1, \end{cases} \tag{45}$$

where the constants L_0, L are taken from (33), (34), and the constant N_p is taken from (38). By definition, put

$$B_R = \{u \in L^p(Q_{0,T}) \ | \ ||u; L^p(Q_{0,T})|| \leq R\}.$$

We will show that $\mathcal{A}(B_R) \subset B_R$. Take a function $u \in B_R$. Using the monotonicity of the function $S_{1/h}$, from estimations (34), (43), (38), (44), and (45), we have

$$||\mathcal{A}u; L^p(Q_{0,T})|| \leq ||(\mathcal{D} + \mathcal{K} - \mathcal{J} \circ \mathcal{N})(u); L^p(Q_{0,T})|| \leq$$

$$\leq ||\mathcal{D}u; L^p(Q_{0,T})|| + ||\mathcal{K}u; L^p(Q_{0,T})|| + ||\mathcal{J}(\mathcal{N}u); L^p(Q_{0,T})|| \leq$$

$$\leq ||\mathcal{D}u; L^p(Q_{0,T})|| + ||\mathcal{K}u; L^p(Q_{0,T})|| + L||\mathcal{N}u; L^p(Q_{0,T})|| \leq$$

$$\leq ||\widehat{d}; L^p(Q_{0,T})|| + L_0||\widehat{d}|_{t=0}; L^p(\Omega)|| + L||\widehat{d}_t - \Delta\,\widehat{d}; L^p(Q_{0,T})|| +$$

$$+ L_0||u_0; L^p(\Omega)|| + L||f; L^p(Q_{0,T})|| + LN_p\Big\{S_{1/h}\Big(||u; L^p(Q_{0,T})||^p\Big)\Big\}^{\frac{1}{p}} \leq$$

$$\leq 5R^{2\varepsilon} + R^{\varepsilon}N_p\Big\{S_{1/h}\Big(R^p\Big)\Big\}^{\frac{1}{p}},$$

where $h(x,t) = \frac{1}{q(x,t)-1}$. Taking into account (35), from inequality $R > 1$ it follows that

$$S_{1/h}\Big(R^p\Big) = \Big(R^p\Big)^{\frac{1}{\operatorname{ess\,inf} h(x,t)}} = \Big(R^p\Big)^{\frac{1}{\operatorname{ess\,inf} \frac{1}{q(x,t)-1}}} = \Big(R^p\Big)^{\frac{1}{\frac{1}{q^0-1}}} = R^{p(q^0-1)}.$$

By the choice of R, we get

$$||\mathcal{A}u; L^p(Q_{0,T})|| \leq 5R^{2\varepsilon} + N_pR^{\varepsilon+q^0-1} = \Big(\frac{5}{R^{1-2\varepsilon}} + \frac{N_p}{R^{2-q^0-\varepsilon}}\Big)R \leq R,$$

i.e. $\mathcal{A}(B_R) \subset B_R$. Therefore, the operator $\mathcal{A}$ satisfies the conditions of the Schauder theorem (see Proposition 1), and has a fixed point. Theorem is proved. □

Remark 2. Using monotonicity method, and the additional condition $g(x,t) \geq 0$ it is easy to show the uniqueness of the solution $u \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega))$ of problem (1)-(3) (notice that Theorem 1 does not show that the solution u belongs to $L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega))$).

Remark 3. The results of Theorem 1 can be extended on to the second and third mixed problems for equation (1) and its generalization.

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ПРО РОЗВ'ЯЗНІСТЬ МОДЕЛЬНИХ НЕОДНОРІДНИХ ЗАДАЧ ДЛЯ ПІВЛІНІЙНИХ ПАРАБОЛІЧНИХ РІВНЯНЬ ЗІ ЗМІННИМИ СТЕПЕНЯМИ НЕЛІНІЙНОСТІ

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Досліджено неоднорідну мішану задачу Діріхле для рівняння

$$u_t - \Delta u + g(x,t)|u|^{q(x,t)-2}u = f(x,t)$$

в циліндричній області. За умови $1 < q_0 \leq q(x,t) \leq q^0 < 2$ доведено існування слабкого розв'язку цієї задачі.

Ключові слова: нелінійне параболічне рівняння, неоднорідна задача, мішана задача, змінний показник нелінійності, узагальнені простори Лебега і Соболєва, слабкий розв'язок, функція Гріна.

О РАЗРЕШИМОСТИ МОДЕЛЬНЫХ НЕОДНОРОДНЫХ ЗАДАЧ ДЛЯ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ С ПЕРЕМЕННЫМИ СТЕПЕНЯМИ НЕЛИНЕЙНОСТИ

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Исследовано смешанную задачу Дирихле для уравнения

$$u_t - \Delta u + g(x,t)|u|^{q(x,t)-2}u = f(x,t)$$

в цилиндрической области. При условии $1 < q_0 \leq q(x,t) \leq q^0 < 2$ доказано существование слабого решения этой задачи.

Ключевые слова: нелинейное параболическое уравнение, смешанная задача, переменный степень нелинейности, обобщённые пространства Лебега и Соболева, слабое решение, функция Грина.