*ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2011. Випуск 75. С. 137–150*
*Visnyk of the Lviv Univ. Series Mech. Math. 2011. Issue 75. P. 137–150*

УДК 517.95

# CLASSICAL SOLVABILITY OF A SINGULAR HYPERBOLIC PROBLEM WITH INTEGRAL BOUNDARY CONDITION

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We construct a distributional solution to a linear initial-boundary hyperbolic problem with integral boundary condition whose initial and boundary data are strongly singular distributions. Our result covers the Lotka McKendrick problem with a variable coefficient in the differential part.

*Key words:* generalized Lotka McKendrick problem, singular data, distributional solution.

**1. Introduction and main result.** The paper concerns a singular initial-boundary problem for the general linear first-order one-dimensional hyperbolic equation of the following type:

$$(\partial_t + \lambda(x,t)\partial_x)u = p(x,t)u + g(x,t), \quad (x,t) \in \Pi, \tag{1}$$

$$u|_{t=0} = a_r(x) + \delta^{(m)}(x - x_1^*), \qquad x \in [0, L), \tag{2}$$

$$u|_{x=0} = (c_r(t) + \delta^{(j)}(t - t_1)) \int_0^L (b_r(x) + \delta^{(n)}(x - x_1))u\,dx, \; t \in [0, \infty), \tag{3}$$

where $x_1 > 0$, $x_1^* > 0$, $t_1 > 0$, $m, j, n \in \mathbb{N}_0$, and

$$\Pi = \{(x,t) \in \mathbb{R}^2 \;\mid\; 0 < x < L, \;\; t > 0\}.$$

This problem can be easily generalized to the case when the initial and the boundary data have singular supports at finitely many points. Without loss of generality we assume that $x_1^* < x_1$. The system (1)-(3) arises in the theory of population dynamics (see [1, 7, 9, 10]), where $u$ denotes the distribution of individuals having age $x > 0$ at time $t > 0$, $a_r(x) + \delta^{(m)}(x - x_1^*)$ is the initial distribution, $-p(x,t)$ denotes the mortality rate, $b_r(x) + \delta^{(n)}(x - x_1)$ denotes the age-dependent fertility rate, $c_r(t) + \delta^{(j)}(t - t_1)$ is the specific fertility rate of females, $g(x,t)$ is the distribution of migrants, $L$ is the maximum age attained by individuals.

The evolution of $u$ is governed by (1)-(3). Since (1)-(3) is a continuous model of a discrete structure, in many problems of such a kind it is natural to consider strongly singular initial and boundary data (see [7]).

A partial case of the problem (1)-(3) where $\lambda \equiv 1$ is investigated in [4]. Here we develop our approach from [4] and construct a distributional solution to (1)-(3) in the case of the variable coefficient $\lambda$, what has meaning for applications to problems of population dynamics (see, e.g. [2]).

Initial-boundary semilinear hyperbolic problems with distributional data were studied in [8, 5, 6], where the authors constructed delta-wave solutions. In contrast to these papers, we here show that the problem (1)-(3) is solvable in the distributional sense and construct the distributional solution by means of multiplication of distributions in the sense of Hörmander [3].

We impose the following conditions for (1)-(3):

*A1.* $\lambda(x,t) > 0$ for all $(x,t) \in \mathbb{R}^2$.

*A2.* $a_r^{(i)}(0) = 0, c_r^{(i)}(0) = 0$ for all $i \in \mathbb{N}_0$.

*A3.* $b_r^{(i)}(L) = 0$ for all $i \in \mathbb{N}_0$ and there exists $\varepsilon > 0$ such that $b_r(x) = 0$ for $x \in [0, \varepsilon]$.

*A4.* The functions $\lambda$, $p$, and $g$ are smooth in $\mathbb{R}^2$, $a_r$ is smooth on $[0, L)$, $b_r$ is smooth on $[0, L]$, and $c_r$ is smooth on $[0, \infty)$.

These assumptions are not particularly restrictive from the practical point of view. In particular, Assumption A3 is a consequence of the fact that $[0, L]$ covers the fertility period of females. Note that Assumption A2 ensures the arbitrary order compatibility between (2) and (3).

It is well known that, under the assumptions A1 and A4, for every $(x,t) \in \mathbb{R}^2$ there exists a unique characteristic of (1) expressed in one of two forms $\xi = \omega(\tau; x, t)$ or $\tau = \tilde{\omega}(\xi; x, t)$, where $\omega$ and $\tilde{\omega}$ are smooth functions with respect to all their arguments. Below we will use both of the forms.

**Definition 1.** *Let* $I_+ = \bigcup_{n \ge 0} I_+[n]$, *where* $I_+[n]$ *are subsets of* $\mathbb{R}^2$ *defined by induction as follows.*

- $I_+[0]$ *is the union of the characteristics* $\omega(t; x_1^*, 0)$ *and* $\omega(t; 0, t_1)$.
- *Let* $n \ge 1$. *If* $I_+[n-1]$ *includes the characteristic* $\omega(t; x_1, \tilde{t})$, *then* $I_+[n]$ *includes the characteristic* $\omega(t; 0, \tilde{t})$.

For characteristics contributing into $I_+$ denote their intersection points with the positive semiaxis $x = 0$ by $t_1^*, t_2^*, \dots$. We assume that $t_j^* < t_{j+1}^*$ for $j \ge 1$. The union of all singular characteristics of the initial problem, as it will be shown, is included into the set $I_+$. In fact, we will show that $\operatorname{sing\,supp} u \subset I_+$. Assume that

*A5.* $\omega(0; x_1, t_1) \ne x_1^*, \quad \tilde{\omega}(0; x_1, t_1) \ne t_s^*$ for all $t_s^* < t_1$.

Without this assumption the distributional solution does not exist, because there appears multiplication of the delta function onto itself.

We proceed similarly to [4] and start with the distributional solution in the domain of influence (or determinacy) of the problem (1)-(3):

$$\Omega = \{(x,t) \in \mathbb{R}^2 \mid x < \omega(t; L, 0)\}.$$

**Definition 2.** *A distribution $u$ is called a $\mathcal{D}'(\Omega)$-solution to the problem (1)-(3) if the following conditions are met.*
*1. The equation (1) is satisfied in $\mathcal{D}'(\Omega)$: for every $\varphi \in \mathcal{D}(\Omega)$*

$$\langle(\partial_t + \lambda(x,t)\partial_x - p(x,t))u, \varphi\rangle = \langle g(x,t), \varphi\rangle.$$

*2. $u$ is restrictable to $[0, L) \times \{0\}$ in the sense of Hörmander* [3].
*3. $u|_{t=0} = a_r(x) + \delta^{(m)}(x - x_1^*), \quad x \in [0, L)$.*
*4. The product $[(b_r(x) + \delta^{(n)}(x - x_1)) \otimes 1(t)] \cdot u(x,t)$ exists in $\mathcal{D}'(\Pi)$ in the sense of Hörmander* [3].
*5. $\int\limits_0^L [(b_r(x) + \delta^{(n)}(x - x_1)) \otimes 1(t)] u\, dx$ for $t > 0$ is a distribution $v(t) \in \mathcal{D}'(\mathbb{R}_+)$ defined by*

$$\langle v(t), \psi(t)\rangle = \langle[(b_r(x) + \delta^{(n)}(x - x_1)) \otimes 1(t)] \cdot u(x,t), 1(x) \otimes \psi(t)\rangle, \quad \psi(t) \in \mathcal{D}(\mathbb{R}_+),$$

*where $b_r(x) = 0$, $x \notin [0, L]$.*
*6. There exists $\varepsilon > 0$ such that $v(t)$ restricted to $(t_1 - \varepsilon, t_1 + \varepsilon)$ is a smooth function.*
*7. $u$ is restrictable to $\{0\} \times [0, \infty)$ in the sense of Hörmander* [3].
*8. $u|_{x=0} = (c_r(t) + \delta^{(j)}(t - t_1)) \cdot v(t), \quad t \in [0, \infty)$.*
*9. $\operatorname{sing\,supp} u \subset \Omega \setminus \{(x,t) \,|\, x = \omega(t; 0, 0)\}$.*

Our next objective is to define the solution concept for (1)-(3) on $\Pi$. To define the restriction of $u \in \mathcal{D}'(\Pi)$ to the boundary of $\Pi$ so that the initial and the boundary conditions are meaningful, let us make the following observation: Note that $\overline{\Pi} \setminus \{(L, 0)\} \subset \Omega$. Let $\Omega_0 \subset \Omega$ be a domain such that $\overline{\Pi} \setminus \{(L, 0)\} \subset \Omega_0$ and $u$ be a $\mathcal{D}'(\Omega)$-solution to the problem (1)-(3) in the sense of Definition 2. Then $u$ restricted to $\Omega_0$ is a $\mathcal{D}'(\Omega_0)$-solution to the problem (1)-(3) in the sense of the same definition. This suggests the following definition.

**Definition 3.** *Let $u$ be a $\mathcal{D}'(\Omega)$-solution to the problem (1)-(3) in the sense of Definition 2. Then $u$ restricted to $\Pi$ is called a $\mathcal{D}'(\Pi)$-solution to the problem (1)-(3).*

Set

$$\Omega_+ = \{(x,t) \in \Omega \,|\, x > 0, t > 0\}.$$

We are now prepared to state the existence result.

**Theorem 1.** *1. Let Assumptions A1–A5 hold. Then there exists a $\mathcal{D}'(\Omega)$-solution $u$ to the problem (1)-(3) in the sense of Definition 2 such that:*

*the restriction of $u$ to any domain $\Omega'_+ \supset \Omega_+$ such that any characteristic of (1) intersects $\partial\Omega'_+$ at a single point does not depend on the values of the functions $\lambda$, $p$ and $g$ on $\Omega \setminus \Omega'_+$.* (4)

*2. Let Assumptions A1–A5 hold. Then there exists a $\mathcal{D}'(\Pi)$-solution to the problem (1)-(3) in the sense of Definition 3.*

**2. Construction of the $\mathcal{D}'$-solution.** It is sufficient to solve the problem in the domain

$$\Omega^T = \{(x,t) \in \Omega \ | \ \omega(t; 0, T) < x, \ \ -T < t < T\}$$

for an arbitrary fixed $T > 0$. Observe that $\Omega^T$ is the intersection of the strip $\mathbb{R} \times (-T, T)$ with the domain of determinacy of (1) with respect to the set $([0, L) \times \{0\}) \cup (\{0\} \times [0, T))$. Fix $T > 0$ and start with a subdomain

$$\Omega_0^T = \{(x,t) \in \Omega^T \ | \ \omega(t;0,0) < x < \omega(t;L,0)\}.$$

In the case that the initial data are functions, a unique solution to the problem (1)-(2) on $\Omega_0^T$ can be written in the form

$$u(x,t) = S(x,t)a_r(\omega(0;x,t)) + S_1(x,t) + S(x,t)\delta^{(m)}(\omega(0;x,t) - x_1^*), \tag{5}$$

where

$$S(x,t) = \exp\left\{ \int\limits_{\theta(x,t)}^{t} p(\tau + x - t, \tau)\, d\tau \right\}, \tag{6}$$

$\theta(x,t) = \tilde{\omega}(0;x,t)$, and

$$\begin{gathered} S_1(x,t) = \int\limits_{\theta(x,t)}^{t} g(\omega(\tau;x,t),\tau)\, d\tau + \\ + \int\limits_{\theta(x,t)}^{t} p(\omega(\tau;x,t),\tau)\, d\tau \int\limits_{\theta(x,t)}^{\tau} g(\omega(\tau_1;\omega(\tau;x,t),\tau),\tau_1)\, d\tau_1 + \dots. \end{gathered} \tag{7}$$

Observe that $S$ and $S_1$ are smooth and the equalities

$$\begin{gathered} S(x,t) = 1 + \int\limits_{\theta(x,t)}^{t} p(\omega(\tau;x,t),\tau) S(\omega(\tau;x,t),\tau)\, d\tau, \\ S_1(x,t) = \int\limits_{\theta(x,t)}^{t} g(\omega(\tau;x,t),\tau)\, d\tau + \int\limits_{\theta(x,t)}^{t} p(\omega(\tau;x,t),\tau) S_1(\omega(\tau;x,t),\tau)\, d\tau \end{gathered} \tag{8}$$

hold. Let $A_i(x,t) = \delta^{(i)}(x) \otimes 1(t)$ and $B_i(x,t) = 1(x) \otimes \delta^{(i)}(t)$ be the distributions defined by the equalities

$$\langle A_i(x,t), \varphi(x,t)\rangle = (-1)^i \int \varphi_x^{(i)}(0,t)\, dt,$$

$$\langle B_i(x,t), \varphi(x,t)\rangle = (-1)^i \int \varphi_t^{(i)}(x,0)\, dx$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$.

Let $f$ be the smooth map

$$f : (x,t) \to (x, \omega(0;x,t) - x_1^*).$$

The inverse

$$f^{-1} : (x,t) \to (x, r(x,t))$$

is unique and maps the $x$-axis to the curve $t = r(x, 0)$ and the $t$-axis onto itself. Here $\tau = r(x, t)$, is a smooth function for which we have $t \equiv \omega(0; x, r(x, t)) - x_1^*$. Furthermore,

$$f'(x,t) = \begin{pmatrix} 1 & 0 \\ \omega'_x(0;x,t) & \omega'_t(0;x,t) \end{pmatrix}$$

with

$$\begin{aligned} \omega'_x(0;x,t) &= \exp\left(\int_t^0 \lambda'_\xi(\xi(\tau),\tau)\,d\tau\right), \\ \omega'_t(0;x,t) &= -\lambda(x,t)\exp\left(\int_t^0 \lambda'_\xi(\xi(\tau),\tau)\,d\tau\right). \end{aligned} \tag{9}$$

By A1, $\lambda(x, t) \neq 0$ for all $(x, t) \in \mathbb{R}^2$. Hence

$$J(f) = |f'| = \omega'_t(0; x, t) \neq 0,$$

where $J(f)$ is the Jacobian of $f$. Hence $f^* B_m = \delta^{(m)}(\omega(0; x, t) - x_1^*)$, the pullback of $B_m$ by $f$ (see [3]), is well defined. Similarly, $f^* B_m$ is the $m$-th derivative of the delta function supported along the curve $t = r(x, 0)$.

**Definition 4.** *A distribution $u$ is called a $\mathcal{D}'(\Omega_0^T)$-solution to the problem (1), (2) if Items 1–3 of Definition 2 with $\Omega$ replaced by $\Omega_0^T$ hold.*

**Lemma 1.** *$u(x, t)$ given by the formula (5) is a $\mathcal{D}'(\Omega_0^T)$-solution to the problem (1)-(2).*

*Proof.* From the classical theory of first-order linear partial differential operators it follows that the sum of the first two summands in (5) is a unique smooth solution to the problem (1)-(2) with the singular part of the initial condition (2) identically equal to 0. It is obvious that this solution satisfies the latter problem in a distributional sense. Our goal is now to prove that the third summand in (5) is a distributional solution to the homogeneous equation (1) with singular initial condition $\delta^{(m)}(x - x_1^*)$. Indeed, for all $\varphi \in \mathcal{D}(\Omega_0^T)$, we have

$$\langle(\partial_t + \lambda\partial_x)(S\delta^{(m)}(\omega(0;x,t) - x_1^*)), \varphi\rangle = -\langle S\delta^{(m)}(\omega(0;x,t) - x_1^*), \varphi_t + (\lambda\varphi)_x\rangle =$$

$$= -\langle\delta^{(m)}(\omega(0;x,t) - x_1^*), S\varphi_t + S(\lambda\varphi)_x\rangle =$$

$$= -\langle\delta^{(m)}(\omega(0;x,t) - x_1^*), (S\varphi)_t + (\lambda S\varphi)_x - S_t\varphi - \lambda S_x\varphi\rangle.$$

Since $w = \delta^{(m)}(\omega(0; x, t) - x_1^*)$ is a distribution in $\omega(0; x, t)$, this is a weak solution to the equation $(\partial_t + \lambda\partial_x)w = 0$. Note that $S\varphi \in \mathcal{D}(\Omega_0^T)$. Therefore

$$\langle\delta^{(m)}(\omega(0;x,t) - x_1^*), (S\varphi)_t + (\lambda S\varphi)_x\rangle = 0.$$

Using (8) and (9), we obtain

$$S_t + \lambda S_x = pS + \int_0^t \left(\frac{\partial\omega(\tau;x,t)}{\partial t} + \lambda(x,t)\frac{\partial\omega(\tau;x,t)}{\partial x}\right)(p_x S + pS_x)\,d\tau = pS,$$

as desired.

It remains to prove that $S(x,t)\delta^{(m)}(\omega(0;x,t)-x_1^*)$ may be restricted to the initial interval $X=[0,L)\times\{0\}$. Observe that $f$ restricted to $\Omega_0^T$ is a diffeomorphism. We have to check the condition (see [3])

$$\mathrm{WF}(Sf^*B_m)\cap N(X)=\emptyset, \tag{10}$$

where the normal bundle $N(X)$ to $X$ is defined by the formula

$$N(X)=\{(x,t,\xi,\eta)\ \mid\ (x,t)\in X,\ \ \langle T_{(x,t)}(X),\ \ (\xi,\eta)\rangle=0\}$$

and $T_{(x,t)}(X)$ is the space of all tangent vectors to $X$ at $(x,t)$. It is clear that in our case

$$N(X)=\{(x,0,0,\eta),\ \ \eta\neq 0\}.$$

Let us now look at $\mathrm{WF}(Sf^*B_m)$. By Proposition 2, we have

$$\mathrm{WF}(Sf^*B_m)\subset \mathrm{WF}(f^*B_m).$$

Recall that by definition

$$\mathrm{WF}(f^*B_m)=\{(x,t,df_x^t\cdot(\xi,\eta))\ \ :\ \ (f(x,t),\xi,\eta)\in \mathrm{WF}(B_m)\}.$$

Moreover,

$$\mathrm{WF}(B_m)\subset \mathrm{WF}(B_0)=\{(x,0,0,\eta),\ \ \eta\neq 0\}.$$

It follows that $f(x,t)$ is equal to $(x,0)$. Therefore $(x,t)=(x,r(x,0))$. Furthermore,

$$df_x^t=\begin{pmatrix}1 & \omega_x'(0;x,t)|_{t=r(x,0)}\\ 0 & \omega_t'(0;x,t)|_{t=r(x,0)}\end{pmatrix},\qquad df_x^t\cdot(0,\eta)=\begin{pmatrix}\omega_x'(0;x,t)|_{t=r(x,0)}\eta\\ \omega_t'(0;x,t)|_{t=r(x,0)}\eta\end{pmatrix}.$$

As a consequence,

$$\mathrm{WF}(Sf^*B_m)\subset\{(x,r(x,0),\omega_x(0;x,t)|_{t=r(x,0)}\eta,\omega_t(0;x,t)|_{t=r(x,0)}\eta),\eta\neq 0\}.$$

This means that $S(x,t)\delta^{(m)}(\omega(0;x,t)-x_1^*)$ is restrictable to $X$. Considering the distribution $\delta^{(m)}(\omega(0;x,t)-x_1^*)$ to be smooth in $t$ with distributional values in $x$, the initial condition (10) follows from (5). This finishes the proof.

We have proved that $u$ defined by (5) satisfies Items 1–3 of Definition 2 with $\Omega$ replaced by $\Omega_0^T$. Items 5–9 on $\Omega_0^T$ do not need any proof. Item 4 will be given by Lemma 3 below.

Now we extend the solution over

$$\Omega_1^T=\{(x,t)\in\Omega^T\ \mid\ \omega(t;0,T)<x<\omega(t;0,0)\}.$$

We use the fact that any $u$ satisfying Item 9 of Definition 2 on $\Omega^T$ is representable as

$$u(x,t)=u_0(x,t)+u_1(x,t), \tag{11}$$

where $u_0=u$ in $\mathcal{D}'(\Omega_0^T)$, $u_0$ is identically equal to 0 on $\overline{\Omega_1^T}$, $u_1=u$ in $\mathcal{D}'(\Omega_1^T)$, and $u_1$ is identically equal to 0 on $\overline{\Omega_0^T}$. Indeed, Item 9 implies that the solution is smooth in a neighborhood of $\{(x,t)\,|\,x=\omega(t;0,0)\}$. For an arbitrary $\varphi\in\mathcal{D}(\Omega^T)$ consider a representation

$$\varphi(x,t)=\varphi_1(x,t)+\varphi_2(x,t)+\varphi_3(x,t)$$

such that $\varphi_i(x,t)\in\mathcal{D}(\Omega^T)$, $\operatorname{supp}\varphi_1\subset\Omega_0^T$, $\operatorname{supp}\varphi_2\cap\operatorname{sing\,supp}u=\emptyset$, and $\operatorname{supp}\varphi_3\subset\Omega_1^T$. Then

$$\begin{aligned}\langle u_0+u_1,\varphi\rangle&=\langle u_0,\varphi_1+\varphi_2\rangle+\langle u_1,\varphi_2+\varphi_3\rangle=\\ =\langle u,\varphi_1\rangle+\langle u_0,\varphi_2\rangle+\langle u_1,\varphi_2\rangle+\langle u,\varphi_3\rangle&=\langle u,\varphi_1+\varphi_2+\varphi_3\rangle=\langle u,\varphi\rangle.\end{aligned}$$

Using (11), we rewrite $v(t)$ defined by Items 5 of Definition 2 in the form:

$$v(t) = \int_0^L b(x)u_0(x,t)\,dx + \int_0^L b(x)u_1(x,t)\,dx.$$

Our task now is to compute the first integral

$$I_0(t) = \int_0^L b(x)u_0(x,t)\,dx, \quad 0 < t < T, \tag{12}$$

that will be used in the construction. We have to tackle the multiplication of distributions involved in the integrand. For technical reasons we extend $a_r(x)$ and $b_r(x)$ over all $\mathbb{R}$ defining them to be 0 outside $[0, L]$. Using (5), we rewrite (12) as follows

$$I_0(t) = \int_{\omega(t;0,0)}^L b_r(x)(S(x,t)a_r(\omega(0;x,t)) + S_1(x,t))\,dx+$$
$$+\int_0^L \delta^{(n)}(x-x_1)(S(x,t)a_r(\omega(0;x,t)) + S_1(x,t))\,dx+$$
$$+\int_0^L b_r(x)S(x,t)\delta^{(m)}(\omega(0;x,t)) - x_1^*)\,dx+$$
$$+\int_0^L \delta^{(n)}(x-x_1)S(x,t)\delta^{(m)}(\omega(0;x,t) - x_1^*)\,dx.$$

To compute the second integral we take a test function $\psi(t) \in \mathcal{D}(0,T)$ and consider the action (see Definition 2, Item 5)

$$\langle \delta^{(n)}(x-x_1)(S(x,t)a_r(\omega(0;x,t)) + S_1(x,t)), 1(x)\otimes\psi(t)\rangle =$$
$$= \langle \delta^{(n)}(x-x_1)\otimes 1(t), (S(x,t)a_r(\omega(0;x,t)) + S_1(x,t))\psi(t)\rangle =$$
$$= (-1)^n\langle 1(t), (S(x,t)a_r(\omega(0;x,t)) + S_1(x,t))_x^{(n)}|_{x=x_1}\psi(t)\rangle =$$
$$= (-1)^n\langle (S(x,t)a_r(\omega(0;x,t)) + S_1(x,t))_x^{(n)}|_{x=x_1}, \psi(t)\rangle.$$

To compute the third integral, consider the bijective map

$$q : (x,t) \to (\omega(0;x,t) - x_1^*, t)$$

that is smooth both in $x$ and $t$. The inverse of $q$ is unique and has form

$$q^{-1} : (x,t) \to (\varrho(x,t), t),$$

where $\eta = \varrho(x,t)$ is a smooth function, for which it holds $x \equiv \omega(0;\varrho(x,t),t) - x_1^*$. Hence

$$\langle S(x,t)b_r(x)\delta^{(m)}(\omega(0;x,t) - x_1^*), 1(x)\otimes\psi(t)\rangle = \langle q^*\delta^{(m)}(x), S(x,t)b_r(x)\psi(t)\rangle =$$
$$= \left\langle \delta^{(m)}(x), \frac{S(x,t)b_r(x)\psi(t)}{\omega'_x(0;x,t)} \circ q^{-1} \right\rangle =$$

$$= (-1)^m \left\langle 1(t), \left(\frac{S(\varrho(x,t),t)b_r(\varrho(x,t))}{\omega'_\varrho(0;\varrho(x,t),t)}\right)^{(m)}_x \Big|_{x=0} \psi(t)\right\rangle =$$

$$= (-1)^m \left\langle \left(\frac{S(\varrho(x,t),t)b_r(\varrho(x,t))}{\omega'_\varrho(0;\varrho(x,t),t)}\right)^{(m)}_x \Big|_{x=0}, \psi(t)\right\rangle.$$

To compute the last integral in the expression for $I_0(t)$ we need the following fact.

**Lemma 2.** *The product of two distributions* $v = \delta^{(n)}(x - x_1) \otimes 1(t)$ *and* $w = \delta^{(m)}(\omega(0;x,t) - x_1^*)$ *exists in the sense of Hörmander* [3].

*Proof.* By [3], it suffices to recall that

$$\mathrm{WF}(v) = \{(x_1, t, \xi_1, 0),\ \ \xi_1 \neq 0\},$$

$$\mathrm{WF}(w) \subset \{x,\ \ r(x,0),\ \ \omega'_x(0;x,t)|_{t=r(x,0)}\xi_2,\ \ \omega'_t(0;x,t)|_{t=r(x,0)}\xi_2,\ \ \xi_2 \neq 0\},$$

where $\omega'_t(0;x,t)|_{t=r(x,0)} \neq 0$.

We have proved the following lemma.

**Lemma 3.** *A distribution* $u$ *defined by (5) satisfies Item 4 of Definition 2 with* $\Pi$ *replaced by* $\Pi \cap \Omega_0^T$.

Turning back to computing the last integral in $I_0(t)$, consider the map

$$H : (x,t) \to (x - x_1, \omega(0;x,t) - x_1^*)$$

and the inverse map

$$H^{-1} : (x,t) \to (x + x_1, r(x + x_1, t)).$$

Then define $H^*A_n = \delta^{(n)}(x - x_1) \otimes 1(t)$ and $H^*B_m = \delta^{(m)}(\omega(0;x,t) - x_1^*)$.

We are now in a position to define the product of two distributions $\delta^{(n)}(x - x_1)$ and $\delta^{(m)}(\omega(0;x,t) - x_1^*)$. For all $\varphi \in \mathcal{D}(\mathbb{R}^2)$ we compute the action

$$\left\langle S(x,t)\delta^{(n)}(x - x_1)\delta^{(m)}(\omega(0;x,t) - x_1^*), \varphi(x,t)\right\rangle = \langle H^*A_n H^*B_m, S(x,t)\varphi(x,t)\rangle =$$

$$= \langle H^*(A_n B_m), S(x,t)\varphi(x,t)\rangle = \left\langle A_n B_m, \frac{(S\varphi)(x + x_1, r(x + x_1, t))}{\omega'_r(0; x + x_1, r(x + x_1, t))}\right\rangle =$$

$$= \left\langle \delta^{(n)}(x) \otimes \delta^{(m)}(t), \frac{(S\varphi)(x + x_1, r(x + x_1, t))}{\omega'_r(0; x + x_1, r(x + x_1, t))}\right\rangle =$$

$$= (-1)^{n+m} \left(\frac{(S\varphi)(x + x_1, r(x + x_1, t))}{\omega'_r(0; x + x_1, r(x + x_1, t))}\right)^{(n+m)}_{x,t} \Bigg|_{x=0,t=0} =$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{n+m} F_{ji}(x,t)\varphi^{(j+i)}_{x,t}(x + x_1, r(x + x_1, t))\Big|_{x=0,t=0} = \sum_{j=0}^{n} \sum_{i=0}^{n+m} F_{ji}(0,0)\varphi^{(j+i)}_{x,t}(x_1, t_1^*) =$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{n+m} F_{ji}(0,0) \left\langle \delta^{(j)}(x - x_1) \otimes \delta^{(i)}(t - t_1^*), \varphi(x,t)\right\rangle.$$

Here $F_{ji}(x,t)$ are known smooth functions of $\lambda$, $S$, and of all their derivatives up to the order $n+m$. Furthermore, for $\psi(t) \in \mathcal{D}(0,T)$

$$\left\langle \int_0^L \delta^{(n)}(x-x_1)S(x,t)\delta^{(m)}(\omega(0;x,t)-x_1^*)\,dx, \psi(t)\right\rangle =$$

$$= \sum_{j=0}^{n}\sum_{i=0}^{n+m} F_{ji}(0,0)\left\langle \int_0^L \delta^{(j)}(x-x_1)\otimes\delta^{(i)}(t-t_1^*)\,dx, \psi(t)\right\rangle =$$

$$= \sum_{j=0}^{n}\sum_{i=0}^{n+m} F_{ji}(0,0)\left\langle \delta^{(i)}(t-t_1^*)\otimes\delta^{(j)}(x-x_1), 1(x)\otimes\psi(t)\right\rangle =$$

$$= \sum_{i=0}^{n+m} F_{0i}(0,0)\langle\delta^{(i)}(t-t_1^*),\psi(t)\rangle.$$

As a consequence,

$$I_0(t) = \int_{\omega(t;0,0)}^{L} b_r(x)(S(x,t)a_r(\omega(0;x,t)) + S_1(x,t))\,dx +$$
$$+(-1)^n(S(x,t)a_r(\omega(0;x,t)) + S_1(x,t))_x^{(n)}|_{x=x_1} + \tag{13}$$
$$+(-1)^m\left(\frac{S(\varrho(x,t),t)b_r(\varrho(x,t))}{\omega'_\varrho(0;\varrho(x,t),t)}\right)_x^{(m)}\Bigg|_{x=0} + \sum_{i=0}^{n+m}F_{0i}(0,0)\delta^{(i)}(t-t_1^*).$$

Observe that, by Assumption A2 and Assumption A3, the first three summands in (13) are smooth for $t>0$.

Further construction is based on the splitting of $\Omega_1^T$ into subdomains

$$\Omega(i) = \{(x,t)\in\Omega_1^T \mid \omega(t;0,t_i^*) < x < \omega(t;0,t_{i-1}^*)\}$$

and constructing the solution separately in each $\Omega(i)$ and in a neighborhood of each border between $\Omega(i)$ and $\Omega(i+1)$. Here $t_0^*=0$, $1\le i\le k(T)$, where $k(T)$ is defined by inequalities $t_{k(T)}^* < T$ and $t_{k(T)+1}^* \ge T$. The finiteness of $k(T)$ is ensured by A1 and A4 for $\lambda$.

**Lemma 4.** *There exists a smooth solution to the problem (1)-(3) on* $\Omega(1)$.

*Proof.* Under the assumption that $x_1^* < x_1$, we have $\omega(t_1^*;0,0) < L$. Hence $(x_1,t_1^*)\in\Omega_0$. Therefore any solution which is given by (5) on $\Omega_0^T$ and is smooth on $\Omega(1)$, by Item 9 of Definition 2 satisfies the integral equation

$$u(x,t) = S_3(x,t) + S_2(x,t)\int_0^{\omega(\theta(x,t);0,0)} b_r(\xi)u(\xi,\theta(x,t))\,d\xi, \tag{14}$$

where

$$S_2(x,t) = S(x,t)c_r(\theta(x,t))$$

and

$$S_3(x,t) = S_2(x,t)I_0(\theta(x,t)) + S_1(x,t)$$

are known by (13). The smoothness of $I_0(\theta(x,t))$ if $(x,t) \in \Omega(1)$ follows from the facts that $\theta(x,t) < t_1^*$ and that $I_0(t)$ restricted to the interval $(0,t_1^*)$ is smooth. Therefore $S_2$ and $S_3$ are smooth. Observe that

$$\omega(\theta(x,t);0,0) \le \theta(x,t) \max_{(x,t)\in\overline{\Omega(1)}} \lambda \le t \max_{(x,t)\in\overline{\Omega(1)}} \lambda.$$

Hence (14) is the Volterra integral equation of the second kind.

The existence follows from the proof of Theorem 3 in [6].

From the formulas (5) and (14), Lemma 4, and Assumption A2 it follows that $u$ is smooth in a neighborhood of the characteristic curve $x = \omega(t;0,0)$. This ensures that $u$ we construct satisfies Item 9 of Definition 2.

Under the assumption that $\Omega(2)$ is nonempty, below we give the formula of the solution on

$$\Omega^\varepsilon(1) = \Omega(1) \cup \{(x,t) \in \overline{\Omega(2)} \;\mid\; x > \omega(t;0,t_1^*+\varepsilon)\}$$

for a fixed $\varepsilon > 0$ such that $t_1^* - \varepsilon > 0$ and

$$b_r(x) = 0, \quad x \in [0, \omega(t_1^*+\varepsilon;0,t_1^*-\varepsilon)]. \tag{15}$$

Such $\varepsilon$ exists by A3.

Write now

$$v(t) = \int_0^L (b_r(x) + \delta^{(n)}(x-x_1))u\,dx = v_r(t) + v_s(t), \tag{16}$$

where $v_r(t)$ and $v_s(t)$ are, respectively, regular (smooth) and singular parts of $v(t)$. On the account of (11), (13), (15), and the fact that $x_1^* < x_1$, we have on $[0,t_1^*+\varepsilon]$:

$$\begin{aligned} v_r(t) = \int_{\omega(t;0,t_1^*-\varepsilon)}^{\omega(t;0,0)} b_r(x)u(x,t)\,dx + \int_{\omega(t;0,0)}^{L} b_r(x)[S(x,t)a_r(\omega(0;x,t)) + S_1(x,t)]\,dx + \\ +(-1)^n(S(x,t)a_r(\omega(0;x,t)) + S_1(x,t))_x^{(n)}|_{x=x_1} + \\ +(-1)^m\left(\frac{S(\varrho(x,t),t)b_r(\varrho(x,t))}{\omega'_\varrho(0;\varrho(x,t),t)}\right)_x^{(m)}\Bigg|_{x=0} \end{aligned} \tag{17}$$

and

$$v_s(t) = \sum_{i=0}^{n+m} F_{0i}(0,0)\delta^{(i)}(t-t_1^*). \tag{18}$$

Note that the first summand in (17) is a known smooth function. This follows from the inclusion $[\omega(t;0,t_1^*-\varepsilon), \omega(t;0,0)] \times \{t\} \subset \Omega(1) \cup \{(x,t)\,|\,x = \omega(t;0,0)\}$, Lemma 4 and Assumption A2. Hence $u(x,t)$ is smooth on $\Omega(1) \cup \{(x,t)\,|\,x=\omega(t;0,0)\}$.

We consider two cases.

*Case 1.* $t_1^* = t_1$. As easily seen from (16), (17), and (18), $v(t) = v_r(t)$ on $[0, t_1^* + \varepsilon]$. As a consequence, Item 6 of Definition 2 for $u$ we construct is fulfilled. Furthermore

$$u(0,t) = (\delta^{(j)}(t - t_1^*) + c_r(t))v_r(t) = \sum_{i=0}^{j} C_i \delta^{(i)}(t - t_1^*) + c_r(t)v_r(t) \tag{19}$$

for $t \in (0, t_1^* + \varepsilon)$. Here $C_i$ are constants depending on $v_r^{(k)}(t_1^*)$ for $0 \le k \le j$. These constants are known due to (17).

*Case 2.* $t_1^* \ne t_1$. Then $\tilde{\omega}(x_1; x_1^*, 0) = t_1^*$. Using (16) and (17), we derive a similar formula for $u(0,t)$ on $(0, t_1^* + \varepsilon)$:

$$u(0,t) = c_r(t) \sum_{i=0}^{n+m} F_{0i}(0,0)\delta^{(i)}(t - t_1^*) + c_r(t)v_r(t) = \sum_{i=0}^{n+m} E_i \delta^{(i)}(t - t_1^*) + c_r(t)v_r(t), \tag{20}$$

where $E_i$ are constants depending on $F_{0,k}(0,0)$ and $c_r^{(k)}(t_1^*)$ for $0 \le k \le n+m$.

Set

$$Q(t) = \sum_{i=0}^{j} C_i \delta^{(i)}(t - t_1^*) \quad \text{if} \quad t_1^* = t_1$$

and

$$Q(t) = \sum_{i=0}^{n+m} E_i \delta^{(i)}(t - t_1^*) \quad \text{if} \quad t_1^* \ne t_1.$$

**Lemma 5.** *$u(x,t)$ given by the formula*

$$u(x,t) = S(x,t)c_r(\theta(x,t))v_r(\theta(x,t)) + S_1(x,t) + S(x,t)Q(\theta(x,t)), \tag{21}$$

*where $v_r(t)$ is determined by (17), is a $\mathcal{D}'(\Omega)$-solution to the problem (1)-(3) restricted to $\Omega^\varepsilon(1)$.*

*Proof.* On the account of (19), (20), and the construction of the solution on $\Omega(1)$, it is enough to prove that the restriction of $S(x,t)Q(\theta(x,t))$ to $Y = \{0\} \times (0, t_1^* + \varepsilon)$ is well defined and that $S(x,t)Q(\theta(x,t))$ satisfies (1) with $g(x,t) \equiv 0$ on $\Omega^\varepsilon(1)$ in a distributional sense.

Consider the smooth bijective map

$$\Phi : (x,t) \to (x, \tilde{\omega}(0; x, t) - t_1^*)$$

with

$$\Phi^{-1} : (x,t) \to (x, \pi(x,t)),$$

where $t \equiv \omega(0; x, \pi(x,t)) - t_1^*$. Observe that $\Phi$ restricted to $\Omega^\varepsilon(1)$ is a diffeomorphism. Since

$$\mathrm{WF}(\Phi^* B_i) \subset \{(0, \pi(0,t), \tilde{\omega}_x(0; 0, \pi(0,t))\eta, \tilde{\omega}_\pi(0; 0, \pi(0,t))\eta), \eta \ne 0\}$$

and

$$N(Y) = \{(0, t, \xi, 0)\},$$

we have

$$\mathrm{WF}(\Phi^* B_i) \cap N(Y) = \emptyset \quad \text{for all} \quad 0 \le i \le n+m.$$

This means that the restriction of $S(x,t)Q(\theta(x,t))$ to $Y$ is well defined.

Similarly to the proof of Lemma 1, one can show that $S(x,t)Q(\theta(x,t))$ satisfies (1) with $g(x,t) \equiv 0$ in a distributional sense. This finishes the proof.

To shorten notation, without loss of generality we assume that $\max \overline{\Omega_1^T} \cap \{(x,t) \,|\, x = 0\} \ge t_2^*$.

**Lemma 6.** *There exists a smooth solution to the problem (1)-(3) on* $\Omega(2)$.

*Proof.* We start from the general formula of a smooth solution on $\Omega(2)$:

$$u(x,t) = S(x,t)u(0,\theta(x,t)) + S_1(x,t). \tag{22}$$

Since $S$ and $S_1$ are smooth, our task is to prove that there exists a smooth function identically equal to $u(0,\theta(x,t))$ on $\Omega(2)$. Since $\theta(x,t)$ is smooth, $t_1^* < \theta(x,t) < t_2^*$ if $(x,t) \in \Omega(2)$, and $c(t) = c_r(t)$ if $t \in (t_1^*, t_2^*)$, it suffices to show the existence of a smooth function $v_r(t)$ identically equal to $v(t)$ on $(t_1^*, t_2^*)$. From the formula (18) for $v_s(t)$ on $(0, t_1^* + \varepsilon)$ it follows that $v(t) = v_r(t)$ if $t \in (t_1^*, t_1^* + \varepsilon)$, where $\varepsilon$ is as above and $v_r(t)$ is known and determined by (17). To prove the lemma, it is sufficient to show that there exists a smooth extension of $v_r(t)$ from $(0, t_1^* + \varepsilon)$ to $[t_1^* + \varepsilon, t_2^*)$ such that $v_r(t) = v(t)$ if $t \in [t_1^* + \varepsilon, t_2^*)$. If a such extension exists, then by (21) it satisfies the following integral equation on $[t_1^* + \varepsilon, t_2^*)$:

$$v_r(t) = \int\limits_0^{\omega(t;0,t_1^*+\varepsilon)} b_r(x)S(x,t)c_r(\theta(x,t))v_r(\theta(x,t))\,dx + R(t), \tag{23}$$

where

$$\begin{aligned} R(t) = &\int\limits_{\omega(t;0,t_1^*+\varepsilon)}^{P(t)} b_r(x)S(x,t)c_r(\theta(x,t))v_r(\theta(x,t))\,dx + \int\limits_0^{P(t)} b_r(x)S_1(x,t)\,dx \\ &+ I_0(t) + \int\limits_0^L b_r(x)S(x,t)Q(\theta(x,t))\,dx, \end{aligned} \tag{24}$$

$$P(t) = \begin{cases} \omega(t;0,0) & \text{if } \ \tilde\omega(L;0,0) \le t, \\ L & \text{if } \ \tilde\omega(L;0,0) \ge t, \end{cases}$$

$b_r(x)$ is defined to be 0 outside $[0,L]$, and $v_r$ in the formula (24) is known and defined by (17). One can easily see that the first three summands in (24) are smooth functions on $[t_1^* + \varepsilon, t_2^*)$. We now show that the last summand is a $C^\infty[t_1^* + \varepsilon, t_2^*)$-function as well. Let us consider the smooth map

$$h : (x,t) \to (\theta(x,t) - t_1^*, t)$$

that is bijective and therefore has the inverse map

$$h^{-1} : (x,t) \to (\zeta(x,t), t),$$

for which it holds $\theta(\zeta(x,t),t) - t_1^* \equiv x$. The function $\zeta(x,t)$ is smooth with respect to $x$ and $t$, what is clear from (9).

Since $\theta(x,t) = \tilde\omega(0;x,t)$, for $\psi(t) \in \mathcal{D}(t_1^* + \varepsilon/2, t_1^*)$ we have

$$\left\langle \int\limits_0^L b_r(x)S(x,t)\delta^{(j)}(\theta(x,t) - t_1^*)\,dx, \psi(t) \right\rangle = \left\langle \delta^{(j)}(\theta(x,t) - t_1^*), b_r(x)S(x,t)\psi(t) \right\rangle =$$

$$= \left\langle h^*\delta^{(j)}(x), b_r(x)S(x,t)\psi(t) >=< \delta^{(j)}(x) \otimes 1(t), \frac{b_r(\zeta(x,t))S(\zeta(x,t),t)\psi(t)}{\tilde{\omega}'_\zeta(0;\zeta(x,t),t)} \right\rangle =$$

$$= (-1)^j \left\langle \left( \frac{b_r(\zeta(x,t))S(\zeta(x,t),t)}{\tilde{\omega}'_\zeta(0;\zeta(x,t),t)} \right)^{(j)}_x \Bigg|_{x=0}, \psi(t) \right\rangle.$$

From this equality we conclude that, irrespective of whether $t_1 = t_1^*$ or $t_1 \neq t_1^*$, the last summand in (24) is a known smooth function. As follows from (15), the functions $v_r(t)$ defined by (17) and (23) coincide at $t = t_1^* + \varepsilon$. The same is true with respect to all the derivatives of $v_r$.

Therefore our task is reduced to show that there exists a $C^\infty[t_1^* + \varepsilon, t_2^*)$-function $v_r(t)$ satisfying the equation (23). This follows from the fact that (23) is the integral Volterra equation of the second kind with respect to $v_r(t)$ (for details see the proof of Lemma 4). The proof is complete.

Continuing our construction in this fashion, we extend $u$ over a neighborhood of each subsequent border between $\Omega(i-1)$ and $\Omega(i)$ and over $\Omega(i)$ for all $3 \le i \le k(T)$. Eventually we construct $u$ on $\Omega^T$ for any $T > 0$ in the sense of Definition 2 with $\Omega$ replaced by $\Omega^T$ and $\Pi$ replaced by $\Pi^T = \{(x,t) \in \Pi \,|\, t < T\}$. As easily seen from our construction, the condition (4) is fulfilled with $\Omega_+$ and $\Omega'_+$ replaced by $\Omega^T \cap \Omega_+$ and $\Omega^T \cap \Omega'_+$, respectively. Since $T$ is arbitrary, the proof of Item 1 of Theorem 1 is complete. On the account of Definition 3 and the definition of the restriction $u \in \mathcal{D}'(\Omega)$ to a subset of $\Omega$ (see [3, Section 5]), Item 2 of Theorem 1 is a straightforward consequence of Item 1. Theorem 1 is proved.

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*Стаття: надійшла до редакції 11.08.2011*
*прийнята до друку 14.12.2011*

## КЛАСИЧНА РОЗВ'ЯЗНІСТЬ СИНГУЛЯРНОЇ ГІПЕРБОЛІЧНОЇ ЗАДАЧІ З ІНТЕГРАЛЬНОЮ КРАЙОВОЮ УМОВОЮ

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Розглянуто лінійну мішану гіперболічну задачу з інтегральною крайовою умовою, вихідні дані якої є сильно сингулярними розподілами. Основний результат полягає в побудові розв'язку цієї задачі в сенсі розподілів. Як частковий випадок розв'язано задачу Лотки МакКендріка зі змінними коефіцієнтами в диференціальній частині.

*Ключові слова:* узагальнена задача Лотки МакКендріка, сингулярні вихідні дані, розв'язок у сенсі розподілів.

## КЛАССИЧЕСКАЯ РАЗРЕШИМОСТЬ СИНГУЛЯРНОЙ ГИПЕРБОЛИЧЕСКОЙ ЗАДАЧИ С ИНТЕГРАЛЬНЫМ ГРАНИЧНЫМ УСЛОВИЕМ

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Рассмотрено линейную смешанную гиперболическую задачу с интегральным граничным условием, исходные данные которой являются сильно сингулярными распределениями. Основной результат заключается в построении решения этой задачи в смысле распределений. Как частный случай решено задачу Лотки МакКендрика с изменяющимися коэффициентами в дифференциальной части.

*Ключевые слова:* обобщенная задача Лотки МакКендрика, сингулярные исходные данные, решение в смысле распределений.