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ON FUZZY GROMOV-HAUSDORFF HYPERSPACE OF THE UNIT SEGMENT

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The main result of this note is to prove that the fuzzy Gromov-Hausdorff space of the unit segment is homeomorphic to the Hilbert cube.

Key words: fuzzy space, Gromov-Hausdorff space.

1. Introduction. The notion of Gromov-Hausdorff distance (see, e.g., [1]) finds numerous applications in different areas of mathematics as well as in the field of computer graphics and computational geometry. Using this notion, one can naturally define the so called Gromov-Hausdorff hyperspace of any metric space. Some questions concerning this hyperspace are formulated in [2].

The fuzzy Gromov-Hausdorff space is first considered in [7]. Remark that the notion of fuzzy metric space traces back to the notions of probabilistic metric space. Nowadays, this notion is widely investigated and finds numerous applications in different areas of mathematics.

The following problem is a natural modification of problems concerning the Gromov-Hausdorff spaces: describe the topology of the fuzzy Gromov-Hausdorff space (see the definition below) of the unit segment I. An answer to the corresponding for problem the Gromov-Hausdorff space is announced in [9]. The main result states that the mentioned fuzzy Gromov-Hausdorff space is homeomorphic to the Hilbert cube.

We also formulate some open questions. Note that the most problems that concern the fuzzy Gromov-Hausdorff space are open.

2. Preliminaries.

2.1. Fuzzy metric spaces. We start with the definition of fuzzy metric spaces (see, e.g., [5]). A continuous t-norm is a continuous map $(x, y) \mapsto x * y\colon [0,1]\times[0,1]\to[0,1]$ which satisfies the following properties:

(1) $(x * y) * z = x * (y * z)$;
(2) $x * y = y * x$;
(3) $x * 1 = x$;
(4) if $x \le x'$ and $y \le y'$, then $x * y \le x' * y'$.

In other words, a continuous t-norm is a continuous Abelian monoid with unit 1 and with the monotonic operation. The following are examples of continuous t-norms:

(1) $x * y = \min\{x, y\}$;
(2) $x * y = \max\{0, x + y - 1\}$ (Łukasiewicz t-norm).

In this paper we use the notion of fuzzy metric space defined in [5].

Definition 1. *A* fuzzy metric space *is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set of $X \times X \times (0, \infty)$ (i.e. M is a map from $X \times X \times (0, \infty)$ to $[0, 1]$) satisfying the following properties:*

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ *if and only if* $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, s) * M(y, z, t) \leq M(x, z, s + t)$;
(v) *the function* $M(x, y, -)\colon (0, \infty) \to [0, 1)$ *is continuous.*

Note that every metric d on a set X generated the fuzzy metric M_d on X by the formula:

$$M_d(x, y, t) = \frac{t}{d(x, y) + t}.$$

In a fuzzy metric space $(X, M, *)$, we say that the set

$$B_M(x, r, t) = \{y \in X \quad | \quad M(x, y, t) > 1 - r\}, \quad x \in X, \quad r \in (0, 1), \quad t \in (0, \infty),$$

is the *open ball* of radius $r > 0$ centered at x for t. It is proved in [5] that the family of all open balls is a base of a topology on X; this topology is denoted by τ_M.

If we speak on a fuzzy (pseudo)metric on a topological space, we always assume that this metric is compatible with the topology of this space.

2.2. Hausdorff and Gromov-Hausdorff metric. Let $\exp X$ denote the hyperspace of X, i.e. the set of all nonempty compact subsets of X. This space is endowed with the Vietoris topology, i.e. the topology whose base consists of the sets of the form

$$\langle U_1, \dots, U_n\rangle = \{A \in \exp X \quad | \quad A \subset \cup_{i=1}^n A_i, \quad A \cap U_i \neq \emptyset, \quad i = 1, \dots, n\}.$$

If (X, d) is a metric space, then the Vietoris topology is generated by the Hausdorff metric d_H:

$$d_H(A, B) = \max\{\sup_{a\in A} d(a, B), \quad \sup_{b\in B} d(b, A)\}$$

(see, e.g., [4]).

Let (X_i, d_i), $i = 1, 2$, be compact metric spaces. The *Gromov-Hausdorff distance* between these spaces is the number

$$d_{GH}((X_1, d_1), (X_2, d_2)) = \inf\{d_H(f_1(X_1), f_2(X_2)) \quad | \quad f_i\colon X_i \to (Y, d) \quad \text{is an isometric embedding}\}.$$

Given a metric space (X, d), we denote by $\exp_{GH}(X)$ the space of nonempty closed subsets in X endowed with the Gromov-Hausdorff metric.

Note that in the sequel we identify the isometric compact metric spaces and therefore the Gromov-Hausdorff distance is the distance between the classes of equivalence of compact metric spaces up to isometry.

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2.3. Fuzzy Hausdorff and fuzzy Gromov-Hausdorff metric. For every $a \in X$ i $t > 0$, let

$$M(a, B, t) = \sup\{M(a, b, t) \ \mid \ b \in B\}$$

(see. [10, Definition 2.4]).

Following [10] define the function $M_H\colon \exp X \times \exp X \to (0, \infty)$ by the formula:

$$M_H(A, B, t) = \min\left\{\inf_{a\in A} M(a, B, t), \ \ \inf_{b\in B} M(A, b, t)\right\}$$

for every $A, B \in \exp X$ i $t > 0$. The pair $(M_H, *)$ is a fuzzy metric on the space $\exp X$ (see [10, theorem 1]). This metric is called *the fuzzy Hausdorff metric* and is also known to generate the Vietoris topology on $\exp X$.

Let $(X_i, M_i, *)$, $i = 1, 2$, be fuzzy metric spaces. The number

$$\begin{aligned}&M_{GH}((X_1, M_1, *), (X_2, M_2, *), t) = \sup\{M_H(F_1(X_1), F_2(X_2), t) \mid \\ &F_i\colon X_i \to Z \quad \text{is an isometric embedding into a fuzzy metric space} \quad Z\}\end{aligned}$$

is called the fuzzy Gromov-Hausdorff distance between $(X_1, M_1, *)$ and $(X_2, M_2, *)$ at t.

Remark that the number $M_{GH}((X_1, M_1, *), (X_2, M_2, *), t)$ is well defined, because for any two fuzzy metric spaces there exists a fuzzy metric space that contains their isometric copies. Namely, the bouquet of these spaces can serve as an example (see [11]).

By $\mathcal{FM}_{GH}(X, M, *)$ we denote the family of compact subspaces of a fuzzy metric space $(X, M, *)$ endowed with the fuzzy Gromov-Hausdorff metric.

Proposition 1. *Let $(X_i, d_i)_{i=1}^{\infty}$ be a sequence of compact metric spaces converging to (X, d) with respect to the Gromov-Hausdorff metric. Then the sequence $(X_i, M_{d_i}, *)_{i=1}^{\infty}$ converges to $(X, M_d, *)$ with respect to the topology generated by the fuzzy Gromov-Hausdorff metric.*

Proof. Let $t \in (0, \infty)$ and $r \in (0, 1)$. There exists $N \in \mathbb{N}$ such that, for every $n > N$, we have $d_{GH}((X_n, d_n), (X, d)) < \frac{t}{1-r} + t$. Without loss of generality, one may assume that there exists a metric space (Z, ϱ) containing both X_n and X such that $d_{GH}((X_n, d_n), (X, d)) = \varrho_H(X_n, X)$. Simple calculations demonstrate that $(M_\varrho)_H(X_n, X) > 1 - r$, and therefore $M_{GH}((X_n, M_{d_n}, *), (X, M_d, *), t) > 1 - r$.

Corollary 1. *For any compact fuzzy metric space $(X, M, *)$, the space $\mathcal{FM}_{GH}(X, M, *)$ is compact.*

We are going to show that the obtained space is Hausdorff. Indeed, otherwise, there exist fuzzy metric spaces $(Y, M_\varrho, *)$, $(Y', M_{\varrho'}, *)$ and $(X_i, M_i, *)$ such that there exists a fuzzy metric space $(Z_i, N_i, *)$, $i \in \mathbb{N}$, containing these spaces and satisfying the property: $N_{iH}(Y, X_i, 1/i) > 1 - (1/i)$, $N_{iH}(Y', X_i, 1/i) > 1 - (1/i)$. Now, let $a, b \in Y$. There exist $a_i, b_i \in X_i$ and $a_i', b_i' \in X'$ such that

$$N_i(a, a_i, 1/i) > 1 - (1/i), \quad N_i(a_i, a_i', 1/i) > 1 - (1/i),$$

$$N_i(b, b_i, 1/i) > 1 - (1/i), \quad N_i(b_i, b_i', 1/i) > 1 - (1/i).$$

Then, for every $t > 0$,

$$(1 - (1/i)) * M_{\varrho'}(a', b', t) * (1 - (1/i)) \le M_\varrho(a, b, t + (2/i)).$$

Since X' is compact, without loss of generality, one may assume that the sequences $(a'_i), (b'_i)$ converge to a' and b' respectively. Then passing to the limit we obtain that $M_{\varrho'}(a', b', t) \leq M_\varrho(a, b, t)$ and similarly $M_{\varrho'}(a', b', t) \geq M_\varrho(a, b, t)$. One can easily prove that a' and b' do not depend on the chosen sequences (a_i) and (b_i) and thus the map $a \mapsto a'$ is well-defined. Actually, this map isometrically embeds Y into Y'.

One can similarly prove that there exists an isometric embedding of Y' into Y. Since Y and Y' are compact, we conclude that they are isometric, i.e. they represent the same isometry class.

We have therefore proven the following statement.

Proposition 2. *For every compact metrizable space X, the spaces $\exp_{GH}(X, d)$ and $\mathcal{FM}_{GH}(X, M_d, *)$ are homeomorphic.*

2.4. Hilbert cube. By AR we denote the class of absolute retracts in the class of metrizable spaces.

We say that a compact metric space (X, d) satisfies the disjoint approximation property (DAP) if, for every $\varepsilon > 0$, there exist maps $f, g\colon X \to X$ such that $d(f, 1_X) < \varepsilon$, $d(g, 1_X) < \varepsilon$ and $f(X) \cap g(X) = \emptyset$.

The Hilbert cube $Q = [0, 1]^\omega$ can be characterized as follows.

Theorem 1 (Toruńczyk's characterization theorem)**.** *A compact metrizable space X is homeomorphic to the Hilbert cube if X is an AR space and satisfies the DAP.*

3. Main result. The aim of this note is to prove the following theorem. Here d is the standard metric on the unit segment $I = [0, 1]$.

Theorem 2. *The space $\mathcal{FM}_{GH}(I, M_d, *)$ is homeomorphic to the Hilbert cube.*

Proof. Consider the circle $S^1 = \{e^{2\pi i t} \mid t \in [0, 1]\}$. The group Spin(2) naturally acts on this space and also on its hyperspace $\exp S^1$. Recall that the group Spin(2) is the double cover of SO(2) and there exists an exact sequence

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(2) \to \mathrm{SO}(2) \to 1.$$

Let I be embedded into S^1 as a segment lying in a half-circle. Without loss of generality, one may assume that the invariant metric on S^1 extends the standard metric d on I. Therefore, $\exp_{GH}(I)$ is naturally embedded into $(\exp S^1)/\mathrm{Spin}(2)$. In turn, $(\exp S^1)/\mathrm{Spin}(2) = ((\exp S^1)/S^1)/(\mathbb{Z}/2)$. We will identify $\exp_{GH}(I)$ with the subspaces in the mentioned orbit spaces.

First, we will show that $\exp_{GH}(I)$ is an absolute retract. Let $J \supset I$ be an open interval containing I in S^1. We assume that J lies in a half-circle. Denote by U the subset in $(\exp S^1)/S^1$ consisting of the orbits with an element in J. Clearly, U is and open contractible subset in $(\exp S^1)/S^1$ and therefore an absolute retract. One can define an equivariant retraction r of U onto $(\exp S^1)/S^1$ by the condition: $r(A)$ is a homothetic copy of A with the coefficient equal to $\frac{\min\{\mathrm{diam}(A), 1\}}{\mathrm{diam}(A)}$. Next, a retraction of $(\exp I)/S^1$ onto $\exp_{GH}(I) \subset (\exp S^1)/\mathrm{Spin}(2)$ is given by the following construction: the image of the orbit containing A is $A \cup A'$, where A' is a symmetric copy of A with respect to the center of the minimal segment containing A. This proves that $\exp_{GH}(I)$ is an absolute retract.

Next, we are going to show that the space $\mathcal{FM}_{GH}(I, M_d, *)$ satisfies the DAP. Given $\varepsilon > 0$, define $f(A)$ as the closed $\varepsilon/2$-neighborhood of A (symmetrically truncated, if necessary, so that its length does not exceed 1). Further, define $g(A)$ as the union of two endpoints of $f(A)$ and the homothetic copy of $f(A)$ with respect to its center of symmetry. If the scale factor of the homothety is close enough to 1, we are done. This proves the DAP and completes the proof of the theorem.

4. Remarks and open problems. Note first that the result of the previous section can be extended over another fuzzy Gromov-Hausdorff spaces. In particular, a counterpart of Theorem 2 holds for the spaces defined as follows.

Let $* = \min$. Given $k, m > 0$ and $n \geq 1$, define the fuzzy metric M_i, $i = 1, 2$, on I as follows:

$$M_1(x,y,t) = \frac{kt^n}{kt^n + |x-y|}, \quad M_2(x,y,t) = e^{-\frac{|x-y|}{t^n}}$$

(see [7]).

The following question remains open. Describe the topology of the fuzzy Gromov-Hausdorff space of the n-dimensional cube I^n, $n \geq 2$.

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ПРО РОЗМИТИЙ ГІПЕРПРОСТІР ГРОМОВА-ГАУСДОРФА ОДИНИЧНОГО ВІДРІЗКА

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З основного результату випливає, що розмитий простір Громова-Гаусдорфа одиничного сегмента гомеоморфний гільбертовому кубові.

Ключові слова: розмитий простір, простір Громова-Гаусдорфа.

О НЕЧЕТКОМ ГИПЕРПРОСТРАНСТВЕ ГРОМОВА-ГАУСДОРФА ЕДИНИЧНОГО ОТРЕЗКА

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С главного результата следует, что размытое пространство Громова-Гаусдорфа единичного сегмента гомеоморфное гильбертовому кубу.

Ключевые слова: размытое пространство, пространство Громова-Гаусдорфа.