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AN INITIAL VALUE PROBLEM $x' = f(t, x, x')$, $x(0) = 0$: SOLVABILITY, NUMBER OF SOLUTIONS, ASYMPTOTICS

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We find a nonempty set of continuously differentiable solutions $x\colon (0, \rho] \to \mathbb{R}$ each of which possesses required asymptotical properties when $t \to +0$. Also we establish uniqueness conditions.

Key words: implicit differential equation, initial value problem, solvability, uniqueness, asymptotic property.

The general solvability and solutions number problem for implicit ordinary differential equations was under consideration in [1], [2], [3], [6]. In [7], [9], [10] conditions for convergence of successive approximations to implicit equations solutions were found. At the same time asymptotic properties of implicit differential equations are still only partially understood; there are only isolated results obtained, for example, [11]. This article presents an investigation of the initial value problem $x' = f(t, x, x')$, $x(0) = 0$. The asymptotic behaviour of solutions is being discussed. We describe an approach which makes it possible to consider implicit initial value problems. Our approach to the problem seems to be very much different from the usual ones. We use qualitative methods (see, for instance, [4], [5], [8], and also [11]) together with fixed point methods. We establish general schemes of investigation which may be applied to many various problems of local analysis. In this paper existence of continuously differentiable solutions is being proved. Asymptotic properties of each of these solutions is discussed and if certain conditions are fulfilled then the uniqueness of solution is established.

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1. First the following initial value problem

$$x'(t) = f(t, x(t), x'(t)), \tag{1}$$

$$x(0) = 0 \tag{2}$$

is under consideration, where $t \in (0, \tau)$ is a real variable, $x : (0, \tau) \to \mathbb{R}$ is a real unknown function, $f : \mathcal{D} \to \mathbb{R}$ is a continuous function,

$$\mathcal{D} = \{(t, x, y) : t \in (0, \tau), \ |x - \xi(t)| < r_1 t\alpha(t), \ |y - \xi'(t)| < r_2\alpha(t)\};$$

here $\xi : (0, \tau) \to (0, +\infty)$, $\alpha : (0, \tau) \to (0, +\infty)$ are continuously differentiable functions,

$$|\xi'(t) - f(t, \xi(t), \xi'(t))| \le \alpha(t), \ t \in (0, \tau),$$

$$\lim_{t \to +0} \alpha(t) = 0, \ \lim_{t \to +0} \xi(t) = 0, \ \lim_{t \to +0} \xi'(t) = \xi_0, \ 0 \le \xi_0 < +\infty,$$

$$\lim_{t \to +0} \frac{\alpha(t)}{\xi'(t)} = 0, \lim_{t \to +0} t\frac{\alpha'(t)}{\alpha(t)} = \alpha_0, 0 \le \alpha_0 < +\infty.$$

Suppose that

$$|f(t, x, y_1) - f(t, x, y_2)| \le l_y |y_1 - y_2|, \ (t, x, y_i) \in \mathcal{D}, \ i \in \{1, 2\},$$

where l_y is a constant, $0 < l_y < 1$, $(1 - l_y)^{-1} < \min\{(1 + \alpha_0) r_1, l_y r_2\}$.

Definition 1. *For any $\rho \in (0, \tau)$ a continuously differentiable function $x\colon (0, \rho] \to \mathbb{R}$ is said to be a ρ-solution of the problem* (1), (2)*, if*

1) $(t, x(t), x'(t)) \in \mathcal{D}, \ t \in (0, \rho]$;
2) *x identically satisfies equation (1) for all $t \in (0, \rho]$;*
3) $\lim_{t \to +0} x(t) = 0$.

We denote by $\mathcal{U}(\rho, M, q)$ the set of all continuously differentiable functions $u\colon (0, \rho] \to \mathbb{R}$ such that

$$|u(t) - \xi(t)| \le Mt\alpha(t), \ |u'(t) - \xi'(t)| \le qM\alpha(t), \ t \in (0, \rho]; \tag{3}$$

here ρ, M, q are constants, $\rho \in (0, \tau)$, $M > 0$, $q > 0$.

Theorem 1. *Suppose that the following conditions hold:*

$$|f(t_1, x, y) - f(t_2, x, y)| \le l_t(\mu) |t_1 - t_2|, \ (t_i, x, y) \in \mathcal{D}, \ 0 < \mu \le t_1, \ t_2 < \tau, \tag{4}$$

$$|f(t, x_1, y) - f(t, x_2, y)| \le l_x(t) |x_1 - x_2|, \ (t, x_i, y) \in \mathcal{D}, \ i \in \{1, 2\}, \tag{5}$$

where $l_t\colon (0, \tau) \to (0, +\infty)$, $l_x\colon (0, \tau) \to (0, +\infty)$ are continuous functions, $0 < t_1 < t_2 < \tau \Rightarrow l_t(t_1) \ge l_t(t_2)$, $\lim_{t \to +0} t l_x(t) = 0$. Then there exist ρ, M, q such that the problem (1), (2) *has a nonempty set of ρ-solutions $x\colon (0, \rho] \to \mathbb{R}$ each of which belongs to $\mathcal{U}(\rho, M, q)$.*

Theorem 2. *Suppose that the following condition holds:*

$$|f(t, x_1, y) - f(t, x_2, y)| \le l_x |x_1 - x_2|, (t, x_i, y) \in \mathcal{D}, i \in \{1, 2\}, \tag{6}$$

where l_x is a constant, $l_x + l_y < 1$. Then there exist ρ, M, q such that the problem (1), (2) *has a unique ρ-solution $x\colon (0, \rho] \to \mathbb{R}$ which belongs to $\mathcal{U}(\rho, M, q)$.*

Proof of Theorem 1. First of all we select constants ρ, M, q. Let the following conditions hold:

$$1 + \alpha_0 < q < \frac{m_0 (1 + \alpha_0) - 1}{m_0 l_y},\ (1 + \alpha_0 - q l_y)^{-1} < M < m_0,$$

where $m_0 = ((1 + \alpha_0)(1 - l_y))^{-1}$. We do not present here the conditions for selection of ρ to keep the size of this paper reasonable. We now indicate nothing but ρ is small enough, M, q are large enough and our selection of ρ, M, q ensures the validity of all our reasoning given below. Let $\mathcal{B}$ be the space of continuously differentiable functions $x\colon (0, \rho] \to \mathbb{R}$ with the norm

$$\|x\|_{\mathcal{B}} = \max_{t \in [0,\rho]} (|x(t)| + |x'(t)|). \tag{7}$$

Let $\mathcal{U}$ be the subset of $\mathcal{B}$ such that every its element $u : [0, \rho] \to \mathbb{R}$ satisfies inequalities (3), and also $u(0) = 0$, $u'(0) = \xi_0$ and, moreover,

$$\forall \mu \in (0, \rho],\ \forall t_1,\, t_2 \in [\mu, \rho] : |u'(t_1) - u'(t_2)| \le K(\mu) |t_1 - t_2|, \tag{8}$$

where

$$K(\mu) = (1 - l_y)^{-1} \left(l_t(\mu) + \mu^{-1}\right).$$

It is easy to see that $\mathcal{U}$ is a closed, bounded and convex set. Moreover, $\mathcal{U}$ is a compact set (in view of the Arzelá Theorem). We will consider the differential equation

$$x'(t) = f(t, u(t), u'(t)), \tag{9}$$

where $u \in \mathcal{U}$ is an arbitrary fixed function. Let

$$\mathcal{D}_0 = \{(t, x) : t \in (0, \rho], x \in \mathbb{R}\}.$$

In $\mathcal{D}$ for equation (9) conditions of the Existence and Uniqueness Theorem and conditions of the Continuous Dependence of the Initial Data Theorem are fulfilled. Let

$$\Phi_1 = \{(t, x) : t \in (0, \rho], |x - \xi(t)| = M t \alpha(t)\},$$
$$\mathcal{D}_1 = \{(t, x) : t \in (0, \rho], |x - \xi(t)| < M t \alpha(t)\},$$
$$H = \{(t, x) : t = \rho, |x - \xi(\rho)| < M \rho \alpha(\rho)\}.$$

Let the function $A_1 \colon \mathcal{D}_0 \to [0, +\infty)$ be defined by the equality

$$A_1(t, x) = (x - \xi(t))^2 (t \alpha(t))^{-2}$$

and let $a_1 \colon \mathcal{D}_0 \to \mathbb{R}$ be the derivative of the function A_1 by virtue of equation (9). It is easy to see that $a_1(t, x) < 0$ when $(t, x) \in \Phi_1$. Let us prove that any integral curve of equation (9) which intersects Φ_1 at an arbitrary point (t_0, x_0) for small enough $|t - t_0|$ (where $t \le \rho$) lies in $\mathcal{D}_1$ if $t > t_0$ and lies outside of $\overline{\mathcal{D}}_1$ if $t < t_0$. In fact, let $P(t_0, x_0)$ be an arbitrary point belonging to Φ_1 and let $J_p : (t, x_P(t))$ be an integral curve of equation (9) which passes through the point P. Then

$$A_1(t_0, x_P(t_0)) = M^2,\ a_1(t_0, x_P(t_0)) < 0.$$

Therefore if $t_0 \in (0, \rho)$ then there exists $\delta > 0$ such that

$$\operatorname{sign}\,(A_1(t, x_P(t)) - A_1(t_0, x_P(t_0))) = \operatorname{sign}(\mathrm{t}_0 - t),\ |t - t_0| < \delta,$$

or

$$\operatorname{sign}\left(|x_P(t) - \xi(t)| (t\alpha(t))^{-1} - M\right) = \operatorname{sign}(\mathrm{t}_0 - t),\ |t - t_0| < \delta.$$

What this means is $(t, x_P(t)) \in \mathcal{D}_1$ if $t \in (t_0, t_0+\delta)$ and $(t, x_P(t)) \overline{\in} \overline{\mathcal{D}}_1$ if $t \in (t_0-\delta, t_0)$. If $t_0 = \rho$ then there exists $\delta > 0$ such that

$$A_1(t, x_P(t)) > A_1(t_0, x_P(t_0)),\ t \in (\rho-\delta, \rho),$$

or

$$|x_P(t) - \xi(t)|(t\alpha(t))^{-1} > M,\ t \in (\rho-\delta, \rho),$$

and this means that $(t, x_P(t)) \overline{\in} \overline{\mathcal{D}}_1$, $t \in (\rho-\delta, \rho)$.

This implies that at least one of integral curves of equation (9) which intersect H is defined for all $t \in (0, \rho]$ and lies in $\mathcal{D}_1$ if $t \in (0, \rho]$. In fact, having common points with Φ_1 when t increases is beyond the capabilities of any integral curve of equation (9) which intersects Φ_1. That is why all these curves have to intersect $\overline{H}$. Let the mapping $\psi\colon \Phi_1 \to \overline{H}$ be defined by the following way: the point $\psi(P) \in \overline{H}$ is assigned to $P \in \Phi_1$ if both P and $\psi(P)$ belong to the common integral curve of equation (9). Let

$$\psi(\Phi_1) = \{\psi(P) : P \in \Phi_1\}.$$

The set $\overline{H}\backslash\psi(\Phi_1)$ is nonempty ($\overline{H}$ is a closed set, but $\psi(\Phi_1)$ is not since $\psi(\Phi_1)$ is the image of the nonclosed set Φ_1). Let $J_u : (t, x_u(t))$ be an integral curve of equation (9) such that $(\rho, x_u(\rho)) \in \overline{H}\backslash\psi(\Phi_1)$. It is clear that $J_u : (t, x_u(t))$ has no common point with Φ_1. Therefore $J_u : (t, x_u(t))$ is defined for all $t \in (0, \rho]$ and $J_u : (t, x_u(t))$ comes into the point $(0, 0)$ if $t \to +0$ and, moreover, $J_u : (t, x_u(t))$ lies in $\mathcal{D}_1$ if $t \in (0, \rho]$. It is easy to see that the following inequalities are fulfilled when $t \in (0, \rho]$:

$$|x_u(t) - \xi(t)| \le Mt\alpha(t),\ |x'_u(t) - \xi'(t)| \le qM\alpha(t). \tag{10}$$

Let $x_u(0) = 0$, $x'_u(0) = \xi_0$. Let us prove that

$$\forall \mu \in (0, \rho]\, \forall t_1,\, t_2 \in [\mu, \rho] : |x'_u(t_1) - x'_u(t_2)| \le K(\mu)\,|t_1 - t_2|. \tag{11}$$

Select $\mu \in (0, \rho]$ and $t_i \in [\mu, \rho]$, $i \in \{1, 2\}$; let $t_1 < t_2$. From the identities

$$x'_u(t_i) = f(t_i, u(t_i), u'(t_i)),\ i \in \{1, 2\} \tag{12}$$

we obtain

$$\begin{aligned} |x'_u(t_1) - x'_u(t_2)| &\le l_t(\mu)\,|t_1 - t_2| + l_x(t_1)\,|u(t_1) - u(t_2)| + l_y\,|u'(t_1) - u'(t_2)| \le \\ &\le \left(l_t(\mu) + \mu^{-1}\right)|t_1 - t_2| + l_y K(\mu)\,|t_1 - t_2| = \\ &= (1 - l_y)\,K(\mu)\,|t_1 - t_2| + l_y K(\mu)\,|t_1 - t_2| = \\ &= K(\mu)\,|t_1 - t_2|. \end{aligned}$$

This means that $x_u \in \mathcal{U}$. Let us prove that if $t \to +0$ then all integral curves of equation (9) leave the set $\overline{D}_1 \backslash \{(0, 0)\}$, with the only exception $J_u : (t, x_u(t))$. Indeed, let

$$\Phi_2(\mu) = \{(t, x) : t \in (0, \rho],\ |x - x_u(t)| = \mu t\alpha(t)(-\ln t)\},$$

$$\mathcal{D}_2(\mu) = \{(t, x) : t \in (0, \rho],\ |x - x_u(t)| < \mu t\alpha(t)(-\ln t)\},$$

where μ is a parameter, $\mu \in (0, 1]$. Let the function $A_2\colon \mathcal{D}_0 \to [0, +\infty)$ be defined by the equality

$$A_2(t, x) = (x - x_u(t))^2 (t\alpha(t)(-\ln t))^{-2}$$

and let $a_2\colon \mathcal{D}_0 \to \mathbb{R}$ be the derivative of the function A_2 by virtue of equation (9). It is easy to see that $a_2(t, x) < 0$ when $(t, x) \in \mathcal{D}_0$, $x \ne x_u(t)$. In particular, $a_2(t, x) < 0$ when $(t, x) \in \Phi_2(\mu)$ for each $\mu \in (0, 1]$. Therefore for each $\mu \in (0, 1]$ an integral curve of equation (9) which intersects $\Phi_2(\mu)$ at an arbitrary point (t_0, x_0), for small enough $|t - t_0|$ (where $t \le \rho$): lies in $\mathcal{D}_2(\mu)$ when $t > t_0$ and lies outside of $\overline{\mathcal{D}_2(\mu)}$ when $t < t_0$ (the proof is similar to that for Φ_1). Let $P_*(t_*, x_*) \in \overline{D}_1 \setminus \{(0,0)\}$, $x_* \ne x_u(t_*)$. Then there exists $\mu_* \in (0,1]$ such that $P_* \in \Phi_2(\mu_*)$. As follows from the above, the integral curve of equation (9) $J_* : (t, x^*(t))$ which passes through P_* lies outside of $\overline{\mathcal{D}_2(\mu_*)}$ if $t \in (t_-, t_*)$, where (t_-, t_*) is the left maximal existence interval for the solution x^*. From the other hand there exists $t_{**} \in (0, \rho)$ such that if $(t,x) \in \overline{D}_1$ and if $t \in (0, t_{**})$ then $(t,x) \in D_2(\mu_*)$. Let

$$t^* = \min\{t_*, t_{**}\}.$$

As appears from the above $J_* : (t, x^*(t))$ lies outside of $\overline{\mathcal{D}_1}$ when $t \in (t_-, t^*)$. Introduce an operator $T \colon \mathcal{U} \to \mathcal{U}$ by $Tu = x_u$. Let us prove that $T \colon \mathcal{U} \to \mathcal{U}$ is a continuous operator. Let $u_i \in \mathcal{U}$, $i \in \{1,2\}$, be arbitrary functions and let $Tu_i = x_i$, $i \in \{1,2\}$. Then $x_i \in \mathcal{U}$, $i \in \{1,2\}$, and if $t \in (0, \rho]$ then the following identities are valid:

$$x_i'(t) = f(t, u_i(t), {u'}_i(t)), i \in \{1,2\}. \tag{13}$$

If $u_1 = u_2$ then $x_1 = x_2$. Suppose $\|u_1 - u_2\|_{\mathcal{B}} = h$, $h > 0$. Let

$$\Phi_3 = \left\{(t,x) : t \in (0,\rho],\ |x - x_2(t)| = h^\nu (t\alpha(t))^{1-\nu}\right\},$$

$$\mathcal{D}_3 = \left\{(t,x) : t \in (0,\rho],\ |x - x_2(t)| < h^\nu (t\alpha(t))^{1-\nu}\right\},$$

where ν is a constant such that $0 < \nu < 0$. Let the function $A_3 \colon \mathcal{D}_0 \to [0, +\infty)$ be defined by the equality

$$A_3(t,x) = (x - x_2(t))^2 (t\alpha(t))^{-2(1-\nu)}$$

and let $a_3 : \mathcal{D}_0 \to \mathbb{R}$ be the derivative of the function A_3 by virtue of equation

$$x'(t) = f(t, u_1(t), {u'}_1(t)). \tag{14}$$

Since

$$\begin{aligned} |u_1(t) - u_2(t)| &= |u_1(t) - u_2(t)|^\nu |u_1(t) - u_2(t)|^{1-\nu} \le \\ &\le \|u_1 - u_2\|_{\mathcal{B}}^\nu (|u_1(t) - \xi(t)| + |u_2(t) - \xi(t)|)^{1-\nu} \le \\ &\le h^\nu (2Mt\alpha(t))^{1-\nu}, \qquad t \in (0,\rho], \end{aligned}$$

$$\begin{aligned} |{u'}_1(t) - {u'}_2(t)| &= |{u'}_1(t) - {u'}_2(t)|^\nu |{u'}_1(t) - {u'}_2(t)|^{1-\nu} \le \\ &\le \|u_1 - u_2\|_{\mathcal{B}}^\nu (|{u'}_1(t) - \xi'(t)| + |{u'}_2(t) - \xi'(t)|)^{1-\nu} \le \\ &\le h^\nu (2qM\alpha(t))^{1-\nu}, \qquad t \in (0,\rho], \end{aligned}$$

it is easy to see that $a_3(t,x) < 0$ when $(t,x) \in \Phi_3$. Therefore an integral curve of equation (14) which intersects Φ_3 at an arbitrary point (t_0, x_0), for small enough $|t - t_0|$ (where $t \le \rho$): lies in $\mathcal{D}_3$ when $t > t_0$ and lies outside of $\overline{\mathcal{D}_3}$ when $t < t_0$ (the proof is similar to that for Φ_1). Moreover, we obtain

$$|x_1(t) - x_2(t)| \le |x_1(t) - \xi(t)| + |x_2(t) - \xi(t)| \le 2Mt\alpha(t) < h^\nu (t\alpha(t))^{1-\nu},$$

when $t \in (0, t(h)]$, where $t(h) \in (0, \rho]$ is small enough. Therefore if $t \in (0, t(h)]$ then the integral curve $J : (t, x_1(t))$ of equation (14) lies in $\mathcal{D}_3$. As follows from the above, if t increases monotonically from $t = t(h)$ to $t = \rho$ then the integral curve $J : (t, x_1(t))$ cannot intersect Φ_3 and therefore this curve remains in $\mathcal{D}_3$ for all $t \in (0, \rho]$. We obtain

$$|x_1(t) - x_2(t)| \le h^{\nu} (t\alpha(t))^{1-\nu}, \qquad t \in (0, \rho]. \tag{15}$$

From (13) we see that

$$|x'_1(t) - x'_2(t)| \le \frac{h^{\nu}}{t} (t\alpha(t))^{1-\nu}, \qquad t \in (0, \rho]. \tag{16}$$

Since ρ is small sufficiently it follows from (15), (16) that

$$|x_1(t) - x_2(t)| + |x'_1(t) - x'_2(t)| \le \frac{h^{\nu}}{t}, \qquad t \in (0, \rho]. \tag{17}$$

We now turn to a direct proof of the continuity of the operator $T : \mathcal{U} \to \mathcal{U}$. Let there be given $\varepsilon > 0$. There exists $t_\varepsilon \in (0, \rho)$ such that

$$2Mt\alpha(t) + 2qM\alpha(t) \le \frac{\varepsilon}{2}, \qquad t \in (0, t_\varepsilon].$$

Then

$$|x_1(t) - x_2(t)| + |x'_1(t) - x'_2(t)| \le |x_1(t) - \xi(t)| + |x_2(t) - \xi(t)| +$$
$$|x_1{}'(t) - \xi'(t)| + |x'_2(t) - \xi'(t)| \le 2Mt\alpha(t) + 2qM\alpha(t) \le \frac{\varepsilon}{2}, \qquad t \in (0, t_\varepsilon]. \tag{18}$$

Suppose $t \in [t_\varepsilon, \rho]$. We find from (17) that

$$|x_1(t) - x_2(t)| + |x'_1(t) - x'_2(t)| \le \frac{h^{\nu}}{t_\varepsilon}, \qquad t \in [t_\varepsilon, \rho]. \tag{19}$$

Let

$$\delta(\varepsilon) = \left(\frac{\varepsilon t_\varepsilon}{2}\right)^{\frac{1}{\nu}}.$$

If $h < \delta(\varepsilon)$ then it follows from (19) that

$$|x_1(t) - x_2(t)| + |x'_1(t) - x'_2(t)| \le \frac{\varepsilon}{2}, \qquad t \in [t_\varepsilon, \rho]. \tag{20}$$

Since $x_i(0) = 0$, $x'_i(0) = \xi_0$, $i \in \{1, 2\}$, it follows from (18), (20) that

$$|x_1(t) - x_2(t)| + |x'_1(t) - x'_2(t)| \le \frac{\varepsilon}{2}, t \in [0, \rho]$$

and therefore

$$\|x_1 - x_2\|_{\mathcal{B}} \le \frac{\varepsilon}{2}.$$

Thus, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $\|u_1 - u_2\|_{\mathcal{B}} = h < \delta(\varepsilon)$ then

$$\|Tu_1 - Tu_2\|_{\mathcal{B}} = \|x_1 - x_2\|_{\mathcal{B}} \le \frac{\varepsilon}{2} < \varepsilon.$$

The reasoning given above is independent of selection $u_i \in \mathcal{U}$, $i \in \{1, 2\}$. Therefore $T : \mathcal{U} \to \mathcal{U}$ is a continuous operator.

To complete the proof of Theorem 1 it suffices to apply the Schauder Fixed Point Theorem to the operator $T : \mathcal{U} \to \mathcal{U}$. □

It may be noted that the condition $\lim\limits_{t \to +0} \frac{\alpha(t)}{\xi'(t)} = 0$ is not necessary; we use this condition only for obtaining asymptotic form of estimates (3).

Proof of Theorem 2. At the beginning we select the constants ρ, M, q identical to those for the proof of Theorem 1. Let $\mathcal{B}$ be the space of continuously differentiable functions $x\colon [0,\rho] \to \mathbb{R}$ with norm (7). Let $\mathcal{U}$ be the subset of $\mathcal{B}$ such that every its element $u\colon [0,\rho] \to \mathbb{R}$ satisfies inequalities (3) and also $u(0) = 0$, $u'(0) = \xi_0$. It is obvious that $\mathcal{U}$ is a bounded closed set. Let us consider the initial value problem (9), (2) where $u \in \mathcal{U}$ is an arbitrary fixed function. Let us consider precisely the same sets $\mathcal{D}_0$, Φ_1, $\mathcal{D}_1$, H as in the proof of Theorem 1. In $\mathcal{D}_0$ for equation (9) conditions of the Existence and Uniqueness Theorem and conditions of the Continuous Dependence of the Initial Data Theorem are fulfilled. By using a reasoning as in the proof of Theorem 1 we make sure that among integral curves of equation (9) which intersect H there exists a unique integral curve (e.g. $J_0 : (t, x_u(t)))$ which is defined for all $t \in (0,\rho]$ and lies in $\mathcal{D}_1$ when $t \in (0,\rho]$. It is easy to see that inequalities (10) are fulfilled if $t \in (0,\rho]$. Let $x_u(0) = 0$, $x_u'(0) = \xi_0$. Then $x_u \in \mathcal{U}$. Introduce an operator $T\colon \mathcal{U} \to \mathcal{U}$ by $Tu = x_u$.

Let us prove that $T : \mathcal{U} \to \mathcal{U}$ is a contraction operator. Let $u_i \in \mathcal{U}$, $i \in \{1,2\}$ be arbitrary functions and let $Tu_i = x_i$, $i \in \{1,2\}$. Then $x_i \in \mathcal{U}$, $i \in \{1,2\}$, and if $t \in (0,\rho]$ then the identities (13) are fulfilled. If $u_1 = u_2$ then $x_1 = x_2$. Suppose that $\|u_1 - u_2\|_{\mathcal{B}} = h$, $h > 0$. Let

$$\Phi_3 = \{(t,x) : t \in (0,\rho],\ |x - x_2(t)| = \eta h t\},$$

$$\mathcal{D}_3 = \{(t,x) : t \in (0,\rho],\ |x - x_2(t)| < \eta h t\},$$

where η is a constant such that $\eta > l_x + l_y$. Let a function $A_3\colon \mathcal{D}_0 \to [0,+\infty)$ be defined by the equality

$$A_3(t,x) = (x - x_2(t))^2 t^{-2}$$

and let $a_3\colon \mathcal{D}_0 \to \mathbb{R}$ be the derivative of the function A_3 by virtue of equation (14). It is easy to see that $a_3(t,x) < 0$ when $(t,x) \in \Phi_3$. Therefore an integral curve of equation (14) which intersects Φ_3 at an arbitrary point (t_0, x_0) for small enough $|t - t_0|$ (where $t \le \rho$) lies in $\mathcal{D}_3$ if $t > t_0$ and lies outside of $\overline{\mathcal{D}_3}$ if $t < t_0$ (the proof is similar to that for Φ_1 in the proof of Theorem 1). Thus

$$|x_1(t) - x_2(t)| \le |x_1(t) - \xi(t)| + |x_2(t) - \xi(t)| \le 2Mt\alpha(t) < \eta h t,$$

if $t \in (0, t(h)]$; here $t(h) \in (0,\rho)$ is small enough. Therefore if $t \in (0, t(h)]$ then the integral curve $J : (t, x_1(t))$ of equation (14) lies in $\mathcal{D}_\ni$. As appears from the above, if t increases monotonically from $t = t(h)$ to $t = \rho$ then the integral curve $J : (t, x_1(t))$ cannot intersect Φ_3. Therefore $J : (t, x_1(t))$ remains in $\mathcal{D}_3$ for all $t \in (0,\rho]$. We obtain

$$|x_1(t) - x_2(t)| \le \eta h t, \qquad t \in (0,\rho]. \tag{21}$$

From (13) we see that

$$|x'_1(t) - x'_2(t)| \le (l_x + l_y)\,h, \qquad t \in (0,\rho]$$

and therefore

$$|x_1(t) - x_2(t)| + |x'_1(t) - x'_2(t)| \le (l_x + l_y + \eta t)\,h, \qquad t \in (0,\rho]. \tag{22}$$

Let $\theta = \frac{1}{2}(1 + l_x + l_y)$; it is obvious that $\theta \in (0,1)$. Since ρ is small enough and $x_i(0) = 0$, $x_i'(0) = \xi_0$, $i \in \{1,2\}$, it follows from (22) that

$$|x_1(t) - x_2(t)| + |x'_1(t) - x'_2(t)| \le \theta h, t \in (0,\rho]$$

and therefore

$$\|x_1 - x_2\|_{\mathcal{B}} \le \theta h,$$

or

$$\|Tu_1 - Tu_2\|_{\mathcal{B}} \le \theta \|u_1 - u_2\|_{\mathcal{B}}, \tag{23}$$

where $\theta \in (0,1)$. The reasoning given above is independent of selection $u_i \in U$, $i \in \{1,2\}$. Therefore $T\colon \mathcal{U} \to \mathcal{U}$ is a contraction operator.

To complete the proof of Theorem 2 it suffices to apply the Banach Contraction Mapping Theorem to the operator $T\colon \mathcal{U} \to \mathcal{U}$. □

2. Next, the initial value problem (1), (2) will be under consideration, where $t \in (0,\tau)$ a real variable, $x\colon (0,\tau) \to \mathbb{R}$ a real unknown function, $f\colon \mathcal{D} \to \mathbb{R}$ is a continuous function,

$$\mathcal{D} = \{(t,x,y) : t \in (0,\tau), |x| < r_1 \xi(t), |y| < r_2 \xi'(t)\};$$

here $\xi\colon (0,\tau) \to (0,+\infty)$ is a continuously differentiable function, $\xi'(t) > 0$, $t \in (0,\tau)$,

$$\lim_{t\to+0} \xi(t) = 0, \quad \lim_{t\to+0} \xi'(t) = 0, \quad \lim_{t\to+0} \frac{\xi(t)}{\xi'(t)} = 0,$$

$$|f(t,0,0)| \le K\xi'(t), \qquad t \in (0,\tau).$$

Suppose that

$$|f(t,x,y_1) - f(t,x,y_2)| \le l_y |y_1 - y_2|, \qquad (t,x,y_i) \in \mathcal{D}, \qquad i \in \{1,2\},$$

where l_y is a constant, $l_y < 1$.

Let us introduce the same definition of ρ-solution of problem (1), (2) as in the first part of the paper.

We denote by $\mathcal{U}(\rho,M,q)$ the set of all continuously differentiable functions $u\colon (0,\rho] \to \mathbb{R}$ such that

$$|u(t)| \le M\xi(t), \quad |u'(t)| \le qM\xi'(t), \quad t \in (0,\rho]; \tag{24}$$

here ρ, M, q are constants, $\rho \in (0,\tau)$, $M > 0$, $q > 0$.

Theorem 3. *Suppose that conditions* (4), (5) *hold, where* $l_t\colon (0,\tau) \to (0,+\infty)$, $l_x\colon (0,\tau) \to (0,+\infty)$ *are continuous nonincreasing functions,*

$$\lim_{t\to+0} \frac{\xi(t)}{\xi'(t)} l_x(t) = L_x, \qquad 0 \le L_x < +\infty$$

and

$$L_x + l_y < 1, \qquad K < (1 - L_x - l_y)\min\{r_1, r_2\}.$$

Then there exist ρ, M, q *such that problem* (1), (2) *has a nonempty set of* ρ*-solutions* $x\colon (0,\rho] \to \mathbb{R}$ *each of which belongs to* $\mathcal{U}(\rho,M,q)$.

Theorem 4. *Suppose that condition* (6) *be fulfilled, where* l_x *is a constant,*

$$l_x + l_y < 1, \qquad K < (1 - l_y)\min\{r_1, r_2\}.$$

Then there exist ρ, M, q *such that problem* (1), (2) *has a unique* ρ*-solution* $x\colon (0,\rho] \to \mathbb{R}$ *which belongs to* $\mathcal{U}(\rho,M,q)$.

Proof of Theorem 3. First of all we select constants ρ, M, q. Let the following conditions hold:

$$1 < q < \frac{(1 - L_x)\min\{r_1, r_2\}}{K + l_y \min\{r_1, r_2\}}, \qquad \frac{K}{1 - L_x - q l_y} < M < \frac{\min\{r_1, r_2\}}{q}.$$

We do not present here the conditions for selection of ρ, because the volume of this paper is resricted. We now note nothing but ρ is small enough, M, q are large enough and selection of ρ, M, q ensures the validity of all our reasoning given below. Let $\mathcal{B}$ be the space of continuously differentiable functions $x\colon [0, \rho] \to \mathbb{R}$ with norm (7). Let $\mathcal{U}$ be the subset of $\mathcal{B}$ such that every its element $u\colon [0, \rho] \to \mathbb{R}$ satisfies inequalities (24) and also $u(0) = 0,\ u'(0) = 0$ and, moreover, condition (8) holds, where

$$K(\mu) = (1 - l_y)^{-1} (l_t(\mu) + l_x(\mu)).$$

It is easy to see that $\mathcal{U}$ is a closed, bounded and convex set. Moreover, $\mathcal{U}$ is a compact set (according to the Arzelá Theorem). We will consider differential equation (9), where $u \in \mathcal{U}$ is an arbitrary fixed function. Let

$$\mathcal{D}_0 = \{(t, x) : t \in (0, \rho], x \in \mathbb{R}\}.$$

In $\mathcal{D}_0$ for equation (9) conditions of the Existence and Uniqueness Theorem and conditions of the Continuous Dependence of the Initial Data Theorem hold. Let

$$\Phi_1 = \{(t, x) : t \in (0, \rho],\ |x| = M\xi(t)\},$$
$$\mathcal{D}_1 = \{(t, x) : t \in (0, \rho],\ |x| < M\xi(t)\},$$
$$H = \{(t, x) : t = \rho,\ |x| < M\xi(\rho)\}.$$

Let a function $A_1\colon \mathcal{D}_0 \to [0, +\infty)$ be defined by the equality

$$A_1(t, x) = x^2 (\xi(t))^{-2}$$

and let $a_1\colon \mathcal{D}_0 \to \mathbb{R}$ be the derivative of the function A_1 by virtue of equation (9). It is easy to see that $a_1(t, x) < 0$ when $(t, x) \in \Phi_1$. By using a reasoning as in the proof of Theorem 1 we make sure that among integral curves of equation (9) which intersect H there exists at least one integral curve (let it be $J_0 : (t, x_u(t))$) which is defined for all $t \in (0, \rho]$ and lies in $\mathcal{D}_1$ for all $t \in (0, \rho]$. Next we will prove that there is only one integral curve of such type; for this purpose we consider the families of sets

$$\Phi_2(\mu) = \left\{(t, x) : t \in (0, \rho],\ |x - x_u(t)| = \mu(\xi(t))^{\frac{1}{2}}\right\},$$
$$\mathcal{D}_2(\mu) = \left\{(t, x) : t \in (0, \rho],\ |x - x_u(t)| < \mu(\xi(t))^{\frac{1}{2}}\right\},$$

where μ is a parameter, $\mu \in (0, 1]$. Let a function $A_2\colon \mathcal{D}_0 \to [0, +\infty)$ be defined by the equality

$$A_2(t, x) = (x - x_u(t))^2 (\xi(t))^{-1}$$

and let $a_2 : \mathcal{D}_0 \to \mathbb{R}$ be the derivative of the function A_2 by virtue of equation (9). It is easy to see that $a_2(t, x) < 0$ when $(t, x) \in \mathcal{D}_0,\ x \neq x_u(t)$. Then we can use a reasoning as in the proof of Theorem 1. It is easy to see that the following inequalities are valid:

$$|x_u(t)| \leq M\xi(t), |x'_u(t)| \leq qM\xi'(t), t \in (0, \rho] \tag{25}$$

and condition (11) is fulfilled. Let $x_u(0) = 0,\ x_u'(0) = 0$. Then $x_u \in \mathcal{U}$. Introduce an operator $T\colon \mathcal{U} \to \mathcal{U}$ by $Tu = x_u$. Let us prove that $T\colon \mathcal{U} \to \mathcal{U}$ is a continuous operator.

Let $u_i \in \mathcal{U}$, $i \in \{1,2\}$ be arbitrary functions and let $Tu_i = x_i$, $i \in \{1,2\}$. Then $x_i \in \mathcal{U}$, $i \in \{1,2\}$ and if $t \in (0,\rho]$ then identities (13) are valid. If $u_1 = u_2$ then $x_1 = x_2$. Assume $\|u_1 - u_2\|_{\mathcal{B}} = h$, $h > 0$. Let

$$\Phi_3 = \left\{(t,x) : t \in (0,\rho]\,,\ |x - x_2(t)| = \eta h^{\nu} (\xi(t))^{1-\nu}\right\},$$
$$\mathcal{D}_3 = \left\{(t,x) : t \in (0,\rho]\,,\ |x - x_2(t)| < \eta h^{\nu} (\xi(t))^{1-\nu}\right\},$$

where ν, η are constants such that

$$0 < \nu < 1, \quad \eta > 2(1-\nu)^{-1}(L_x + 1)(2M)^{1-\nu}.$$

Let a function $A_3 : \mathcal{D}_0 \to [0,+\infty)$ be defined by the equality

$$A_3(t,x) = (x - x_2(t))^2 (\xi(t))^{-2(1-\nu)}$$

and let $a_3 \colon \mathcal{D}_0 \to \mathbb{R}$ be the derivative of the function A_3 by virtue of equation (14). It is easy to see that $a_3(t,x) < 0$ when $(t,x) \in \Phi_3$. Further our reasoning is identical with the corresponding part of the proof of Theorem 1. We obtain

$$|x_1(t) - x_2(t)| \le \eta h^{\nu} (\xi(t))^{1-\nu}, \qquad t \in (0,\rho]\,,$$
$$|x'_1(t) - x'_2(t)| \le \omega(t)\, h^{\nu} (\xi(t))^{1-\nu}, \qquad t \in (0,\rho]\,,$$

where $\omega \colon\ (0,\rho] \to (0,+\infty)$ is a continuous function, $\lim\limits_{t\to+0} \omega(t) = 0$, and, lastly,

$$|x_1(t) - x_2(t)| + |x'_1(t) - x'_2(t)| \le h^{\nu} (\xi(t))^{-1}, \qquad t \in (0,\rho]\,,$$

and

$$\|Tu_1 - Tu_2\|_{\mathcal{B}} \le \frac{\varepsilon}{2} < \varepsilon$$

if

$$\|u_1 - u_2\|_{\mathcal{B}} = h < \left(\frac{\varepsilon}{2}\xi(t_\varepsilon)\right)^{\frac{1}{\nu}}.$$

The reasoning given above is independent of selection of $u_i \in \mathcal{U}$, $i \in \{1,2\}$. Therefore $T \colon \mathcal{U} \to \mathcal{U}$ is a continuous operator.

To complete the proof of Theorem 3 it is sufficient to apply the Schauder Fixed Point Theorem to the operator $T \colon \mathcal{U} \to \mathcal{U}$. □

Proof of Theorem 4. First of all we select constants ρ, M, q. Let the following conditions hold:

$$1 < q < \frac{(1 - L_x)\min\{r_1, r_2\}}{K + l_y \min\{r_1, r_2\}}, \qquad \frac{K}{1 - q l_y} < M < \frac{\min\{r_1, r_2\}}{q}.$$

The conditions for selection of ρ is not presented. ρ is small enough. Let $\mathcal{B}$ be the space of continuously differentiable functions $x \colon\ [0,\rho] \to \mathbb{R}$ with norm (7). Let $\mathcal{U}$ be the subset of $\mathcal{B}$, every element $u \colon\ [0,\rho] \to \mathbb{R}$ of which satisfies inequalities (24), and also $u(0) = 0$, $u'(0) = 0$. It is easy to see that $\mathcal{U}$ is a closed bounded set. Let us consider the initial value problem (9), (2) where $u \in \mathcal{U}$ is an arbitrary fixed function. Further let us consider precisely the same sets $\mathcal{D}_0$, Φ_1, $\mathcal{D}_1$, H and $\Phi_2(\mu)$, $\mathcal{D}_2(\mu)$ as in the proof of Theorem 3. In $\mathcal{D}_0$ for equation (9) conditions of the Existence and Uniqueness Theorem and conditions of the Continuous Dependence of the Initial Data Theorem hold. By using a reasoning as in the proof of Theorem 3 we establish that there is one and only one integral curve of equation (9) (let us denote it by $J_0 : (t, x_u(t))$) which intersects H and lies in $\mathcal{D}_1$ when

$t \in (0, \rho]$. It is easy to see that inequalities (25) hold. Let $x_u(0) = 0$, $x'_u(0) = 0$. Then $x_u \in \mathcal{U}$. Introduce an operator $T : \mathcal{U} \to \mathcal{U}$ by $Tu = x_u$. Let us prove that $T : \mathcal{U} \to \mathcal{U}$ is a contraction operator. Let $u_i \in \mathcal{U}$, $i \in \{1, 2\}$ be arbitrary functions and let $Tu_i = x_i$, $i \in \{1, 2\}$. Then $x_i \in \mathcal{U}$, $i \in \{1, 2\}$, and if $t \in (0, \rho]$ then identities (13) are fulfilled. If $u_1 = u_2$ then $x_1 = x_2$. Suppose that $\|u_1 - u_2\|_{\mathcal{B}} = h$, $h > 0$. Let us consider the same sets $\Phi_3, \mathcal{D}_3$ and the function $A_3 : \mathcal{D}_0 \to [0, +\infty)$ as in the proof of Theorem 2. Let $a_3 : \mathcal{D}_0 \to \mathbb{R}$ be the derivative of the function A_3 by virtue of equation (14). It is easy to see that $a_3(t, x) < 0$ when $(t, x) \in \Phi_3$. Moreover,

$$|x_1(t) - x_2(t)| \le |x_1(t)| + |x_2(t)| \le 2M\xi(t) < \eta h t$$

when $t \in (0, t(h)]$, where $t(h) \in (0, \rho)$ is small enough. In the same way as in the proof of Theorem 2 it is easy to obtain (21), (22) and (23), where

$$\theta = \frac{1}{2}(1 + l_x + l_y).$$

It is obvious that $\theta \in (0, 1)$. The reasoning given above is independent of selection of $u_i \in \mathcal{U}$, $i \in \{1, 2\}$. Therefore $T : \mathcal{U} \to \mathcal{U}$ is a contraction operator.

To complete the proof of Theorem 4 it is sufficient to apply the Banach Contraction Mapping Theorem to the operator $T : \mathcal{U} \to \mathcal{U}$. □

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ЗАДАЧА КОШІ $x' = f(t, x, x')$, $x(0) = 0$: РОЗВ'ЯЗНІСТЬ, КІЛЬКІСТЬ РОЗВ'ЯЗКІВ, АСИМПТОТИКА

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Розглядаємо задачу Коші $x' = f(t, x, x')$, $x(0) = 0$. Доведено існування неперервно диференційовних розв'язків $x\colon (0, \rho] \to \mathbb{R}$ з потрібними асимптотичними властивостями.

Ключові слова: розв'язанність, кількість розв'язків, асимптотика.