ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2011. Випуск 74. С. 41–46
Visnyk of the Lviv Univ. Series Mech. Math. 2011. Issue 74. P. 41–46

УДК 517.95

ON BEHAVIOR OF SOLUTIONS OF HIGHER ORDER DEGENERATE PARABOLIC EQUATIONS

Tahir GADJIEV, Kenyul MAMEDOVA

Institute of Mathematics and Mechanics of NAS of Azerbaijan,
Afgajev Str., 9, Baku, AZ 1141
e-mail: soltanaliev@yahoo.com

In this paper the behavior of the solutions of the initial-boundary problem for degenerate quasilinear parabolic equations in unbounded domains with noncompact boundary is study.

Key words: parabolic nonlinear equation, degeneration, unbounded domain.

1. Introduction. The goal of the paper is to study behavior of solutions of the initial-boundary problem for degenerate quasi-linear parabolic equations in unbounded domains with noncompact boundary.

For linear elliptic and parabolic equations on behavior of solution were studied in the paper of O.A. Oleinik [1], [2]. For quasilinear equations, similar results were obtained in the papers of A.F. Tedeev, A.E. Shishkov [3], T.S. Gadjiev [4]. S. Bonafade [5] is studied quality properties of solutions for degenerate equations. Also we mention papers [6], [7], [10]-[12].

We obtained some estimations that analogies of Saint-Venant's principle known in theory of elasticity. By means of these estimations we obtained estimation on behavior of solution of type Fragmen-Lindelyof.

In unbounded domain Q which contains in layer $H_T = \{(x,t) : 0 < t < T < \infty\}$ of Euclid space $\mathbb{R}^{n+1}_{x,t}$ consider initial-boundary problem

$$\frac{\partial u}{\partial t} - \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}\left(x,t,u,Du,...,D^m u\right) = 0, \tag{1}$$

$$u\,|_{t=0} = 0, \tag{2}$$

$$D^{\alpha} u\,|_{\Gamma} = 0, \;\; |\alpha| \le m-1, \tag{3}$$

where $D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_m^{\alpha_m}}$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_m$, $m \ge 1$, $D^m = (D^{\alpha} u)_{|\alpha|=m}$.

The domain Q has noncompact boundary $\partial Q = \Gamma_0 \bigcup \Gamma_T \bigcup \Gamma$, where $\Gamma_0 = \partial Q \bigcap \{(x,t) : t = 0\}$, $\Gamma_T = \partial Q \bigcap \{(x,t) : t = T\}$.

Assume that the coefficients $A_\alpha(x,t,\xi)$ are measurable with respect to $(x,t)\in Q$, continuous with respect to $\xi\in\mathbb{R}^M$ (M is the number of different multi-indices of length no more than m) and satisfy the conditions

$$\sum_{|\alpha|\le m} A_\alpha(x,t,\xi)\,\xi_\alpha^m \ge \omega(x)\,|\xi^m|^p - c_1\omega(x)\sum_{i=1}^{m-1}\left|\xi^i\right|^p - f_1(x,t), \tag{4}$$

$$|A_\alpha(x,t,\xi)| \le C_2\omega(x)\sum_{i=0}^{m}\left|\xi^i\right|^{p-1} + f_2(x,t), \tag{5}$$

where $\xi=\left(\xi^0,...,\xi^m\right)$, $\xi^i=\left(\xi^i_\alpha\right)$, $|\alpha|=i$, $p>1$, $f_1\in L_{p'}(0,T\ L_{p,loc}(\Omega_t))$,

$$f_2\in L_{1,loc}(Q),\quad \Omega_\tau = Q\bigcap\{(x,t):\ t=\tau\}.$$

The space $L_p(0,T,\,W^m_{q,\omega}(\Omega_t'))$ defined as $\left\{u(x,t):\ \int_0^T\left(\|u\|_{W^m_{q,\omega}(\Omega_t)}\right)^p dt<\infty\right\}$, where Q'-bounded subdomain Q. $\Omega_t' = Q'\bigcap\{(x,t):\ t=\tau\}$. $W^m_{q,\omega}(\Omega_t')$ is a closure of t the functions from $C^m\left(\bar\Omega_t\right)$ with respect to the norm

$$\|u\|_{W^m_{q,\omega}(\Omega_t')} = \left(\int_{\Omega_t'}\omega(x)\sum_{|\alpha|\le m}|D^\alpha u|^q\,dxdt\right)^{1/q}.$$

Assume that $\omega(x)$, $\varepsilon(x)$, $x\in\Omega$, are measurable non negative function satisfying the conditions:

$$\omega\in L_{1,loc}(Q), \tag{6}$$

where $\Omega_s=\Omega_t\bigcap B_s$, $B_s=\{x:\ |x|<s\}$, C_i are positive constants dependent only on problems data. In particular, it follows from condition (6) that $\omega\in A_\tau$ (see [8]), i.e. for any $\rho>0$

$$\int_{\Omega_\rho}\omega(x)\,dx\left[\int_{\Omega_\rho}\omega(x)^{-\frac{1}{\sigma-1}}dx\right]^{\sigma-1}\le C_4\rho^{n\sigma}. \tag{7}$$

Well describe geometry ∂Q with weight nonlinear basis frequency $\lambda_p(r,\tau)$ of section $\sigma(r,\tau)=S(r)\bigcap\Omega_\tau$, where $S(r)=Q\bigcap\partial Q(r)$, $Q(r)=Q\bigcap\{B_S\times(0,T)\}$,

$$\lambda_p^p(r,\tau)=\inf\left(\int_{\sigma(r,\tau)}\omega(x)\,|\nabla v\,|^p d\tau\right)\left(\int_{\tau(r,\tau)}\omega(x)\,|\nabla v|^p\,d\tau\right)^{-1},$$

where the lower bound is taken by all continuously differentiable functions in the vicinity of $\sigma(r,\tau)$ that vanish on ∂Q; $\nabla_s v(x)$ is a projection of the vector $\nabla_x v(x)$ on a tangential plane to $\sigma(r,\tau)$ at the point x.

The function $u\in L_p(0,T,\ \overset{\circ}{W}{}^m_{p,\omega,loc}(\Omega_t))\bigcap W_2^1(0,T;\ L_{2,loc}(\Omega_t))$ is said to be a generalized solution of problem (1)-(3) if

$$\int_Q\frac{\partial u}{\partial t}\varphi dxdt+\int_Q\sum_{|\alpha|\le m}A_\alpha(x,t,u,Du,D^m u)\,D^\alpha\varphi dxdt=0 \tag{8}$$

is fulfilled for the arbitrary function $\varphi \in L_p(0,T;\, \overset{\circ}{W}{}^{m}_{p,\omega}(\Omega'_t) \bigcap L_2(Q'))$. We will consider classes of domains, for which hold estimate

$$\int\limits_{S_r} \omega(x)\,|u|^p\,dxdt \le \lambda_p^{-p}(r) \int\limits_{S_r} \omega(x)\,|\nabla u|^p\,dxdt. \tag{9}$$

The necessary and sufficiently conditions on domains for holds estimate (9) is given for example [9].

2. Behavior of solutions. Let $k(x) \in C^m_{loc}(\Omega)$ positive function, $k(0) = 0$ and that at $x \in \Omega$ hold estimates

$$|D_x k(x)| \ge h_1 > 0,$$
$$\left|D_x^j k(x)\right| \le h_2\,(k(x))^{-j+1}, \quad h_2 > 0, \quad j = 1, 2, ..., m.$$

Is defined $\lambda^2_{\mu(s)}(r,\tau) = \lambda_2^2(r,\tau) + \mu^{\frac{2}{m}}(s)$, $\forall\, s,\, \tau > 0$.

$$J_{\mu(s),p}(r,\tau) \equiv \int\limits_{\Omega_\tau(s)} \left(|D^m u|^p + \mu^2(s)u^2\right) dx,$$

$J_{\mu(s),2}(r,\tau)$, $\Omega_\tau(s_1) \backslash \Omega_\tau(s_2) = M_\tau(s_1, s_2)$, where $\mu(v)$ function which define later.

Lemma 1. *Let $u \in L_p(0,T;\, W^m_{p,\omega}(\Omega'_t) \bigcap L_2(Q'))$ and $\mu(k(x))$ be a measurable non-negative function locally bounded in Ω. Then the inequality*

$$\int\limits_{M_r(s_1,s_2)} \left|D_x^j u\right|^2 \lambda_{\mu(s)}(k(x),\tau)\, f(k(x))dx \le$$

$$\le \frac{h_2}{h_1} \int\limits_{M_r(s_1,s_2)} \left(|D_x^m u|^2 + \mu^2(s)\left|D^j u\right|^2\right) f(k(x))\,dx, \tag{10}$$

is valid, where $j \le m$.

We introduce shear function $\xi(t)$ be m times continuously differentiable function, $0 < \xi(t) < 1$, $2^{-1} < t < 1$; $\xi(t) = 1$ for $t < \frac{1}{2}$, $\xi(t) = 0$ for $t \ge 1$. Denote $\xi_h^{(t)} = \xi(\frac{t-h}{1-h})$. The following estimations are true for this shear function

$$\left|D_x^j \xi_h\left(\frac{k(x)}{r}\right)\right| \le \frac{C_j}{[r(1-h)]^j}, \quad rh + \frac{r}{2}(1-h) < j(x) < r, \quad j = 0, 1, ..., m.$$

$D_x^j \xi_h\left(\frac{k(x)}{r}\right) = 0$ for $g(x) \le rh + \frac{r}{2}(1-h)$, and $g(x) > r$, $j > 1$.

Lemma 2. *Assume that the continuous non-decreasing on (t,∞) function $I(t)$ satisfies inequality*

$$\begin{aligned} &I(t) \le \theta I(t\psi(t)), \quad 0 < \theta < 1, \\ &\psi(t) = 1 + \varphi(t), \quad \varphi(t) > 0 \end{aligned} \tag{11}$$

and measurable function $\varphi(t)$ satisfy

$$(\varphi(t))^{-1} \inf \varphi(\tau) > \delta > 0, \quad t < \tau < t\psi(t), \tag{12}$$

Then the estimation

$$I(t) \geq \theta \exp\left(\delta \ln \theta^{-1} \int\limits_{t_0}^{t} \frac{d\tau}{\tau\varphi(\tau)}\right) I(t_0)$$

is valid for $I(t)$.

Our main goal is to obtain estimations of behavior of the function $J_{\mu(\tau),p}(\tau)$ at $\tau \to \infty$. We defined function $\psi(\tau)$ and $\mu(\tau)$:

$$\inf_{\substack{\tau<k(x)<\tau\psi(\tau)\\ 0<t<T}} \lambda_{\mu(\tau)}(k(x),t)\,\tau(\psi(\tau)-1) \geq h_0 > 0 \quad \forall\, \tau > \tau_0, \tag{13}$$

$$0 < h \leq \mu(\tau\psi(\tau))(\mu(\tau))^{-1} \leq H < \infty \quad \forall\, i > \tau. \tag{14}$$

We substitute to integral identity (8) of test function

$$\varphi(x,t) = u(x,t)\left[1 - \xi\left(\frac{\varphi(\tau) - k(x)\tau^{-1}}{\psi(\tau)-1}\right)\right] \exp\left(-2\mu^2(\tau)\ t\right).$$

Then by virtue of condition (4), (5) having

$$J_{\mu(\tau,p)}(\tau) \cong \int\limits_{\Omega_\tau} \left(\omega(x)\,|D^m u|^p + \mu^2(\tau)\,u^2\right)\exp\left(-2\mu^2(\tau)t\right)dxdt \leq$$

$$\leq \int\limits_{\Omega_{\tau\psi(\tau)}} \Bigg[c_2\omega(x)\sum_{|\alpha|<m}|D^\alpha u|^{\,p} - c_3\omega(x)\left(\sum_{|\alpha|\leq m}|D^\alpha u|^{p-1}\right)\Bigg(\sum_{|\alpha|<m}|D^\alpha u| +$$

$$+\sum_{|\alpha|<m}\left|f_2(x)\,|D^\alpha u| + \sum_{|\alpha|\leq m}|F_\alpha(x)D^\alpha u|\right|\Bigg)\left[1-\xi\left(\frac{\varphi(\tau)-k(x)\tau^{-1}}{\varphi(\tau)-1}\right)\right]\times$$

$$\times\exp\left(-2\mu^2(\tau)\,t\right)dxdt + \int\limits_{\Omega_\tau\bigcap\Omega_{\tau\psi(\tau)}}\Bigg[c_3k_2\omega(x)\left(\sum_{|\alpha|\leq m}|D^\alpha u|^{p-1}\right)\times$$

$$\times\sum_{|\alpha|\leq m}\sum_{|\beta|\leq|\alpha|}\left|D^{\alpha-\beta}u\right|\left|D^\beta\xi\right| + \sum_{|\alpha|\leq m}\sum_{|\beta|\leq|\alpha|}|f_2(x)|\,|D^{\alpha-\beta}u|\left|D^\beta\xi\right| +$$

$$+\sum_{|\alpha|\leq m}\sum_{|\beta|\leq|\alpha|}|F_\alpha(x)|\,D^{\alpha-\beta}\left|D^\beta\xi\right|\Bigg]\exp\left(-2\mu^2(t)t\right)\,dxdt.$$

Later if we use lemma 1 and 2, also conditions to (13), (14) following basic theorem is obtained.

Theorem 1. *Let* $u(x)$ *be a generalized solution of problem* (1)-(3) *and measurable, locally bounded function* $\mu(\tau)$, $\psi(\tau) > 1$ *satisfy conditions* (13), (14). *Moreover function* $\varphi(\varphi) \equiv \psi(\tau) - 1$ *satisfy condition* (12) *of lemma 2 with some* $\delta > 0$. *Then for integral of energy* $J_{\mu(\tau)}(\tau)$ *alternative*

1) or $\lim\limits_{\tau\to\infty} J_{\mu(\tau)}(\tau)\left(G_{\mu(\tau)}(\tau)\right)^{-1} < C < \infty$;

2) or $J_{\mu(\tau)}(\tau) > (\beta+\varepsilon)\exp\left(\delta\ln(\beta+\varepsilon)^{-1}\int\limits_{\tau_0}^{-1}\frac{d\tau}{(\tau\psi(\tau)-1)}\right)J_{\mu(\tau)}(\tau)$;

is valid, where $G_\mu(\tau) = \int_0^\tau g_{\mu(\tau)}(\tau,t)\exp\left(-2\mu^2(\tau)\ t\right)dt$.

References

1. *Oleinik O.A.* Boundary value problems for second order elliptic equations in unbounded domains and Saint-Venant's principle / *Oleinik O.A., Josifian G.A.* // Ann. Scuola Norm. Super Pisa. – 1977. – Ser. IV, Vol. 2. – P. 269-290.
2. *Oleinik O.A.* On exceptional properties on a boundary and uniqueness of solutions of boundary value problems for second order elliptic and parabolic equations / *Oleinik O.A., Josifian G.A.* // Funk. Anal. – 1977. – Vol. II, Iss. 3. – P. 54-67.
3. *Tedeev A.F.* On quality properties of solutions and subsolutions of quasilinear elliptic equations / *Tedeev A.F., Shishkov A.E.* // Izv. Vuzov. Matematika. – 1984. – №1. – P. 62-68.
4. *Gadjiev T.S.* On behavior of solutions of mixed problems for quasilinear elliptic equations / *Gadjiev T.S.* // Diff. Uravneniya. – 1991. – P. 1031-1036.
5. *Bonafade S.* Quazilinear degenerate elliptic variational inequalities with discontinuous coefficients / *Bonafade S.* // Comment. Math. Univ. Carolinae. – 1993. – Vol. 34, №1. – P. 55-61.
6. *Lancaster K.* On the Asymptotic Behavior of Solutions of Quasilinear Elliptic Equations / *Lancaster K., Stanley J.* // Ann. Univ. Ferrara-Sez. VII-Sc. Mat. – 2003. – Vol. IL. – P. 85-125.
7. *Jorge G.-M.* Boundary behavior for large solutions to elliptic equations with singular weights / *Jorge G.-M.* // Nonlinear Analysis: Theory, Methods & Applications. – 2007. – Vol. 67, Iss. 3. – P. 818-826.
8. *Chanillo S.* Weighted Poincare and Sobolev inequalities and estimates for weighted Peano maximal functions / *Chanillo S., Wheeden R.* // Mer. J. Math. – 1985. – №107. – P. 1191-1226.
9. *Miklyukov V.M.* On asymptotic properties of subsolutions of elliptic type quasilinear equations / *Miklyukov V. M.* // Mat. Sbor. – 1980. – Vol. 111 (145), №1. – P. 42-66.
10. *Galaktionov V.A.* Saint-Venant's principle in blow-up for higher-order quasilinear parabolic equations / *Galaktionov V.A., Shishkov A.E.* // Proc. R. Soc. Edinb. Sect. A, Math. – 2003. – Vol. 133, №5. – P. 1075-1119.
11. *Galaktionov V.A.* Structure of boundary blow-up for higher-order quasilinear parabolic equations / *Galaktionov V.A., Shishkov A.E.* // Proc. R. Soc. Lond. A. – 2004. – Vol. 460. – P. 1-27.
12. *Bokalo M.M.* The unique solvability of a problem without initial conditions for linear and nonlinear elliptic-parabolic equations / *Bokalo M.M.* // Ukrainian Math. Bull. – 2011. – Vol. 8, №1. – P. 54-85.

Стаття: надійшла до редакції 29.04.2011
прийнята до друку 21.09.2011

ПРО ПОВЕДІНКУ РОЗВ'ЯЗКІВ ВИРОДЖЕНИХ ПАРАБОЛІЧНИХ РІВНЯНЬ ВИЩИХ ПОРЯДКІВ

Таір ГАДЖИЕВ, Кенюль МАМЕДОВА

Інститут математики і механіки НАН Азербайджану,
вул. Ф. Афгаева, 9, Баку, AZ 1141
e-mail: soltanaliev@yahoo.com

Вивчено поведінку розв'язків початково-крайової задачі для вироджених квазілінійних параболічних рівнянь у необмежених областях з некомпактною межею.

Ключові слова: параболічне нелінійне рівняння, виродження, необмежена область.

О ПОВЕДЕНИИ РЕШЕНИЙ ВЫРОЖДАЮЩИХСЯ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ ВЫСОКИХ ПОРЯДКОВ

Таир ГАДЖИЕВ, Кёнюль МАМЕДОВА

Институт математики и механика НАН Азербайджана,
ул. Ф. Агаева, 9, Баку, AZ 1141
e-mail: soltanaliev@yahoo.com

Изучено поведение решений начально-краевой задачи для вырождающихся квазилинейных параболических уравнений в неограниченных областях с некомпактной границей.

Ключевые слова: параболические нелинейные уравнения, вырождение, неограниченная область.