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FUNCTIONAL CALCULUS ON A WIENER TYPE ALGEBRA OF ANALYTIC FUNCTIONS OF INFINITY MANY VARIABLES

Olga MYAUS

Lviv Polytechnic National University
12 St. Bandera, 79013 Lviv, Ukraine

For generators of isometric strong continuous operator groups, defined on nuclear Wiener algebras of analytic complex functions on a unit Banach ball, a functional calculus is constructed. Its symbol algebra consists of Fourier-images of exponential type distributions.

Key words: Functional calculus, Wiener type algebra, Banach ball, Fourier-image of exponential type distributions.

1. $\boldsymbol{C}_0$-group $\hat{\boldsymbol{U}}_{\boldsymbol{t}}$ over the Wiener algebra $\boldsymbol{W}_\pi$

Let $\boldsymbol{X}, \boldsymbol{X}'$ be complex Banach reflexive space and its dual, respectively. By $\langle \boldsymbol{X} \mid \boldsymbol{X}' \rangle$ we denote the corresponding duality. We use the main notations and definitions from [1]. For every $\boldsymbol{F}_{\boldsymbol{n}}' \in \boldsymbol{X}_\pi^{'\otimes \boldsymbol{n}}$ there exists [2] a unique $\boldsymbol{n}$-homogeneous polynomials $\boldsymbol{F}_{\boldsymbol{n}}$ such that

$$\boldsymbol{F}_{\boldsymbol{n}}(\boldsymbol{x}) := \left\langle \boldsymbol{x}^{\otimes \boldsymbol{n}} \mid \boldsymbol{F}_{\boldsymbol{n}}' \right\rangle \text{ for all } \boldsymbol{x} \in \boldsymbol{X}.$$

We denote by

$$\mathbf{P}_\pi^{\boldsymbol{n}}(\boldsymbol{X}) = \{\boldsymbol{F}_{\boldsymbol{n}} : \boldsymbol{F}_{\boldsymbol{n}}' \in \boldsymbol{X}_\pi^{'\otimes \boldsymbol{n}}\}$$

the space of so-called nuclear $\boldsymbol{n}$-homogeneous polynomials, where the complete symmetric tensor product $\boldsymbol{X}_\pi^{'\otimes \boldsymbol{n}}$ with the projective norm $\|\cdot\|_\pi$ endowed. It follows from it the isomertry $\mathbf{P}_\pi^{\boldsymbol{n}}(\boldsymbol{X})$ and $\boldsymbol{X}_\pi^{'\otimes \boldsymbol{n}}$, so on $\mathbf{P}_\pi^{\boldsymbol{n}}(\boldsymbol{X})$ we may define the following norm

$$\| \boldsymbol{F}_{\boldsymbol{n}} \| := \| \boldsymbol{F}_{\boldsymbol{n}}' \|_\pi, \boldsymbol{F}_{\boldsymbol{n}}' \in \boldsymbol{X}_\pi^{'\otimes \boldsymbol{n}}.$$

Definition. The $\mathbf{l}_1$-sum

$$\boldsymbol{W}_\pi := \{\boldsymbol{F} = \sum_{\boldsymbol{n} \geq 0} \boldsymbol{F}_{\boldsymbol{n}} : \boldsymbol{F}_{\boldsymbol{n}} \in \mathbf{P}_\pi^{\boldsymbol{n}}(\boldsymbol{X})\}$$

Друкується в авторській редакції.

with the norm $\| F \| = \sum \| F_n \|$ is called the nuclear Wiener type algebra.

Similar to [3], it can be shown that W_π is a Banach algebra of bounded analytic functions in $B = \{x \in X : \| x \| < 1\}$.

It is known ([4], theorem 1) that for C_0-group of linear isometric operator $\mathbb{R} \ni t \mapsto U_t \in \mathrm{L}(X)$ the C_0-group

$$\hat{U}_t F(x) = F(U_t x), (x \in B) \text{ on } W_\pi$$

is contractive.

Applying the isometric property of $\hat{U}_t$ over W_π and [5, theorem 2] we obtain that its (infinitesimal) generator $\hat{A}$ is a closable *conservative* differentiation on W_π.

We will use the fact that the C_0-group $\hat{U}_t$ over the Wiener algebra W_π is well-defined and acted as follows

$$\hat{U}_t F = \sum_{n \geq 0} \hat{U}_t^{\otimes n} F_n, \; F = \sum F_n \in W_\pi,$$

where $\hat{U}_t^{\otimes n}$ is defined in [4, proposition 2] by the equality

$$\hat{U}_t^{\otimes n} F_n(x) = \left\langle x^{\otimes n} \mid U_t'^{\otimes n} F_n' \right\rangle \text{ for all } x \in X,$$

where $U_t'^{\otimes n} = \underbrace{U_t' \otimes \ldots \otimes U_t'}_{n}$ and $\left\langle U_t x \mid y \right\rangle = \left\langle x \mid U_t' y \right\rangle$ for all $x, y \in X$, U_t' is the adjoint group of U_t.

Let $\hat{A}$ be the generator of $\hat{U}_t$ of the form ([4, proposition 3])

$$\hat{A}F(x) = \sum_{n \in \mathbb{Z}_+} \left\langle x^{\otimes n} \mid \sum_{j=0}^{n} A_j' F_n' \right\rangle, A_j' := \underbrace{I' \otimes \ldots \otimes I'}_{j-1} \otimes A' \otimes \underbrace{I' \otimes \ldots \otimes I'}_{n-j+1}, x \in B,$$

defined on a norm dense subspace $\mathrm{D}(\hat{A}) = \{F = \sum F_n : F_n' \in \mathrm{D}(A')^{\otimes n}\}$ in W_π, where $\mathrm{D}(A')$ is the definition's domain of adjoint generator A'.

2. Finite functions of the generator $\hat{A}$

For $\nu > 0$ we consider (see [6]) $\mathrm{E}^\nu := \{\varphi \in L_1(\mathbb{R}) : \| \varphi \|_\nu < \infty\}$ with the norm $\| \varphi \|_\nu = \sup_{k \in \mathbb{Z}_+} \dfrac{\| D^k \varphi(t) \|_{L_1}}{\nu^k}$

- the subspace in $L_1(\mathbb{R})$ of entire analytic complex functions of exponential type ν on $\mathbb{C}$, whose restrictions to $\mathbb{R} \subset \mathbb{C}$ belongs to $L_1(\mathbb{R})$. Let

$$\mathbf{E} := \bigcup_{\nu > 0} \mathbf{E}^{\nu} = \liminf_{\nu \to \infty} \mathbf{E}^{\nu}$$

be the inductive limit of Banach spaces $\mathbf{E}^{\nu}$ under the continuous embeddings $\mathbf{E}^{\nu} \subset \mathbf{E}^{\mu}$ with $\nu \le \mu$. Note that the subspace $\mathbf{E} \subset L_1(\mathbb{R})$ consists of all entire analytic complex functions on $\mathbb{C}$ of exponential type, whose restrictions to the real axis $\mathbb{R}$ belong to $L_1(\mathbb{R})$. Following [6], the functionals of the space $\mathbf{E}'$ are called *exponential type distributions* over $\mathbb{R}$. In the dual pair $\langle \mathbf{E} \mid \mathbf{E}' \rangle$ the space $\mathbf{E}$ play a role of the space test functions and the embedding $\mathbf{E} \subset \mathbf{E}'$ is dense.

It is proven in [6] that $\mathbf{E}'$ is invariant under the differentiations. Consequently, we obtain

$$\langle \boldsymbol{D}^k \boldsymbol{g} \mid \varphi \rangle = (-1)^k \langle \boldsymbol{g} \mid \boldsymbol{D}^k \varphi \rangle, \boldsymbol{k} \in \mathbb{Z}_+, \tag{1}$$

for all functionals $\boldsymbol{g} \in \mathbf{E}'$ and all entire functions $\varphi \in \mathbf{E}$ on $\mathbb{R}$ of exponential type. It follows from [6] that $\mathbf{E}'$ is a locally convex topological algebra with respect to the convolution

$$\mathbf{E}' \times \mathbf{E}' \ni (\boldsymbol{g}, \boldsymbol{h}) \mapsto \boldsymbol{g} * \boldsymbol{h} \in \mathbf{E}'$$

and $\mathbf{E}$ is its convolution subalgebra.

By the well-known Paley-Wiener theorem [7], the Fourier-image $\hat{\mathbf{E}}$ of the space $\mathbf{E}$, endowed with inductive topology under the Fourier transform $\mathbf{F} : \mathbf{E} \ni \varphi \to \hat{\varphi} \in \hat{\mathbf{E}}$, consist of infinitely smooth finite complex functions on $\mathbb{R}$. So,

$$\hat{\mathbf{E}} \subset \mathbf{D}(\mathbb{R}), \tag{2}$$

where $\mathbf{D}(\mathbb{R})$ means the classic Schwartz space of test functions.

Via [6] the Fourier transform $\mathbf{F}$ can be extended from the space $\mathbf{E}$ onto the strong dual space

$$\mathbf{F}^{\#} : \mathbf{E}' \ni \boldsymbol{g} \to \hat{\boldsymbol{g}} \in \hat{\mathbf{E}}',$$

where $\hat{\mathbf{E}}'$ denotes its image, i.e. $\mathbf{F}^{\#}|_{\mathbf{E}} = \mathbf{F}$. This extended Fourier transform $\mathbf{F}^{\#}$ has the property

$$\mathbf{F}^{\#}(\boldsymbol{g} * \boldsymbol{h}) = \hat{\boldsymbol{g}} \cdot \hat{\boldsymbol{h}}, \qquad \boldsymbol{g}, \boldsymbol{h} \in \mathbf{E}'.$$

So, the extended Fourier-image $\hat{\mathbf{E}}'$ is a topological algebra with pointwise multiplication and $\hat{\mathbf{E}}$ is its multiplication subalgebra.

Since the embedding (2) is dense, the dense embedding

$$\mathbf{D}'(\mathbb{R}) \subset \hat{\mathbf{E}}'$$

holds, where the dual spaces $\mathbf{D}'(\mathbb{R})$ and $\hat{\mathbf{E}}'$ are endowed with the strong (or weak) topologies under the dualities $\langle \mathbf{D}(\mathbb{R}), \mathbf{D}'(\mathbb{R})\rangle$ and $\langle \hat{\mathbf{E}}, \hat{\mathbf{E}}'\rangle$, respectively. Hence, $\langle \hat{\mathbf{E}} | \hat{\mathbf{E}}'\rangle$ forms a new dual pair, which is a Fourier-image of the dual pair $\langle \mathbf{E} | \mathbf{E}'\rangle$.

The following theorem is a generalization of [7, theorem 3] and may be proven in analogical way.

Theorem 1. For every $\varphi \in \mathbf{E}$ the operator

$$\hat{\varphi}(\hat{A})F(x) = \sum_{n\in\mathbb{Z}_+} \left\langle x^{\otimes n} \mid \sum_{j=0}^{n} \left[\hat{\varphi}(A_j)\right]' F_n' \right\rangle, \qquad x \in B, \tag{3}$$

belongs to the Banach algebra $\mathbf{L}(W_\pi)$ of all bounded linear operators over W_π, where the operators

$$\hat{\varphi}(A) = \int_{\mathbb{R}} U_t \varphi(t)dt, \ \left[\hat{\varphi}(A_j)\right]' := \underbrace{I'\otimes\ldots\otimes I'}_{j-1} \otimes \left[\hat{\varphi}(A)\right]' \otimes \underbrace{I'\otimes\ldots\otimes I'}_{n-j+1}$$

(here $\left[\hat{\varphi}(A)\right]$ is adjoint to $\hat{\varphi}(A) \in \mathbf{L}(X)$) are bounded over X and $X_\pi'^{\otimes n}$, respectively. Moreover, the differential property

$$\widehat{(D\varphi)}(A) = \hat{A} \circ \hat{\varphi}(A), \qquad \varphi \in \mathbf{E}$$

holds and the mapping

$$\hat{E} \ni \hat{\varphi} \mapsto \hat{\varphi}(A) \in \mathbf{L}(W_\pi)$$

is an algebraic homomorphism, particularly

$$\widehat{(\varphi * \psi)}(A) = \hat{\varphi}(A) \cdot \hat{\psi}(A) \qquad \text{for all } \varphi, \psi \in \mathbf{E}.$$

3. Functional calculus for the generator $\hat{A}$ in the symbol algebra $\hat{\mathbf{E}}'$

Following [6] we define the completions $\mathbf{E}(W_\pi) := \mathbf{E} \otimes_\pi W_\pi$ and $\mathbf{E}^\nu(W_\pi) := \mathbf{E}^\nu \otimes_\pi W_\pi$ of the tensor product $\mathbf{E} \otimes W_\pi$ and $\mathbf{E}^\nu \otimes W_\pi$ under the

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corresponding projective tensor topologies. Then

$$\mathsf{E}(W_\pi) = \bigcup_{\nu>0} \mathsf{E}^\nu(W_\pi) = \lim_{\nu\to\infty}\operatorname{ind} \mathsf{E}^\nu(W_\pi) = \left(\lim_{\nu\to\infty}\operatorname{ind} \mathsf{E}^\nu\right) \otimes_\pi W_\pi .$$

Each element $F \in \mathsf{E}(W_\pi)$ is a W_π-valued exponential type entire function

$$\mathbb{R} \ni t \mapsto F(x,t) = \sum_{n\in\mathbb{Z}_+} F_n(x,t) \in W_\pi,$$

which also is an analytic complex function of $x \in B$ for any fixed t.

It follows from the known Grotendieck theorem [8] that for every $F \in \mathsf{E}(W_\pi)$ there exists $\nu > 0$ such that

$$F = \sum_{j\in\mathbb{N}} F_j \otimes \varphi_j \quad \text{with} \quad F_j \in W_\pi, \quad \varphi_j \in \mathsf{E}^\nu \tag{4}$$

is absolutely convergent in $\mathsf{E}^\nu(W_\pi)$. Hence, we can well-define the elements

$$\hat{F} := \sum_{j\in Z^+} \hat{\varphi}_j(\hat{A}) F_j,$$

where $\hat{\varphi}_j(\hat{A})$ is defined by (3). For any $\nu > 0$ the subspace

$$\hat{\mathsf{E}}^\nu(W_\pi) := \left\{\hat{F} : F \in \mathsf{E}^\nu(X)\right\}$$

is complete under the norm induced by the mapping $\mathsf{E}^\nu(W_\pi) \ni F \mapsto \hat{F} \in \hat{\mathsf{E}}^\nu(W_\pi)$ (see [5, lemma 5]).

We define the convolution of an exponential type distribution $g \in \mathsf{E}'$ and W_π-valued exponential type entire function $F \in \mathsf{E}(W_\pi)$, representing by a series (4), as follows

$$(F * g)(x,t) = \sum_{j\in N} F_j(x) \otimes (g * \varphi_j)(t), \qquad x \in B, \quad t \in \mathbb{R}.$$

Denote

$$(F * g)(x,t) := (I \otimes K_g) F(x,t), \quad x \in B, \quad t \in \mathbb{R},$$

where I is identity operator on W_π and the convolution $g * \varphi$ of exponential type distribution $g \in \mathsf{E}'$ and exponential type entire complex function $\varphi \in \mathsf{E}$ is defined in [5].

For any $\nu > 0$ the subspace $\hat{\mathbf{E}}(W_\pi)$ is invariant under each operator $I \otimes K_g$ with $g \in \mathbf{E}'$ (see [5], lemma 6). If we define the inductive limit

$$\hat{\mathbf{E}}(W_\pi) := \bigcup_{\nu>0} \hat{\mathbf{E}}^\nu(W_\pi) = \lim_{\nu\to\infty} \operatorname{ind} \hat{\mathbf{E}}^\nu(W_\pi)$$

then $\hat{\mathbf{E}}(W_\pi)$ also is invariant under each operator $I \otimes K_g$ with $g \in \mathbf{E}'$.

The following theorem is a generalization of [6, theorem 6]. We denote by $\mathbf{L}\left[\hat{\mathbf{E}}(W_\pi)\right]$ the algebra of all bounded linear operators over the space $\hat{\mathbf{E}}(W_\pi)$ endowed with the strong operator topology.

Theorem 2. The mapping $\hat{\mathbf{E}}' \ni \hat{g} \to \hat{g}(\hat{A}) \in \mathbf{L}\left[\hat{\mathbf{E}}(W_\pi)\right]$, where the linear operator $\hat{g}(\hat{A})$ is defined by

$$\hat{g}(\hat{A}) : \hat{\mathbf{E}}(W_\pi) \ni \hat{F} = \sum_{j\in Z^+} \hat{\varphi}_j(\hat{A})F_j \to \hat{g}(\hat{A})\hat{F} := \sum_{j\in Z^+} (g \hat{*} \varphi_j)(\hat{A})F_j \in \hat{\mathbf{E}}(W_\pi),$$

is a continuous homomorphism from the symbol algebra $\hat{\mathbf{E}}'$ into $\mathbf{L}\left[\hat{\mathbf{E}}(W_\pi)\right]$. Moreover,

$$(\hat{D}g)(\hat{A}) = \hat{A} \circ \hat{g}(\hat{A}), \qquad g \in \mathbf{E},$$

where the generalized derivative D is defined by (1).

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ФУНКЦІОНАЛЬНЕ ЧИСЛЕННЯ НА АЛГЕБРІ ТИПУ ВІНЕРА АНАЛІТИЧНИХ ФУНКЦІЙ НЕСКІНЧЕННОЇ КІЛЬКОСТІ ЗМІННИХ

Ольга М'ЯУС

Національний університет „Львівська політехніка”, вул. С. Бандери, 12 79013 Львів, Україна

Для генераторів ізометричних сильно неперервних операторних груп на ядерних алгебрах Вінера аналітичних комплексних функцій на одиничній банаховій кулі побудовано функціональне числення. Його алгебра символів складається з Фур’є-образів розподілів експоненціального типу.

Ключові слова: функціональне числення, алгебра Вінера, банахова куля, Фур’є-образ розподілів експоненціального типу.

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Ольга МЯУС

Национальный университет „Львовская политехника”, ул. С. Бендеры, 12 79013 Львов, Украина

Для генераторов изометрических сильно непрерывных операторных групп на ядерных алгебрах Винера аналитических комплексных функций на единичном банаховом шаре построено функциональное исчисление. Его алгебра символов состоит из Фурье-образов распределений экспоненциального типа.

Ключевые слова: функциональное исчисление, алгебра Винера, банахов шар, Фурье-образ распределений экспоненциального типа.

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