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DEFICIENCIES OF MEROMORPHIC FUNCTIONS IN A PUNCTURED PLANE

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Let f be a meromorphic function defined in the punctured plane $\mathbb{C}\setminus\{0\}$. In this paper we study the problem of possible deficient values and magnitudes of appropriate defects of meromorphic functions in the punctured plane $\mathbb{C}\setminus\{0\}$. The definition of the deficiency for meromorphic function in the punctured plane is given. The certain relationship that concerns deficiencies of meromorphic function f is established. Nevanlinna problem for meromorphic functions in the punctured plane is considered.

Key words: deficiency, deficient value, Nevanlinna problem, couple of veritable orders, Nevanlinna characteristic.

1. Introduction. Extensions of Nevanlinna theory to annuli have been made by many authors [3], [5], [7]-[13]. The main tools the authors used were a lemma on index of meromorphic functions along a circle [3], [5], a decomposition lemma due to G. Valiron [11], and the argument principle. In [5] an approach was given which allows to introduce a one-parameter Nevanlinna characteristic and apply Fourier series methods to the functions meromorphic in the annulus $A_r=\{z: 1/r<|z|<r\}$, $1<r\leqslant+\infty$. The theory of meromorphic functions on annuli is more complicated than those in the disks. It is natural to consider the meromorphic functions in the annulus $A_{\frac{1}{\tau}r}=\left\{z: \frac{1}{\tau}<|z|<r\right\}$, $\tau\geq 1$, $r\geq 1$. In [3], a two-parameter characteristic $T(\tau,r;f)$ was introduced for meromorphic function f in such annulus, which gives a possibility to describe the behavior of such function at approaching to the inner and outer boundary circles of the annulus $A_{\frac{1}{\tau}r}$. In this paper, the meromorphic function f in $\mathbb{C}\setminus\{0\}$ with $T(\tau,r;f)$ and couple of veritable orders is considered. In [2] it was established that if for a meromorphic function f the set

$$K(T)=\left\{(\alpha,\beta): \quad \exists(\tau_0,r_0) \quad \forall(\tau,r) \quad \tau>\tau_0, \quad r>r_0: \quad T(\tau,r;f)\leqslant\tau^{\alpha}+r^{\beta}\right\}$$

is nonempty, then this set is a quadrant. A vertex of this quadrant is denoted by (ρ_1, ρ_2) and is called a couple of veritable orders of f.

In the direction of the development of Nevanlinna theory for meromorphic in $\mathbb{C}\setminus\{0\}$ function f, the question appears about a deficiency of such function. In this paper a deficiency of f is introduced. It is studied the problem of possible deficient values and magnitudes of appropriate defects of f, and connections between a couple of veritable orders of such function and possible deficient values. Also Nevanlinna problem for meromorphic functions in the punctured plane is considered.

Let f be a non-constant meromorphic function in the complex plane, and

$$K(f) = \lim_{r\to\infty}\sup \frac{N(r,0,f) + N(r,\infty,f)}{T(r,f)},$$

where the standard notations from Nevanlinna theory are used. The number $K(f)$ was introduced by R. Nevanlinna. He posed the problem of finding the greatest lower bound of $K(f)$ for functions of a given order ρ. The following Theorem A answers this question for $0 \le \rho < 1$.

Theorem A [1]. *Let $f(z)$ be a meromorphic function of order ρ, $0 \le \rho < 1$. Then*

$$K(f) \ge 1, \qquad 0 \le \rho \le 1/2$$
$$K(f) \ge \sin\pi\rho, \qquad 1/2 \le \rho < 1.$$

These inequalities are best possible.

A variant of Theorem A is showed in this paper for meromorphic function defined in the punctured plane $\mathbb{C}\setminus\{0\}$ with couple of veritable orders (ρ_1, ρ_2) [2], where $0 \le \rho_1 < 1$, $0 \le \rho_2 < 1$.

2. Definitions and notations. Let f be a meromorphic function in $A_{\frac{1}{\tau}r} = \left\{z : \frac{1}{\tau} < |z| < r\right\}$, $\tau \ge 1$, $r \ge 1$. Denote

$$m(\tau, r; f) = m\left(\frac{1}{\tau}, f\right) + m(r,f) - 2m(1,f),$$

where $m(t,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(te^{i\theta}\right)\right| d\theta$.

Definition 1. [3]. *The function*

$$T(\tau, r; f) = N(\tau, r; f) + m(\tau, r; f) + c_f \log\tfrac{\tau}{r},\ \tau \ge 1,\ r \ge 1,$$

where $N(\tau, r; f) = \int_1^\tau \frac{n(1/t,1;f)}{t}dt + \int_1^r \frac{n(1,t;f)}{t}dt + n(\mathrm{T}, f)\log\sqrt{\tau r}$, T *is the unit circle,* $n(1/\tau, r; f)$ *is the number of poles of* f *in* $A_{\frac{1}{\tau}r}$, $n(\mathrm{T}, f)$ *is the number of poles of* f *on* T, $c_f = \frac{1}{2\pi}\int_{E_f^+} \mathrm{Im}\left(\frac{f'}{f}dz\right) + \frac{1}{4\pi}\int_{E_f^0} \mathrm{Im}\left(\frac{f'}{f}dz\right)$, $E_f^+ = \{z \in \mathrm{T} : \ |f(z)| > 1\}$, $E_f^0 = \{z \in \mathrm{T} : \ |f(z)| = 1\}$, *is called the Nevanlinna characteristic of* f.

By Theorem 2 [2] the function $T(\tau, r; f)$ has a couple of veritable orders.

Definition 2. *Let $f(z)$ be a meromorphic function in the punctured plane $\mathbb{C}\setminus\{0\}$. A couple of veritable orders of f is called the couple of veritable orders of $T(\tau, r; f)$* [2].

Definition 3. *Let $f(z)$ be a non-constant meromorphic function in the punctured plane $\mathbb{C}\setminus\{0\}$. Denote*

$$\delta_0(a,f) = 1 - \varlimsup_{\substack{\tau\to+\infty \\ r\to+\infty}} \frac{N\left(\tau, r; \frac{1}{f-a}\right)}{T(\tau, r; f)}$$

The value $\delta_0(a,f)$ is called the defect or deficiency of the function f for the value a. The value $\delta_0(a,f)$ is also defined at $a=\infty$ by taking f instead of $\frac{1}{f-a}$.

If there is no doubt as to which $f(z)$ is referred to, we write $N(\tau, r, a)$, $T(\tau, r)$ instead of $N(\tau, r; \frac{1}{f-a})$, $T(\tau, r; f)$ if a is finite, and $N(\tau, r, \infty)$ instead of $N(\tau, r; f)$.

3. Main results. Let $f(z)$ be a meromorphic function in $\mathbb{C}\setminus\{0\}$ with a couple of veritable orders (ρ_1, ρ_2). We start with formulation of the following Theorem 1, which answers the following questions for $0<\rho_1<1$, $0<\rho_2<1$:

1. Let $\delta_0(0)$ and $\delta_0(\infty)$ be two real numbers: $0 \leqslant \delta_0(0) \leqslant 1$, $0 \leqslant \delta_0(\infty) \leqslant 1$. When can one find a meromorphic in $\mathbb{C}\setminus\{0\}$ function f with a couple of veritable orders (ρ_1, ρ_2) such that $\delta_0(0) = \delta_0(0, f)$, $\delta_0(\infty) = \delta_0(\infty, f)$?
2. Let $f(z)$ be a meromorphic function with couple of veritable orders (ρ_1, ρ_2), where $0<\rho_1<1$, $0<\rho_2<1$, and such that $\delta_0(a) = \delta_0(a,f)$, $\delta_0(b) = \delta_0(b,f)$. What can one say about the possible values of the pair of numbers $\delta_0(a)$, $\delta_0(b)$?

Theorem 1. *Let $f(z)$ be a meromorphic function in the punctured plane $\mathbb{C}\setminus\{0\}$ with a couple of veritable orders (ρ_1, ρ_2), where $0<\rho_1<1$, $0<\rho_2<1$. Let*

$$u = 1-\delta_0(0,f), \quad v = 1-\delta_0(\infty, f).$$

***I.** Then, in addition to trivial inequalities*

$$0 \le u \le 1, \qquad 0 \le v \le 1, \tag{1}$$

u and v satisfy

$$\begin{cases} u^2+v^2-2uv\cos\pi\rho_1 \ge \sin^2\pi\rho_1 \\ u^2+v^2-2uv\cos\pi\rho_2 \ge \sin^2\pi\rho_2 \end{cases} \tag{2}$$

$$If\ u < \min\{\cos\pi\rho_1; \cos\pi\rho_2\},\ then\ v=1;\ if\ v<\min\{\cos\pi\rho_1;\cos\pi\rho_2\},\ then\ u=1. \tag{3}$$

***II.** All values u, v compatible with these restrictions are actually possible.*

Note that the equations

$$u^2+v^2-2uv\cos\pi\rho_1 = \sin^2\pi\rho_1, \quad u^2+v^2-2uv\cos\pi\rho_2 = \sin^2\pi\rho_2$$

in the uv-plane represent ellipses with symmetry axes $\{u=v\}$ and $\{u=-v\}$ that are inscribed in the square S defined by (1). These ellipses touch the lines $\{u=1\}$ and $\{v=1\}$ at the points $(1, \cos\pi\rho_1)$ and $(\cos\pi\rho_1, 1)$, $(1, \cos\pi\rho_2)$ and $(\cos\pi\rho_2, 1)$ respectively. If $0<\rho_1\le\frac{1}{2}$ the first equation of (2) shows that the point (u,v) lies either on one of the sides $\{u=1\}$, $\{v=1\}$ of the square S or at the corner of this square cut off by the arc joining $(\cos\pi\rho_1, 1)$ to $(1, \cos\pi\rho_1)$. This set of (u,v) is denoted by $S^1_{\rho_1}$. If $\frac{1}{2}<\rho_1<1$ then the first equation of (2) divides S into two parts. The point (u,v) lies in the part of S which contains the sides $\{u=1\}$, $\{v=1\}$ of S. This set of (u,v) is denoted by $S^2_{\rho_1}$. The same considerations for ρ_2 and the second equation of (2), and consequently we have $S^1_{\rho_2}$ and $S^2_{\rho_2}$. Thus, Theorem 1 shows that the point (u,v) belongs to one of intersections

$S^1_{\rho_1}\cap S^1_{\rho_2}$, $S^1_{\rho_1}\cap S^2_{\rho_2}$, $S^2_{\rho_1}\cap S^1_{\rho_2}$, $S^2_{\rho_1}\cap S^2_{\rho_2}$ that is admissible depending on values ρ_1 and ρ_2.

As $\rho_1\to 0$, the first ellipse of (2) tends to a limit position which is the linear segment joining $(1,1)$ and $(-1,-1)$. Similarly, as $\rho_1\to 1$ the limit position is the segment joining $(-1,1)$ and $(1,-1)$. The same for ρ_2.

These facts suggest that Theorem 1 can be supplemented by the following propositions:

If $\rho_1=0$ and $0\le\rho_2\le 1$ or $\rho_2=0$ and $0\le\rho_1\le 1$ then the point (u,v) lies on one of sides $\{u=1\}$ or $\{v=1\}$. Every point of these sides is admissible.

If $\rho_1=1$ and $\rho_2=1$, the point (u,v) may be any point of the square S.

If $\rho_1=1$ and $0<\rho_2<1$, the point (u,v) either belong to $S^1_{\rho_2}$ or $S^2_{\rho_2}$ depending on value ρ_2. If $\rho_2=1$ and $0<\rho_1<1$, the point (u,v) either belong to $S^1_{\rho_2}$ or $S^2_{\rho_2}$ depending on value ρ_1.

To prove Theorem 1 we need the following Lemmas.

Lemma 1. *Let f be a meromorphic function in $\mathbb{C}\setminus\{0\}$ with a couple of finite veritable orders (ρ_1,ρ_2), let $\{a_j\}$ and $\{b_k\}$ be the sequences of their zeros, $|a_j|\ge 1$, $|b_k|<1$, and $\{c_j\}$, $\{d_k\}$ be the sequences of its poles, $|c_j|\ge 1$, $|d_k|<1$. Let p_1, p_2, q_1 and q_2 be genera of the sequences $\{a_j\}$, $\{b_k\}$, $\{c_j\}$ and $\{d_k\}$ respectively. Then*

$$f(z)=z^m\exp\left(z^{-\nu_1}P(z)\right)\frac{\prod\limits_{|b_k|<1}E\left(\frac{b_k}{z},p_1\right)\prod\limits_{|a_j|\ge 1}E\left(\frac{z}{a_j},p_2\right)}{\prod\limits_{|d_k|<1}E\left(\frac{d_k}{z},q_1\right)\prod\limits_{|c_j|\ge 1}E\left(\frac{z}{c_j},q_2\right)},\tag{4}$$

where $m\in\mathbb{Z}$, $\nu_1\in\mathbb{Z}_+$, $P(z)$ is a polynomial, $\deg P(z)=\nu_1+\nu_2$, *$\nu_2\in\mathbb{Z}_+$, and $\nu_1\le[\rho_1]$, $\nu_2\le[\rho_2]$, $E(z,p)$ is the Weierstrass elementary factor.*

The genus of the sequence $\{a_j\}$ and respectively of $\{c_j\}$ is defined as usual. The genus of the sequence $\{b_k\}$ and respectively of $\{d_k\}$ is defined as the lowest non-negative integer p such that

$$\sum_k |b_k|^{p+1}<+\infty.\tag{5}$$

An important property of meromorphic in the punctured plane $\mathbb{C}\setminus\{0\}$ function with a couple of veritable orders (ρ_1,ρ_2), where both ρ_1 and ρ_2 are less than one is the following

Lemma 2. *Let*

$$f(z)=\frac{\prod\limits_{|b_k|<1}\left(1-\frac{b_k}{z}\right)\prod\limits_{|a_j|\ge 1}\left(1-\frac{z}{a_j}\right)}{\prod\limits_{|d_k|<1}\left(1-\frac{d_k}{z}\right)\prod\limits_{|c_j|\ge 1}\left(1-\frac{z}{c_j}\right)}\tag{6}$$

and

$$h(z)=\frac{\prod\limits_{|b_k|<1}\left(1+\frac{|b_k|}{z}\right)\prod\limits_{|a_j|\ge 1}\left(1+\frac{z}{|a_j|}\right)}{\prod\limits_{|d_k|<1}\left(1-\frac{|d_k|}{z}\right)\prod\limits_{|c_j|\ge 1}\left(1-\frac{z}{|c_j|}\right)}\tag{7}$$

The function $h(z)$ is called the function associated with f. Then

$$T(\tau, r; f) \leq T(\tau, r; h) + (c_f - c_h) \log \frac{\tau}{r}. \tag{8}$$

To prove Lemma 2 one can use the same idea as in the classical version [[4], p. 294-296, Lemma 4.4] but with two additional products, and Definition 1.

Lemma 3. *Let $g(z)$ be a holomorphic function in $\mathbb{C}\setminus\{0\}$ without zeros and*

$$\varliminf_{\tau\to+\infty} \frac{T(\tau, 1; g)}{\tau^{\mu_1}} = 0 \quad for\ \ some \quad \mu_1 > 0, \tag{9}$$

$$\varliminf_{r\to+\infty} \frac{T(1, r; g)}{r^{\mu_2}} = 0 \quad for\ \ some \quad \mu_2 > 0. \tag{10}$$

Then

$$g(z) = z^m \exp\left(z^{-\nu_1} P(z)\right), \tag{11}$$

where $m \in \mathbb{Z}$, $\nu_1 \in \mathbb{Z}_+$, $P(z)$ is a polynomial, $\deg P(z) = \nu_1 + \nu_2$, *$\nu_2 \in \mathbb{Z}_+$, and $\nu_1 \leq [\mu_1]$, $\nu_2 \leq [\mu_2]$.*

Proof. Let $m = \frac{1}{2\pi i} \int\limits_{|z|=1} \frac{g'(z)}{g(z)} dz$. Consider the function $G(z) = z^{-m} g(z)$. By virtue of ([5], Lemma 3.2) a branch of $\log G(z)$ is determined in $\mathbb{C}\setminus\{0\}$. Considering the Laurent expansion of $\log G(z)$ with the coefficients $\{c_k\}$ we obtain

$$\log|G(z)| = \frac{1}{2} \sum_{k\in\mathbb{Z}} \left(c_k r^k + \bar{c}_{-k} r^{-k}\right) e^{ik\theta}, \quad |z| = r \geq 1,$$

$$\frac{1}{2}\left(c_k r^k + \bar{c}_{-k} r^{-k}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log\left|G\left(re^{i\theta}\right)\right| e^{-ik\theta} d\theta, \tag{12}$$

$$\log|G(z)| = \frac{1}{2} \sum_{k\in} \left(c_{-k} \tau^k + \bar{c}_k \tau^{-k}\right) e^{-ik\theta}, \quad |z| = 1/\tau, \quad \tau \geqslant 1,$$

$$\frac{1}{2}\left(c_{-k} \tau^k + \bar{c}_k \tau^{-k}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log\left|G\left(e^{i\theta}/\tau\right)\right| e^{ik\theta} d\theta. \tag{13}$$

Since $|\log|G(z)|| \leq |\log|g(z)|| + |m|\,|\log|z||$, we get from (10), (12) as $r \to +\infty$ that $c_k = 0$ for $k \geq \mu_2$, and from (9), (13) as $\tau \to +\infty$ that $c_{-k} = 0$ for $k \geq \mu_1$. Therefore

$$\log G(z) = \sum_{k=0}^{\nu_2} c_k z^k + \sum_{k=1}^{\nu_1} c_{-k} z^{-k} = z^{-\nu_1} \sum_{k=-\nu_1}^{\nu_2} c_k z^{k+\nu_1} = z^{-\nu_1} P(z),$$

$\nu_1 \leq [\mu_1]$, $\nu_2 \leq [\mu_2]$, and $g(z) = z^m \exp(z^{-\nu_1} P(z))$. This completes the proof.

Theorem 2. *A couple of veritable orders of the function*

$$F(z) = \prod_{|b_k|<1} E\left(\frac{b_k}{z}, p_1\right) \prod_{|a_j|\geq 1} E\left(\frac{z}{a_j}, p_2\right)$$

is equal to the couple of veritable orders of function $N(\tau, r, 0)$ termwise.

Proof. Let (ρ_1', ρ_2') be the couple of veritable orders of $N(\tau, r, 0)$ [[2], Lemma 2]. From the first fundamental theorem [[3], FFT] $N(\tau, r, 0) \leq T(\tau, r, F) + 4\log 2 + C \cdot \log\frac{\tau}{r}$, where C is a constant depending on the function F, we have $\rho_1' \leq \rho_1$ and $\rho_2' \leq \rho_2$, where (ρ_1, ρ_2) is the couple of veritable orders of $F(z)$. As p_1 is the lowest integer number such that $\sum\limits_k |b_k|^{p_1+1} < +\infty$ and p_2 is the lowest integer number such that $\sum\limits_k |a_k|^{-p_2-1} < +\infty$, then $p_1 \leq \rho_1' \leq p_1 + 1$ and $p_2 \leq \rho_2' \leq p_2 + 1$. Using [[4], p. 78] one can get

$$\log M(r, F) \leq \log M\left(r, \prod_{|b_k|<1} E\left(\frac{b_k}{z}, p_1\right)\right) + \log M\left(r, \prod_{|a_k|\geq 1} E\left(\frac{z}{a_k}, p_2\right)\right) <$$
$$< C(p_1)\int_r^\infty \frac{n(r/s, 1, 0; F)}{s^{p_1+1}}\frac{ds}{1+s} + C(p_2)\int_{1/r}^\infty \frac{n(1, rs, 0; F)}{s^{p_2+1}}\frac{ds}{1+s}, \tag{14}$$

$$\log M\left(\frac{1}{\tau}, F\right) \leq \log M\left(\frac{1}{\tau}, \prod_{|b_k|<1} E\left(\frac{b_k}{z}, p_1\right)\right) + \log M\left(\frac{1}{\tau}, \prod_{|a_k|\geq 1} E\left(\frac{z}{a_k}, p_2\right)\right) <$$
$$< C(p_1)\int_{1/\tau}^\infty \frac{n(1/\tau s, 1, 0; F)}{s^{p_1+1}}\frac{ds}{1+s} + C(p_2)\int_{\tau}^\infty \frac{n(1, s/\tau, 0; F)}{s^{p_2+1}}\frac{ds}{1+s}. \tag{15}$$

Note that $\log M(r, F) = o\left(r^{p_2+1}\right)$, $r \to \infty$ and $\log M\left(\frac{1}{\tau}, F\right) = o\left(\tau^{p_1+1}\right)$, $\tau \to \infty$. Since the inequalities $\rho_1 \leq p_1 + 1$, $\rho_2 \leq p_2 + 1$ are valid, so if $\rho_1' = p_1 + 1$ and $\rho_2' = p_2 + 1$ then $\rho_1' \geq \rho_1$ and $\rho_2' \geq \rho_2$. Let now $\rho_1 < p_1 + 1$, $\rho_2 < p_2 + 1$. Then for any ρ_1'', $\rho_1' < \rho_1'' < p_1 + 1$ and for any ρ_2'', $\rho_2' < \rho_2'' < p_2 + 1$ the $n(1, t, 0; F) < C_2 r^{\rho_2''}$ and $n(1/t, 1, 0; F) < C_1 \tau^{\rho_1''}$ hold for all r and $\tau \geq 1$, where C_1, C_2 are some constants. From here and from (14), (15) we obtain $\log M(r, F) < C_3 r^{\rho_2''}$ and $\log M\left(\frac{1}{\tau}, F\right) < C_4 \tau^{\rho_1''}$. Hence $\rho_1 \leq \rho_1''$, $\rho_2 \leq \rho_2''$, and therefore $\rho_1 \leq \rho_1'$, $\rho_2 \leq \rho_2'$.

Proof of Lemma 1. Denote the function in the numenator of (4) by $F_1(z)$ and in the denominator by $F_2(z)$. Since $N(\tau, r; f) \leq T(\tau, r; f) - c_f \log\frac{\tau}{r}$ and $N(\tau, r; 1/f) \leq$ $\leq T(\tau, r; f) - c_{1/f} \log\frac{\tau}{r}$ the couples of veritable orders of $N(\tau, r; f)$ and $N(\tau, r, 1/f)$ do not exceed termwise the couple of veritable orders (ρ_1, ρ_2). The same is true for the couple of veritable orders of $F_i(z)$, $i = 1, 2$, by the Theorem 2. Consider the zero-free holomorphic function $g(z) = \frac{f(z)F_2(z)}{F_1(z)}$. The function $g(z)$ satisfies the conditions of Lemma 3. By Lemma 3, the proof is completed.

Proof of part I of Theorem 1. Consider the associated function $h(z) = h_1(z)h_2(z)$, where

$$h_1(z) = \frac{\prod\limits_{|b_k|<1}\left(1 + \frac{|b_k|}{z}\right)}{\prod\limits_{|d_k|<1}\left(1 - \frac{|d_k|}{z}\right)} \quad \text{and} \quad h_2(z) = \frac{\prod\limits_{|a_j|\geq 1}\left(1 + \frac{z}{|a_j|}\right)}{\prod\limits_{|c_j|\geq 1}\left(1 - \frac{z}{|c_j|}\right)},$$

and start from the well-known representations

$$\log h_1(z) = z\int_1^\infty N(t,1,0)\,\frac{dt}{(zt+1)^2} + z\int_1^\infty N(t,1,\infty)\frac{dt}{(zt-1)^2} \tag{16}$$

$$\log h_2(z) = z\int_1^\infty N(1,t,0)\,\frac{dt}{(z+t)^2} + z\int_1^\infty N(1,t,\infty)\,\frac{dt}{(z-t)^2} \tag{17}$$

valid for $0 < \arg z < \pi$. It is sufficient to focus our attention on the values of r and τ for which, simultaneously

$$m(r,h) > 0, \quad m(r,1/h) > 0 \tag{18}$$

$$m(1/\tau,h) > 0, \quad m(1/\tau,1/h) > 0. \tag{19}$$

Assume that $\delta_0(0,f) > 0$, $\delta_0(\infty,f) > 0$. Using Lemma 2 we obtain the inequalities

$$\varliminf_{\substack{\tau\to+\infty\\ r\to+\infty}} \frac{T(\tau,r;f)}{N(\tau,r,f)} \le \varliminf_{\substack{\tau\to+\infty\\ r\to+\infty}} \left(\frac{T(\tau,r;h) + (c_f - c_h)\log\frac{\tau}{r}}{N(\tau,r,h)}\right) \le$$

$$\le \varliminf_{\substack{\tau\to+\infty\\ r\to+\infty}} \frac{T(\tau,r;h)}{N(\tau,r,h)} + \varlimsup_{\substack{\tau\to+\infty\\ r\to+\infty}} \frac{(c_f - c_h)\log\frac{\tau}{r}}{N(\tau,r,h)}.$$

In view of the fact that $\varlimsup_{\substack{\tau\to+\infty\\ r\to+\infty}} \frac{(c_f-c_h)\log\frac{\tau}{r}}{N(\tau,r,h)} = 0$, because the function h is the function with couple of genera which equal zero, we obtain $\varlimsup_{\substack{\tau\to+\infty\\ r\to+\infty}} \frac{N(\tau,r,f)}{T(\tau,r;f)} \ge \varlimsup_{\substack{\tau\to+\infty\\ r\to+\infty}} \frac{N(\tau,r,h)}{T(\tau,r;h)}$. Thus $\delta_0(0,h) > 0$ and the same is true for $\delta_0(\infty,h)$. Hence, inequalities (18),(19) are valid for all sufficiently large values of r and τ. The function $\log\left|h\left(re^{i\theta}\right)\right|$ is an even function of θ decreasing when θ varies from 0 to π. Hence, in view of (18) $\log|h(r)| > 0$, $\log|h(-r)| < 0$, and there exists a number $\beta_1 = \beta_1(r)$, uniquely determined and such that $0 < \beta_1(r) < \pi$, $\log\left|h\left(re^{i\beta_1}\right)\right| = 0$. The above considerations are also valid for $\log\left|h\left(\frac{e^{i\theta}}{\tau}\right)\right|$ with uniquely determined $\beta_2 = \beta_2(\tau)$. For sufficiently large values $r \ge 1$ and $\tau \ge 1$ using (16), (17) we have

$$m(r,h) \le \frac{1}{\pi}\int_0^{\beta_1} \log\left|h_2\left(re^{i\theta}\right)\right| d\theta + C_1 = \lim_{\delta\to 0}\frac{1}{\pi}\int_\delta^{\beta_1} \log\left|h_2\left(re^{i\theta}\right)\right| d\theta + C_1 \le$$

$$\le \int_1^\infty N(1,t,0)\,P(t,r,\beta_1)dt + \int_1^\infty N(1,t,\infty)\,P(t,r,\pi-\beta_1)dt - \tag{20}$$

$$-\lim_{\delta\to 0}\int_1^\infty N(1,t,0)\,P(t,r,\delta)dt - \lim_{\delta\to 0}\int_1^\infty N(1,t,\infty)\,P(t,r,\pi-\delta)dt + C_1,$$

$$m(1/\tau,h) \le \frac{1}{\pi}\int_0^{\beta_2} \log\left|h_1\left(\frac{e^{i\theta}}{\tau}\right)\right| d\theta + C_2 = \lim_{\delta\to 0}\frac{1}{\pi}\int_\delta^{\beta_2} \log\left|h_1\left(\frac{e^{i\theta}}{\tau}\right)\right| d\theta + C_2 \le$$

$$\leq \int_1^\infty N(t,1,0)\,P(t,\tau,\beta_2)dt + \int_1^\infty N(t,1,\infty)\,P(t,\tau,\pi-\beta_2)dt - \tag{21}$$

$$-\lim_{\delta\to 0}\int_1^\infty N(t,1,\infty)\,P(t,\tau,\pi-\delta)dt - \lim_{\delta\to 0}\int_1^\infty N(t,1,0)\,P(t,\tau,\delta)dt + C_2,$$

where $P(t,s,\gamma) = \frac{1}{\pi}\frac{s\sin\gamma}{t^2+2ts\cos\gamma+r^2}$ and C_1, C_2 are constants that only depends on the function $h(z)$. The limits $\lim\limits_{\delta\to 0}\int_1^\infty N(1,t,0)\,P(t,r,\delta)dt$, $\lim\limits_{\delta\to 0}\int_1^\infty N(t,1,0)\,P(t,\tau,\delta)dt$ are equal to zero. Using well-known properties of the Poisson integral for the half-plane [[4], p. 298] we have $\lim\limits_{\delta\to 0}\int_1^\infty N(1,t,\infty)\,P(t,r,\pi-\delta)dt = N(1,r,\infty)$,
$\lim\limits_{\delta\to 0}\int_1^\infty N(t,1,\infty)\,P(t,\tau,\pi-\delta)dt = N(\tau,1,\infty)$. By Lemma 2,

$$T(\tau,1) \leq \int_1^\infty N(t,1,0)\,P(t,\tau,\beta_2)dt + \int_1^\infty N(t,1,\infty)\,P(t,\tau,\pi-\beta_2)dt + c_f\log\tau + C_1, \tag{22}$$

$$T(1,r) \leq \int_1^\infty N(1,t,0)\,P(t,r,\beta_1)dt + \int_1^\infty N(1,t,\infty)\,P(t,r,\pi-\beta_1)dt + c_f\log\frac{1}{r} + C_2. \tag{23}$$

Let U, V denote two positive quantities such that $U > u = 1-\delta_0(0,f)$,
$V > v = 1-\delta_0(\infty,f)$. The definition of deficiency (Definition 3) implies

$$N(t,r_1,0) < UT(t,r_1), \quad N(t,r_1,\infty) < VT(t,r_1) \quad \text{for some fixed} \quad r_1 > r_0, \tag{24}$$

$$N(\tau_1,s,0) < UT(\tau_1,s), \quad N(\tau_1,s,\infty) < VT(\tau_1,s) \quad \text{for some fixed} \quad \tau_1 > \tau_0. \tag{25}$$

Choose ε so that $0 < \varepsilon < \min\{\rho_1;\rho_2\}$, $\varepsilon + \max\{\rho_1;\rho_2\} < 1$. By Polya's Lemma [1, p. 237] there exists sequences $\{r_n\}$ $\left(\lim\limits_{n\to\infty} r_n = +\infty\right)$ (depending on ε) and $\{\tau_m\}$ $\left(\lim\limits_{m\to\infty}\tau_m = +\infty\right)$ (depending on ε) such that

$$\left[\begin{array}{ll} \frac{T(t,r_1)}{t^{\rho_1-\varepsilon}} \leq \frac{T(\tau_m,r_1)}{\tau_m^{\rho_1-\varepsilon}}, & (\tau_0 \leq t \leq \tau_m), \\ \frac{T(t,r_1)}{t^{\rho_1+\varepsilon}} \leq \frac{T(\tau_m,r_1)}{\tau_m^{\rho_1+\varepsilon}}, & (\tau_m \leq t), \end{array}\right. \tag{26}$$

$$\left[\begin{array}{ll} \frac{T(\tau_1,s)}{s^{\rho_2-\varepsilon}} \leqslant \frac{T(\tau_1,r_n)}{r_n^{\rho_2-\varepsilon}}, & (r_0 \leqslant t \leqslant r_n), \\ \frac{T(\tau_1,s)}{s^{\rho_2+\varepsilon}} \leqslant \frac{T(\tau_1,r_n)}{r_n^{\rho_2+\varepsilon}}, & (r_n \leqslant t). \end{array}\right. \tag{27}$$

Using (24), (26) in (22), and (25), (27) in (23), we obtain

$$T(\tau_m,r_1) \leq UT(\tau_m,r_1)\left\{\int_0^{\tau_m}\left(\frac{t}{\tau_m}\right)^{\rho_1-\varepsilon} P(t,\tau_m,\beta_2)\,dt + \int_{\tau_m}^\infty \left(\frac{t}{\tau_m}\right)^{\rho_1+\varepsilon} P(t,\tau_m,\beta_2)\,dt\right\} +$$

$$+VT\left(\tau_m,r_1\right)\left\{\int\limits_0^{\tau_m}\left(\frac{t}{\tau_m}\right)^{\rho_1-\varepsilon}P\left(t,\tau_m,\pi-\beta_2\right)dt+\int\limits_{\tau_m}^{\infty}\left(\frac{t}{\tau_m}\right)^{\rho_1+\varepsilon}P\left(t,\tau_m,\pi-\beta_2\right)dt\right\}+ \quad (28)$$

$$+\eta_1\left(\tau_m\right)+T\left(1,r_1\right)+c_f\log\tau_m+C_1,$$

$$\eta_1\left(\tau_m\right)\le\left\{N\left(\tau_0,1,0\right)+N\left(\tau_0,1,\infty\right)\right\}\frac{\tau_m\left(\tau_0-1\right)}{\left(\tau_m-\tau_0\right)^2}.$$

$$T\left(\tau_1,r_n\right)\le UT\left(\tau_1,r_n\right)\left\{\int\limits_0^{r_n}\left(\frac{t}{r_n}\right)^{\rho_2-\varepsilon}P\left(t,r_n,\beta_1\right)dt+\int\limits_{r_n}^{\infty}\left(\frac{t}{r_n}\right)^{\rho_2+\varepsilon}P\left(t,r_n,\beta_1\right)dt\right\}+$$

$$+VT\left(\tau_1,r_n\right)\left\{\int\limits_0^{r_n}\left(\frac{t}{r_n}\right)^{\rho_2-\varepsilon}P\left(t,r_n,\pi-\beta_1\right)dt+\int\limits_{r_n}^{\infty}\left(\frac{t}{r_n}\right)^{\rho_2+\varepsilon}P\left(t,r_n,\pi-\beta_1\right)dt\right\}+ \quad (29)$$

$$+\eta_2\left(r_n\right)+T\left(\tau_1,1\right)+c_f\log\frac{1}{r_n}+C_2,\eta_2\left(r_n\right)\le\left\{N\left(1,r_0,0\right)+N\left(1,r_0,\infty\right)\right\}\frac{r_n\left(r_0-1\right)}{\left(r_n-r_0\right)^2}.$$

The following considerations are carried out only for (28), for (29) the considerations are much the same. Via the substitution $t=s\tau_m$ we obtain for $0<\rho_1-\varepsilon<\rho_1+\varepsilon<1$

$$\int\limits_0^{\tau_m}\left(\frac{t}{\tau_m}\right)^{\rho_1-\varepsilon}P\left(t,\tau_m,\beta_2\right)dt+\int\limits_{\tau_m}^{\infty}\left(\frac{t}{\tau_m}\right)^{\rho_1+\varepsilon}P\left(t,\tau_m,\beta_2\right)dt=$$

$$=\int\limits_0^{\infty}s^{\rho_1+\varepsilon}P\left(s,1,\beta_2\right)ds+\int\limits_0^{1}\left(s^{\rho_1-\varepsilon}-s^{\rho_1+\varepsilon}\right)P\left(s,1,\beta_2\right)ds. \quad (30)$$

It is well-known that $\int\limits_0^{\infty}s^{\rho_1+\varepsilon}P\left(s,1,\beta_2\right)ds=\frac{\sin\beta_2(\rho_1+\varepsilon)}{\sin\pi(\rho_1+\varepsilon)}$. For given $\eta>0$ and for $0<\varepsilon<\varepsilon_0\left(\eta\right)$ we have $0<\int_0^1\left(s^{\rho_1-\varepsilon}-s^{\rho_1+\varepsilon}\right)P\left(s,1,\beta_2\right)ds<\eta$. Hence and from (28) after division by $T\left(\tau_m,r_1\right)$ for $\varepsilon<\varepsilon_0\left(\eta\right)$ we obtain

$$1\le\sup_{0\le\gamma\le\pi}\left\{U\frac{\sin\gamma\left(\rho_1+\varepsilon\right)}{\sin\pi\left(\rho_1+\varepsilon\right)}+V\frac{\sin\left(\pi-\gamma\right)\left(\rho_1+\varepsilon\right)}{\sin\pi\left(\rho_1+\varepsilon\right)}\right\}+$$
$$+\left(U+V\right)\eta+O\left(\frac{1}{\tau_mT\left(\tau_m,r_1\right)}\right)+\frac{c_f\log\tau_m}{T\left(\tau_m,r_1\right)}. \quad (31)$$

Similarly we obtain from (29)

$$1\le\sup_{0\le\sigma\le\pi}\left\{U\frac{\sin\sigma\left(\rho_2+\varepsilon\right)}{\sin\pi\left(\rho_2+\varepsilon\right)}+V\frac{\sin\left(\pi-\sigma\right)\left(\rho_2+\varepsilon\right)}{\sin\pi\left(\rho_2+\varepsilon\right)}\right\}+$$
$$+\left(U+V\right)\eta+O\left(\frac{1}{r_nT\left(\tau_1,r_n\right)}\right)+\frac{c_f\log\frac{1}{r_n}}{T\left(\tau_1,r_n\right)}. \quad (32)$$

Let $n\to\infty$, $m\to\infty$ in (31), (32), and then make the proceeding to the limit $\varepsilon\to0$, $\eta\to0$, $U\to u$, $V\to v$ we can see that the deficiencies u and v satisfy the system of inequalities

$$\begin{cases} \sin \pi\rho_1 \le \sup\limits_{0\le\gamma\le\pi} \left\{u \sin\gamma\rho_1 + v\sin\left(\pi-\gamma\right)\rho_1\right\}, \\ \sin \pi\rho_2 \le \sup\limits_{0\le\sigma\le\pi} \left\{u \sin\sigma\rho_2 + v\sin\left(\pi-\sigma\right)\rho_2\right\}. \end{cases}$$

Since $u\sin\gamma\rho_1 + v\sin(\pi-\gamma)\rho_1$ is a continuous function of γ, we can find a value of γ for which the sup is attained. For this γ,

$$\sin\pi\rho_1 \le u\sin\gamma\rho_1 + v\sin(\pi-\gamma)\rho_1 = (u - v\cos\pi\rho_1)\sin\gamma\rho_1 + v\sin\pi\rho_1\cos\gamma\rho_1.$$

Hence, by Schwarz's inequality we get (2).

The immediate consequence of Theorem 1 is the following

Theorem 3. ***I.*** *Let f be a meromorphic function in the punctured plane $\mathbb{C}\setminus\{0\}$ with a couple of veritable orders (ρ_1, ρ_2), where $0 \le \rho_1 \le 1/2$ and $0 \le \rho_2 \le 1/2$. If $\delta_0(a,f) \ge$ $\ge 1 - \min\{\cos\pi\rho_1, \cos\pi\rho_2\}$ under the condition that either $0 \le \rho_1 \le 1/2$ and $0 < \rho_2 \le 1/2$, or $0 \le \rho_2 \le 1/2$ and $0 < \rho_1 \le 1/2$, then a is a unique deficiency of meromorphic function $f(z)$.*

II. *Let f be a meromorphic function in the punctured plane $\mathbb{C}\setminus\{0\}$ with a couple of veritable orders (ρ_1,ρ_2), where $0 \leqslant \rho_1 < 1/2$ and $0 \leqslant \rho_2 < 1/2$. If $\delta_0(a,f) > 0$ under the condition that either $\rho_1 = 0$, $0 \leqslant \rho_2 < 1/2$ or $\rho_2 = 0$, $0 \leqslant \rho_1 < 1/2$, then a is a unique deficient value of meromorphic function $f(z)$. In particular, meromorphic function with couple of veritable orders (ρ_1,ρ_2), where at least either ρ_1 or ρ_2 is equal to zero, cannot have more than one deficient value.*

Proof. If $\delta_0(a,f) > 1 - \min\{\cos\pi\rho_1, \cos\pi\rho_2\}$ then the result of the theorem immediately follows from (3). If $0 \le \rho_1 \le 1/2$, $0 < \rho_2 \le 1/2$ or $0 \le \rho_2 \le 1/2$, $0 < \rho_1 \le 1/2$ and $\delta_0(a,f) = 1 - \min\{\cos\pi\rho_1, \cos\pi\rho_2\}$, this means that $u = \min\{\cos\pi\rho_1, \cos\pi\rho_2\}$ in Theorem 1. Without loss of generality let $u = \cos\pi\rho_1$, then from (2) it follows that $v = 1$ or $v \le \cos 2\pi\rho_1$. In the last case from (3) we obtain that $u = 1$ because $\cos 2\pi\rho_1 < \cos\pi\rho_1$ when $0 < \rho_1 \le 1/2$, this leads to the contradiction. Thus we should have $v = 1$.

Now we are interested in finding a lower-bound estimate of quantity

$$\varlimsup_{\substack{\tau\to+\infty \\ r\to+\infty}} \frac{N(\tau,r,0;f) + N(\tau,r,\infty;f)}{T(\tau,r;f)},$$

that is to say to prove a variant of Theorem A for meromorphic functions in the punctured plane $\mathbb{C}\setminus\{0\}$. Thus, a slight modification the proof of Theorem 1 leads to

Theorem 4. *Let f be a meromorphic function in the punctured plane $\mathbb{C}\setminus\{0\}$ with a couple of veritable orders (ρ_1,ρ_2), where $0 \le \rho_1 < 1$, $0 \le \rho_2 < 1$. Then*

$$\varlimsup_{\substack{\tau\to+\infty \\ r\to+\infty}} \frac{N(\tau,r,0;f) + N(\tau,r,\infty;f)}{T(\tau,r,f)} \geqslant$$

$$\geqslant \begin{cases} 1, & \text{if } \rho_1 = 0, \quad 0 \le \rho_2 < 1, \\ & \text{or } \rho_2 = 0, \quad 0 \le \rho_1 < 1, \\ \dfrac{1}{\min\left(\sup\limits_{0\le\gamma\le\pi} \frac{\sin\gamma\rho_1}{\sin\pi\rho_1};\ \sup\limits_{0\le\sigma\le\pi} \frac{\sin\sigma\rho_2}{\sin\pi\rho_2} \right)}, & \text{if } 0<\rho_1<1, \quad 0<\rho_2<1. \end{cases} \tag{33}$$

The inequality above is best possible.

Proof. For the couple of veritable orders (ρ_1, ρ_2) such that $\rho_1 = 0$, $0 \le \rho_2 < 1$ or $\rho_2 = 0$, $0 \le \rho_1 < 1$, Theorem 4 follows from the fact that a function with a couple of veritable orders (ρ_1, ρ_2), where at least either ρ_1 or ρ_2 is equal to zero has at most one deficient value, so that

$$\varlimsup_{\substack{\tau\to+\infty \\ r\to+\infty}} \frac{N(\tau, r, 0; f) + N(\tau, r, \infty; f)}{T(\tau, r, f)} \ge$$

$$\ge \max\left\{ \varlimsup_{\substack{\tau\to+\infty \\ r\to+\infty}} \frac{N(\tau, r, 0; f)}{T(\tau, r, f)}, \quad \varlimsup_{\substack{\tau\to+\infty \\ r\to+\infty}} \frac{N(\tau, r, \infty; f)}{T(\tau, r, f)} \right\} = 1.$$

In the sequel, $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$. The (22) and (23) imply the following inequality

$$T(\tau, r, f) \le \int_1^\infty N(t, 1, 0)\, P(t, \tau, \beta_2) dt + \int_1^\infty N(t, 1, \infty)\, P(t, \tau, \pi - \beta_2) dt +$$

$$+ \int_1^\infty N(1, t, 0)\, P(t, r, \beta_1) dt + \int_1^\infty N(1, t, \infty)\, P(t, r, \pi - \beta_1) dt + c_f \log\frac{\tau}{r} + C_1 + C_2, \tag{34}$$

which is valid for all sufficiently large values $r \ge 1$, $\tau \ge 1$. If $\lambda_1 = \lambda_1(\tau) = \max\{\beta_2(\tau), \pi - \beta_2\}$ and $\lambda_2 = \lambda_2(r) = \max\{\beta_1(r), \pi - \beta_1\}$, then obviously $\lambda_1 \ge \pi/2$, $\lambda_2 \ge \pi/2$, and $\max\{P(t, r, \beta_1), P(t, r, \pi - \beta_1)\} = P(t, r, \lambda_2)$,
$\max\{P(t, \tau, \beta_2), P(t, \tau, \pi - \beta_2)\} = P(t, \tau, \lambda_1)$. Hence, by (34)

$$T(\tau, r, f) \le \int_1^\infty \{N(t, 1, 0) + N(t, 1, \infty)\} P(t, \tau, \lambda_1)\, dt +$$

$$+ \int_1^\infty \{N(1, t, 0) + N(1, t, \infty)\}\, P(t, r, \lambda_2) dt + c_f \log\frac{\tau}{r} + C_1 + C_2.$$

Apply Polya's Lemma to the functions in square brackets in both integrals above, as in the proof of Theorem 1. We obtain the inequality

$$T(\tau_m, r_n, f) \le N_1(\tau_m)\, A(\rho_1) + N_2(r_n)\, A(\rho_2) + \eta_1(\tau_m) + \eta_2(r_n) +$$

$$+ 2\eta + c_f \log\frac{\tau_m}{r_n} + C_1 + C_2,$$

where $N_1(\tau_m) = N(\tau_m, 1, 0) + N(\tau_m, 1, \infty)$ and $N_2(r_n) = N(1, r_n, 0) + N(1, r_n, \infty)$, $A(\rho_1) = \frac{\sin\lambda_1(\rho_1+\varepsilon)}{\sin\pi(\rho_1+\varepsilon)}$, $A(\rho_2) = \frac{\sin\lambda_2(\rho_2+\varepsilon)}{\sin\pi(\rho_2+\varepsilon)}$. After division the last inequality by

$N(\tau_m, r_n, 0) + N(\tau_m, r_n, \infty)$, where $N(\tau_m, r_n, 0) + N(\tau_m, r_n, \infty) = N_1(\tau_m) + N_2(r_n)$ we get

$$\frac{T(\tau_m, r_n; f)}{N(\tau_m, r_n, 0) + N(\tau_m, r_n, \infty)} \leq \frac{A(\rho_1)N_1(\tau_m) + A(\rho_2)N_2(r_n)}{N_1(\tau_m) + N_2(r_n)} + \frac{c_f \log \frac{\tau_m}{r_n}}{N_1(\tau_m) + N_2(r_n)} +$$
$$+O\left(\frac{1}{\tau_m(N_1(\tau_m) + N_2(r_n))} + \frac{1}{r_n(N_1(\tau_m) + N_2(r_n))}\right) + \frac{2\eta}{N_1(\tau_m) + N_2(r_n)} +$$
$$+\frac{C_1 + C_2}{N_1(\tau_m) + N_2(r_n)}.$$

Passing to the lower limit in the previous inequality when $n \to +\infty$, $m \to +\infty$ we obtain

$$\varliminf_{\substack{\tau \to +\infty \\ r \to +\infty}} \frac{T(\tau, r; f)}{N(\tau, r, 0) + N(\tau, r, \infty)} \leq$$
$$\leq \min\{A(\rho_1), A(\rho_2)\} \leq \min\left\{\sup_{0 \leq \gamma \leq \pi} \frac{\sin \gamma(\rho_1 + \varepsilon)}{\sin \pi(\rho_1 + \varepsilon)}; \quad \sup_{0 \leq \sigma \leq \pi} \frac{\sin \sigma(\rho_2 + \varepsilon)}{\sin \pi(\rho_2 + \varepsilon)}\right\},$$

since $\varliminf_{\substack{n \to +\infty \\ m \to +\infty}} \frac{A(\rho_1)N_1(\tau_m) + A(\rho_2)N_2(r_n)}{N_1(\tau_m) + N_2(r_n)} = \min\{A(\rho_1), A(\rho_2)\}$. Then passing to the limit when $\varepsilon \to 0$ completes the proof of (33).

The following function with couple of veritable orders (λ_2, λ_1), where $0 < \lambda_1 < 1$, $0 < \lambda_2 < 1$, will be used to show that the result of Theorem 4 is best possible,

$$H_{\alpha_1,\alpha_2}(z) = \prod_{\nu=1}^{\infty}\left(1 + \frac{\alpha_1^{1/\lambda_1}(z + c_1)}{a_\nu}\right)\prod_{n=2}^{\infty}\left(1 + b_n \alpha_2^{1/\lambda_2}\left(\frac{1}{z} + c_2\right)\right), \tag{35}$$

where $a_\nu = \nu^{1/\lambda_1}$, $(\nu = 1, 2, 3, ...)$ $b_n = \frac{1}{n^{1/\lambda_2}}$ $(n = 2, 3, ...)$, $0 \leq \alpha_1$ and $0 \leq \alpha_2$, $c_1 \geq 0$, $c_2 \geq 0$. Now we will look at the asymptotic behavior of $H_{\alpha_1,\alpha_2}(z)$ when $z \to \infty$ and $z \to 0$. The asymptotic behavior of $f(z; \lambda) = \prod_{\nu=1}^{\infty}\left(1 + \frac{z}{c_\nu}\right)$ (Lindelöf functions), where $c_\nu = \nu^{1/\lambda}$ $(\nu = 1, 2, 3, ...)$, $0 < \lambda < 1$ is well-known [1], [6] and it implies

$$\log H_{\alpha_1,\alpha_2}(z) = \frac{\pi\alpha_1}{\sin \pi\lambda_1} z^{\lambda_1}(1 + \varepsilon_1(z)), \tag{36}$$

where $\varepsilon_1(z) \to 0$, uniformly, as $z \to \infty$ in the angle $|\arg z| \leq \pi - \theta$ $(0 < \theta < \pi)$, and

$$\log H_{\alpha_1,\alpha_2}(z) = \frac{\pi\alpha_2}{\sin \pi\lambda_2} \frac{1}{z^{\lambda_2}}\left(1 + \varepsilon_2\left(\frac{1}{z}\right)\right), \tag{37}$$

where $\varepsilon_2\left(\frac{1}{z}\right) \to 0$, uniformly, as $z \to 0$ in the angle $|\arg z| \leq \pi - \theta$ $(0 < \theta < \pi)$. It is easy to infer from (36) and (37) that, as $r \to \infty$ and $\tau \to \infty$,

$$\begin{cases} m(r, H_{\alpha_1,\alpha_2}(z)) \sim \frac{\alpha_1 r^{\lambda_1}}{\lambda_1 \sin \pi\lambda_1} & \left(\frac{1}{2} < \lambda_1 < 1\right), \\ m(r, H_{\alpha_1,\alpha_2}(z)) \sim \frac{\alpha_1 r^{\lambda_1}}{\lambda_1} & \left(0 < \lambda_1 \leq \frac{1}{2}\right), \end{cases} \tag{38}$$

$$\begin{cases} m\left({}^1\!/_\tau, H_{\alpha_1,\alpha_2}(z)\right) \sim \frac{\alpha_2 \tau^{\lambda_2}}{\lambda_2 \sin \pi\lambda_2} & \left(\frac{1}{2} < \lambda_2 < 1\right), \\ m\left({}^1\!/_\tau, H_{\alpha_1,\alpha_2}(z)\right) \sim \frac{\alpha_2 \tau^{\lambda_2}}{\lambda_2} & \left(0 < \lambda_2 \leq \frac{1}{2}\right). \end{cases} \tag{39}$$

Let $n(1/t, 1; 0)$, $n(1, t; 0)$ denote the counting-functions associated with $H_{\alpha_1,\alpha_2}(z)$. Evidently, $n(1/\tau, 1; 0) \sim \alpha_2 \tau^{\lambda_2}$ $(\tau \to \infty)$ and $n(1, r; 0) \sim \alpha_1 r^{\lambda_1}$ $(r \to \infty)$. And hence

$$N(\tau, r, 0) \sim \frac{\alpha_1 r^{\lambda_1}}{\lambda_1} + \frac{\alpha_2 \tau^{\lambda_2}}{\lambda_2} \quad (\tau \to \infty, \quad r \to \infty). \tag{40}$$

Under different values of $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$ asymptotic relations (38), (39) and (40) imply the equality in (33).

Proof of part II of Theorem 1. In proving this Theorem the constructed examples are modification of examples given by Albert Edrei and Wolfgang H. J. Fuchs [1] in the classical case. According to the geometrical discussion following the statement of Theorem 1, we must show that for every point (u, v) in the corner of the square $0 \le u \le 1$, $0 \le v \le 1$, cut off by the arc of one of ellipse (2), which lies above, there exists a meromorphic function $f(z)$ with a couple of veritable orders (λ_2, λ_1), such that

$$u = 1 - \delta_0(0, f), \quad v = 1 - \delta_0(\infty, f)$$

If $0 < \lambda_i \leqslant \frac{1}{2}$ $(i = 1, 2)$, the lines $u = 1$ and $v = 1$ have to be added to this corner. Let $P_1(\beta) : (\cos(\pi - \beta)\lambda_1, \cos\pi\lambda_1)$, $P_2(\beta) : (\cos(\pi - \beta)\lambda_2, \cos\pi\lambda_2)$ $(0 \le \beta \le \pi)$. Either the point $P_1(\beta)$ or $P_2(\beta)$ describes the arc which lies above depending on values λ_1 and λ_2. Then let L_β be the closed line segment joining the point $P_1(\beta)$ or $P_2(\beta)$ to $(1, 1)$. Let C_β be the part of L_β contained in the square $0 \le u \le 1$, $0 \le v \le 1$. First it will be proved that every point of C_β is a possible position of $(1 - \delta_0(0, f), 1 - \delta_0(\infty, f))$. As β varies from 0 to π, one of the points $P_1(\beta)$, $P_2(\beta)$ describes the arc which lies above and C_β sweeps over the whole corner described above. Therefore, every point in the corner is a possible position of $(1 - \delta_0(0, f), 1 - \delta_0(\infty, f))$. Consider the function

$$f(z) = \frac{H_{\alpha_1,\alpha_2}(z)\, H_{\alpha_3,\alpha_4}(-z)}{H_{\alpha_5,\alpha_6}(z)\, H_{\alpha_7,\alpha_8}(-z)} \tag{41}$$

where $H_{\alpha_i,\alpha_j}(z)$ is function (35). For every specific choice of the quantities α_i $\left(i = \overline{1,8}\right)$, we may choose the constants c_1, c_2 so as to prevent cancellation of zeros between the various functions $H_{\alpha_i,\alpha_j}(z)$ in (41). We take, for instance, $c_1 = 0$, $c_2 = 0$ in $H_{\alpha_1,\alpha_2}(z)$ and $H_{\alpha_3,\alpha_4}(-z)$ and $c_1 = \gamma_1 > 0$, $c_2 = \gamma_2 > 0$ in $H_{\alpha_5,\alpha_6}(z)$ and $H_{\alpha_7,\alpha_8}(-z)$. The α_i $\left(i = \overline{1,8}\right)$ are chosen so that

$$\begin{cases} (\alpha_3 - \alpha_7)\sin\pi\lambda_1 = -\cos\beta\lambda_1, \\ (\alpha_1 - \alpha_5) + (\alpha_3 - \alpha_7)\cos\pi\lambda_1 = \sin\beta\lambda_1 \\ (\alpha_4 - \alpha_8)\sin\pi\lambda_2 = -\cos\beta\lambda_2, \\ (\alpha_2 - \alpha_6) + (\alpha_4 - \alpha_8)\cos\pi\lambda_2 = \sin\beta\lambda_2, \\ (\alpha_3 + \alpha_5)\sin\pi\lambda_1 = X, \\ (\alpha_4 + \alpha_6)\sin\pi\lambda_2 = X, \end{cases} \tag{42}$$

where $X \geqslant \max\{A, B\}$,

$$A = \max\{0, -\cos(\pi - \beta)\lambda_1, -\cos\beta\lambda_1\}, \; B = \max\{0, -\cos(\pi - \beta)\lambda_2, -\cos\beta\lambda_2\}. \tag{43}$$

Using considerations as in [1] one can easily show that the system (42) may be satisfied by non-negative α. All the α are non-negative and one, at least, is positive. By (36), (37) and (41) we have for every η, $0 < \eta < \pi$,

$$\log f\left(re^{i\theta}\right) = \frac{\pi r^{\lambda_1}}{\sin \pi\lambda_1}\left[(\alpha_1 - \alpha_5)\, e^{i\lambda_1\theta} + (\alpha_3 - \alpha_7)\, e^{i\lambda_1(\theta-\pi)}\right] + \varepsilon'(z)\, r^{\lambda_1}, \tag{44}$$

$$\log f\left(\frac{e^{i\theta}}{\tau}\right) = \frac{\pi\tau^{\lambda_2}}{\sin \pi\lambda_2}\left[(\alpha_2 - \alpha_6)\, e^{-i\lambda_2\theta} + (\alpha_4 - \alpha_8)\, e^{-i\lambda_2(\theta-\pi)}\right] + \varepsilon''\left(\frac{1}{z}\right)\tau^{\lambda_2}, \tag{45}$$

$$0 < \eta \le \theta \le \pi - \eta; \quad \varepsilon'(z) \to 0, \quad \text{uniformly as} \quad |z| = r \to \infty$$

and

$$\varepsilon''\left(\frac{1}{z}\right) \to 0 \quad \text{uniformly as} \quad |z| = \frac{1}{\tau}, \quad \tau \to \infty.$$

Using (44), (45) and the four first equations of (42), we obtain

$$\log\left|f\left(re^{i\theta}\right)\right| = \frac{\pi r^{\lambda_1}}{\sin \pi\lambda_1}\sin\lambda_1(\beta - \theta) + \varepsilon'(r,\theta)\, r^{\lambda_1}, \tag{46}$$

$$\log\left|f\left(\frac{e^{i\theta}}{\tau}\right)\right| = \frac{\pi\tau^{\lambda_2}}{\sin \pi\lambda_2}\sin\lambda_2(\beta - \theta) + \varepsilon''(\tau,-\theta)\,\tau^{\lambda_2}, \tag{47}$$

where the quantities $\varepsilon'(r,\theta)$, $\varepsilon''(\tau,-\theta)$ tend to zero uniformly as $r \to \infty$, $\tau \to \infty$ and $\eta \le \theta \le \pi - \eta$. Although (46) and (47) are not valid with θ replaced by $-\theta$, the asymptotic behavior of $f\left(re^{-i\theta}\right)$ and $f\left(\frac{e^{-i\theta}}{\tau}\right)$ are known since, by (41), the function $f(z)$ is real for real values of z and hence

$$\left|f\left(re^{i\theta}\right)\right| = \left|f\left(re^{-i\theta}\right)\right|, \qquad \left|f\left(\frac{e^{i\theta}}{\tau}\right)\right| = \left|f\left(\frac{e^{-i\theta}}{\tau}\right)\right|.$$

Using the result from ([1], P. 244) we obtain

$$\int_{\omega}^{\omega+\eta}\left|\log\left|H_{\alpha_i,\alpha_j}\left(re^{i\theta}\right)\right|\right| d\theta < Ar^{\lambda_1}\left(\eta + \int_{\omega}^{\omega+\eta}\log\left(\frac{1}{\sin\theta}\right)d\theta\right), \tag{48}$$

$$\int_{\omega}^{\omega+\eta}\left|\log\left|H_{\alpha_i,\alpha_j}\left(e^{i\theta}/\tau\right)\right|\right| d\theta < B\tau^{\lambda_2}\left(\eta + \int_{\omega}^{\omega+\eta}\log\left(\frac{1}{\sin\theta}\right)d\theta\right), \tag{49}$$

$(0 \leqslant \omega < \omega + \eta \leqslant \pi)$, A, B depend only on $H_{\alpha_i,\alpha_j}(z)$. Now from (46), (47) and (48), (49) it easily follows that

$$m(r,f) = \tfrac{r^{\lambda_1}}{\sin\pi\lambda_1}\int_0^{\beta}\sin\lambda_1(\beta-\theta)\,d\theta + o\left(r^{\lambda_1}\right) = \tfrac{r^{\lambda_1}}{\lambda_1\sin\pi\lambda_1}\left(1 - \cos\beta\lambda_1\right) + o\left(r^{\lambda_1}\right)\ (r \to \infty),$$

$$m\left(1/\tau, f\right) = \tfrac{\tau^{\lambda_2}}{\sin\pi\lambda_2}\int_0^{\beta}\sin\lambda_2(\beta-\theta)d\theta + o\left(\tau^{\lambda_2}\right) = \tfrac{\tau^{\lambda_2}}{\lambda_2\sin\pi\lambda_2}\left(1 - \cos\beta\lambda_2\right) + o\left(\tau^{\lambda_2}\right)$$

$(\tau \to \infty)$.

By (40) and (41),

$$N(\tau,r,0) \sim \frac{(\alpha_1+\alpha_3)\, r^{\lambda_1}}{\lambda_1} + \frac{(\alpha_2+\alpha_4)\,\tau^{\lambda_2}}{\lambda_2}, \; N(\tau,r,\infty) \sim \frac{(\alpha_5+\alpha_7)\, r^{\lambda_1}}{\lambda_1} + \frac{(\alpha_6+\alpha_8)\,\tau^{\lambda_2}}{\lambda_2}. \tag{50}$$

Hence, using (42)

$$T(\tau, r, f) \sim \left[\frac{r^{\lambda_1}}{\lambda_1 \sin \pi\lambda_1} + \frac{\tau^{\lambda_2}}{\lambda_2 \sin \pi\lambda_2}\right](1+X). \tag{51}$$

Using (50), (51), and (42) we get

$$u = \varlimsup_{\substack{\tau\to+\infty \\ r\to+\infty}} \frac{N(\tau, r, 0)}{T(\tau, r, f)} = \max\left\{\frac{(\alpha_1+\alpha_3)\sin\pi\lambda_1}{1+X}; \frac{(\alpha_2+\alpha_4)\sin\pi\lambda_2}{1+X}\right\} =$$

$$= \max\left\{\frac{\cos(\pi-\beta)\lambda_1 + X}{1+X}; \frac{\cos(\pi-\beta)\lambda_2 + X}{1+X}\right\} =$$

$$= \frac{\max\{\cos(\pi-\beta)\lambda_1; \cos(\pi-\beta)\lambda_2\} + X}{1+X}. \tag{52}$$

Similarly

$$v = \varlimsup_{\substack{\tau\to+\infty \\ r\to+\infty}} \frac{N(\tau, r, \infty)}{T(\tau, r, f)} = \frac{\max\{\cos\beta\lambda_1; \cos\beta\lambda_2\} + X}{1+X}. \tag{53}$$

As X varies according to (43), the point (u, v) whose coordinates are given by (52) and (53) describes the segment C_β except for the end-point $(1,1)$. But the values $u = 1$, $v = 1$ are attained for any function f with $\delta_0(0, f) = \delta_0(\infty, f) = 0$. To complete the proof we need to consider the cases $\lambda_1 < \frac{1}{2}$, $\lambda_2 < \frac{1}{2}$, $u = 1$, $v < \min\{\cos\pi\lambda_1; \cos\pi\lambda_2\}$; $u < \min\{\cos\pi\lambda_1; \cos\pi\lambda_2\}$, $v = 1$. Consider the function

$$f(z) = \frac{H_{1+\alpha_1, 1+\alpha_1}(z)}{H_{\alpha_1, \alpha_1}(z)},$$

which yields

$$m(r, f) \sim \frac{r^{\lambda_1}}{\lambda_1}, \quad m\left(1/\tau, f\right) \sim \frac{\tau^{\lambda_2}}{\lambda_2},$$

$$N(\tau, r, 0) \sim \frac{(1+\alpha_1)r^{\lambda_1}}{\lambda_1} + \frac{(1+\alpha_1)\tau^{\lambda 2}}{\lambda_2}, \quad N(\tau, r, \infty) \sim \frac{\alpha_1 r^{\lambda_1}}{\lambda_1} + \frac{\alpha_1 \tau^{\lambda 2}}{\lambda_2} \quad (\alpha_1 > 0),$$

$$T(\tau, r, f) \sim \frac{(1+\alpha_1)r^{\lambda_1}}{\lambda_1} + \frac{(1+\alpha_1)\tau^{\lambda 2}}{\lambda_2},$$

$$u = 1, \quad v = \frac{\alpha_1}{1+\alpha_1}.$$

When α_1 varies from 0 to ∞, v attains every value in $(0,1)$. The case $u = 1$, $v = 0$ is trivial since we may take for $f(z)$ any holomorphic in $\mathbb{C}\setminus\{0\}$ function with a couple of veritable orders (λ_2, λ_1), where $\lambda_1 < \frac{1}{2}$ and $\lambda_2 < \frac{1}{2}$. Passing from $f(z)$ to $\frac{1}{f(z)}$, it is clear that the values of u and v are exchanged.

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ДЕФЕКТИ МЕРОМОРФНИХ ФУНКЦІЙ У ПРОКОЛОТІЙ ПЛОЩИНІ

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Нехай f – мероморфна функція в проколотій площині $\mathbb{C}\setminus\{0\}$. Вивчено проблему можливих дефектних значень і величини відповідних дефектів мероморфних функцій у проколотій площині. Доведено деякі співвідношення стосовно дефектів мероморфної фунції f. Розглянуто проблему Неванлінни для мероморфної в проколотій площині функції.

Ключові слова: дефект, дефектне значення, проблема Неванлінни, пара істинних порядків, Неванліннівська характеристика.

ДЕФЕКТЫ МЕРОМОФНЫХ ФУНКЦИЙ В ПРОКОЛОТОЙ ПЛОСКОСТИ

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Пусть f – мероморфная функция в проколотой плоскости $\mathbb{C}\setminus\{0\}$. Рассмотрено проблему возможных дефектных значений и величины соответственных дефектов мероморфных функций в проколотой плоскости. Устанавливаются некоторые соотношения относительно дефектов мероморфной функции f. Рассмотрено проблему Неванлинны для мероморфной функции в проколотой плоскости.

Ключевые слова: дефект, дефектное значение, проблема Неванлинны, пара истинных порядков, характеристика Неванлинны.

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