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CONJUGATE MEASURES ON SEMILATTICES

Oleh NYKYFORCHYN, Oksana MYKYTSEY

Vasyl' Stefanyk Precarpathian National University,
76025 Ivano-Frankivsk, Shevchenka Str., 57
e-mail: oleh.nyk@gmail.com

As a generalization of conjugate lattice-valued capacities on a compactum, a relation of conjugacy for Scott continuous maps from continuous semilattices with 0 to complete continuous lattices is introduced and investigated.

Key words: conjugate capacity, continuous semilattice, Lawson lattice, duality of categories.

Introduction. A proper domain for a real-valued additive measure is a ring or an algebra of sets. If a measure is σ-additive, then the closedness of the domain under countable unions and intersections is usually also required. For a capacity, i.e. a regular non-additive measure, on a compactum, it is sufficient to know only values for closed non-empty subsets [4]. These subsets, ordered by reverse inclusion, form a complete continuous meet semilattice, hence it makes sense to consider capacities with domains which are continuous semilattices, and with values in complete continuous lattices. We suggest a definition for such capacities and discuss a generalization of a notion of conjugate capacities, which proved to be fruitful for set functions. It is shown that conjugacy is related to self-dualities of categories of continuous semigroups and of complete lattices, and is naturally described in terms of Galois connections.

1. Preliminaries. We adopt the following definitions and notation, which are consistent with [1]. All statements in this section are also numbered accordingly to the latter citation. From now on, *semilattice* means *meet semilattice*, if otherwise is not specified. If a poset contains a bottom (a top) element, then it is denoted by 0 (resp. by 1). A top element in a semilattice is also called a *unit*.

For a partial order $\leqslant$ on a set X, the relation $\tilde{\leqslant}$, defined as $x \mathrel{\tilde{\leqslant}} y \iff y \leqslant x$, for $x, y \in X$, is a partial order called *opposite* to $\leqslant$, and $(X, \leqslant)^{op}$ denotes the poset $(X, \tilde{\leqslant})$. If the original order $\leqslant$ is obvious, we write simply X^{op} for the *reversed* poset. We also apply $\tilde{(\ )}$ to all notation to denote passing to the opposite order, i.e. write $\tilde{X} = X^{op}$, $\tilde{\sup} = \inf$, $\tilde{0} = 1$ etc. For a morphism $f : (X \leqslant) \to (Y, \leqslant)$ in a category [3] $\mathcal{P}$oset of posets and isotone (order preserving) mappings, let f^{op} be the same mapping, but regarded as $(X, \tilde{\leqslant}) \to (Y, \tilde{\leqslant})$. It is obvious that f^{op} is isotone as well, thus a functor $(-)^{op} : \mathcal{P}\mathrm{oset} \to \mathcal{P}\mathrm{oset}$ is obtained.

For a subset A of a poset $(X, \leqslant)$, we denote

$$A\!\uparrow = \{x \in X \mid a \leqslant x \text{ for some } a \in A\},\ A\!\downarrow = \{x \in X \mid x \leqslant a \text{ for some } a \in A\}.$$

If $A = A\!\uparrow$ ($A = A\!\downarrow$), then a set A is called *upper* (resp. *lower*).

The *subgraph* (or *hypograph*) of a mapping f from a set T to a poset L is the set

$$\operatorname{sub} f = \{(t, \alpha) \in T \times L \mid \alpha \leqslant f(t)\}.$$

Its *epigraph* is the set

$$\operatorname{epi} f = \{(t, \alpha) \in T \times L \mid f(t) \leqslant \alpha\}.$$

For a poset L, we denote by $L^{\top}$ a poset $L \cup \{\top\}$, where $\top \notin L$ becomes a new top element. Observe that $L^{\top}$ is a complete lattice if and only if L is a complete semilattice.

A *topological meet* (or *join*) *semilattice* is a semilattice L carrying a topology such that the mapping $\wedge : L \times L \to L$ (resp. $\vee : L \times L \to L$) is continuous. A lattice L with a topology such that both $\wedge : L \times L \to L$ and $\vee : L \times L \to L$ are continuous is called a *topological lattice.*

A set A in a poset $(X, \leqslant)$ is *directed* (*filtered*) if, for all $x, y \in A$, there is $z \in A$ such that $x \leqslant z$, $y \leqslant z$ (resp. $z \leqslant x$, $z \leqslant y$). A poset is called *directed complete* (*dcpo* for short) if it has lowest upper bounds for all its directed subsets.

For a topology τ on X define a preorder $\leqslant_\tau$ on X by

$$x \leqslant_\tau y \ \Leftrightarrow\ x \in U \text{ implies } y \in U \text{ for all } x \in \tau.$$

A poset $(X, \leqslant)$ is called *bounded complete* if each bounded from above non-empty subset $A \subset X$ has a least upper bound.

The preorder $\leqslant_\tau$ is called the *specialization order* on X with respect to τ. It is a partial order if and only if τ is T_0-topology.

Fix a partial order $\leqslant$ on a set X. Among the topologies τ on X with specialization order $\leqslant$ those with the property that every $\leqslant$-directed set with a least upper bound in $U \in \tau$ is eventually in U are called order-consistent. The finest such topology is the *Scott topology* $\sigma(X)$. It consists of all those $U \subseteq X$ that satisfy $x \in U \ \Leftrightarrow\ U \cap D \neq \varnothing$ for every $\leqslant -$directed $D \subseteq X$ with a least upper bound x. Note that "$\Leftarrow$" above implies $U = U\uparrow$.

A mapping f between dcpo's X and Y is *Scott continuous*, i.e. continuous w.r.t. $\sigma(X)$ and $\sigma(Y)$, if and only if it preserves suprema of directed sets (cf. Proposition II.2-1).

The *lower topology* $\omega(X)$ on a poset $(X, \leqslant)$ is the least topology such that all sets of the form $\{x\}\!\downarrow$ are closed. The join of (i.e. the least topology that contains) $\sigma(X)$ and $\omega(X)$ is called the *Lawson topology* on X and denoted by $\lambda(X)$. The space $(X, \lambda(X))$ is denoted by ΛX.

In a dcpo X, a lower set is Lawson closed iff it is Scott closed iff it is closed under suprema of directed subsets.

Let L be a poset. We say that x is *way below* y and write $x \ll y$ iff, for all directed subsets $D \subseteq L$ such that $\sup D$ exists, the relation $y \leqslant \sup D$ implies the existence of $d \in D$ such that $x \leqslant d$. "Way-below" relation is transitive and antisymmetric. An element satisfying $x \ll x$ is said to be *compact* or *isolated from below*, and in this case the set $\{x\}\!\uparrow$ is Scott open (hence Lawson open).

Observe that $\top$ is isolated from below in $L^\top$ if and only if each directed set $D \subset L$ has an upper bound in L.

A poset L is called *continuous* if each element $y \in L$ is a least upper bound a direct set of all $x \in L$ such that $x \ll y$. A *domain* is a continuous dcpo. If domain is a semilattice (a lattice), it is called a *continuous semilattice* (resp. a continuous lattice). Obviously a continuous lattice with a bottom element is a complete lattice, and a semilattice S is continuous if and only if $S^\top$ is a continuous semilattice with $\top$ isolated from below.

By Theorem III.1-9 the Lawson topology on a complete semilattice L is a compact T_0-topology. Theorem III.1-10 asserts that, for a domain, the Lawson topology is Hausdorff. Hence the Lawson topology on a complete continuous semilattice is compact Hausdorff, and by Theorems II.1-14, III-2.28 the mapping $\wedge : \Lambda L \times \Lambda L \to \Lambda L$ is continuous, i.e. $(L, \lambda(L))$ is a topological semilattice.

Theorem II.1-14 and Proposition III.2-6 imply that, for a dcpo S and a domain L, the topologies $\lambda(S \times L)$ and $\lambda(S) \times \lambda(L)$ on $S \times L$ are equal.

A topological semilattice is called a *Lawson semilattice* or said to have *small semilattices* if, in each point, it possesses a local base consisting of subsemilattices. A *distributive* topological lattice L is called a *Lawson lattice* if, in each point, it has a local base consisting of sublattices, or, equivalently, if L and L^{op} are Lawson semilattices.

By the Fundamental Theorem on Compact Semilattices (Theorem VI.3-4), each complete continuous semilattice with the Lawson topology is a compact Hausdorff Lawson semilattice, and each compact Hausdorff Lawson semilattice is a complete continuous semilattice such that the given topology agrees with the Lawson topology.

Similarly, by Proposition VII.2-10, a complete distributive lattice L admits a compact Hausdorff topology making it a Lawson lattice if and only if both L and L^{op} are continuous semilattices and the Lawson topologies on L and L^{op} agree (and provide a unique such topology). In this case the Lawson topology on L coincides with the lower topology on L^{op}, and vice versa, hence the topology in question on L is the *interval topology*, i.e. the join of the lower topologies on L and L^{op}.

We regard each subset $P \subset X \times Y$ as a binary relation and write xPy for $(x, y) \in P$. We also denote $xP = \{y \in Y \mid xPy\}$, $Py = \{x \in X \mid xPy\}$ for all $x \in X$, $y \in Y$. The characteristic mapping of a relation will be denoted by the same letter:

$$P(x, y) = \begin{cases} 1, (x, y) \in P, \\ 0, (x, y) \notin P, \end{cases} \qquad x \in X, y \in Y.$$

Let also $\bar{P}$ be the complement $(X \times Y) \setminus P$.

2. Notion of capacity on semilattice. Lawson duals and conjugacy. From now L is a compact Hausdorff Lawson lattice. For a continuous semilattice S, Lemma II.2-5 and Theorem II.2-12 [1] imply that the set $[S \to L]$ all Scott-continuous (hence isotone) mappings from S to L is a complete continuous lattice w.r.t. a natural order: $f \leqslant g$ if $f(s) \leqslant g(s)$ in L for all $s \in S$. Pairwise meets and joins in $[S \to L]$ are calculated argumentwise: $f \wedge g(s) = f(s) \wedge g(s)$, $f \vee g(s) = f(s) \vee g(s)$. Thus $[S \to T]$ with the Lawson topology is a compact Hausdorff Lawson lattice.

It is widely known (cf. Exercise VI.3-18 [1]) that the *Vietoris topology* on the set $\exp X$ of all non-empty closed subsets of a compact Hausdorff space X is the Lawson topology on $(\exp X)^{op}$, i.e. on $\exp X$ ordered by *reverse* inclusion, which we denote by

$\exp_{\supset} X$, and $\exp_{\supset} X$ with this topology is compact Hausdorff Lawson semilattice, hence is a complete continuous (meet) semilattice. Therefore an element $f \in [\exp_{\supset} X \to L]$ is an isotone mapping $f : \exp X \to L^{op}$ which preserves infima of filtered collections of sets, i.e. $f(\bigcap \mathcal{A}) = \inf\{f(A) \mid A \in \mathcal{A}\}$ for any filtered collection $\mathcal{A}$ of closed non-empty subsets of X. It is nothing but a τ-smooth (i.e. upper semicontinuous) L^{op}-valued capacity on X [4].

Thus we suggest the following definition which generalizes the notion of capacity on a compactum in the same manner as σ-additive measure on a complete Boolean algebra is a generalization of σ-additive measure defined on a σ-algebra of sets.

Recall that L is a complete continuous semilattice if and only if L is a compact Hausdorff Lawson meet semilattice with the topology $\lambda(L)$.

Definition 1. *Let S be a join semilattice such that S^{op} is a continuous semilattice, L a compact Hausdorff Lawson lattice. An* L-valued capacity *(or* L-capacity *for brevity) on S is a mapping $c : S \to L$ which preserves infima of filtered sets (and hence is isotone).*

Observe that this is equivalent to being an element of $[S^{op} \to L^{op}]$. We denote the set of all L-capacities on S by $\underline{M}_{[L]}S$ and consider it with the natural order, thus $\underline{M}_{[L]}S = [S^{op}, L^{op}]^{op}$.

Definition 2. *A capacity $c : S \to L$ is* normalized *if S has a unit and c takes it to the unit of L.*

The least normalized capacity $c_0 : S \to L$ is the following one:

$$c_0(s) = \begin{cases} 1, s = 1, \\ 0, s \neq 1, \end{cases} \qquad s \in S.$$

A capacity $c \in \underline{M}_{[L]}S$ is normalized iff $c_0 \leqslant c$, therefore the set $M_{[L]}S$ of all normalized L-capacities on S is equal to $\{c_0\}\uparrow$ in $\underline{M}_{[L]}S$.

Then the following statement is immediate:

Proposition 1. *The posets $\underline{M}_{[L]}S$ and $M_{[L]}S$ are compact Hausdorff Lawson lattices with the respective Lawson topologies.*

A *filter* in a poset is a filtered upper set. For a poset X, by $X^{\triangle}$ we denote by $X^{\triangle}$ denote the *Lawson dual* of X which consists of the non-empty Scott open filters in X ordered by inclusion.

A category which consists of all continuous semilattices with units (top elements) and semilattice morphisms that preserve units and directed suprema (i.e. are Scott continuous) is denoted by $\mathcal{CS}$em. Duality Theorem on Continuous Semilattices [1, Theorem IV-2.16] asserts that, for a continuous semilattice S with a unit, the poset $S^{\triangle}$ is a continuous semilattice with a unit as well, and, for a morphism $f : S \to S'$ in $\mathcal{CS}$em, the formula $f^{\triangle}(F) = f^{-1}(F)$, $F \in S'^{\triangle}$, defines a morphism $f^{\triangle} : S'^{\triangle} \to S^{\triangle}$ in $\mathcal{CS}$em. Thus a functor $(-)^{\triangle} : \mathcal{CS}\mathrm{em} \to \mathcal{CS}\mathrm{em}^{op}$ is obtained, and $(-)^{\triangle} \circ (-)^{\triangle}$ is isomorphic to the identity functor $\mathbf{1}_{\mathcal{CS}\mathrm{em}}$. An isomorphism $\mathcal{U} : \mathbf{1}_{\mathcal{CS}\mathrm{em}} \to (-)^{\triangle} \circ (-)^{\triangle}$ consists of all mappings $\mathcal{U}_S : S \to S^{\triangle\triangle}$ that send each $s \in S$ to the set $\{F \in S^{\triangle} \mid s \in F\}$. Hence by the latter theorem the category $\mathcal{CS}$em is *self-dual* under the contravariant functor $(-)^{\triangle}$. It is obvious that $\max S^{\triangle} = S$.

Let $\mathcal{CS}\text{em}_0$ be the category that consists of all continuous semilattices with *bottom* elements, and all Scott continuous semilattice morphisms which preserve the bottom elements. For each object S of $\mathcal{CS}\text{em}_0$, the poset $S^\top$ is an object of $\mathcal{CS}\text{em}$ such that its top element is isolated from below. For a morphism $f : S \to S'$ in $\mathcal{CS}\text{em}_0$, let $f^\top : S^\top \to S'^\top$ take each $s \in S$ to $f(s) \in S'$, and $\top$ to $\top$. Note that $f^\top$ is a morphism in $\mathcal{CS}\text{em}$, and $(-)^\top : \mathcal{CS}\text{em}_0 \to \mathcal{CS}\text{em}$ is a functor which is an embedding of categories.

Then (cf. Exercise IV.2-21 [1]) the Lawson dual $(S^\top)^\triangle$ is a continuous semilattice with the top element $S^\top$ isolated from below, and with the bottom element $\{\top\}$. Hence a poset

$$S^\wedge = (S^\top)^\triangle \setminus \{S^\top\}.$$

is an object of $\mathcal{CS}\text{em}_0$ as well. This assignment extends to a contravariant functor $(-)^\wedge : \mathcal{CS}\text{em}_0 \to \mathcal{CS}\text{em}_0$ as follows: Each $F \in (S'^\top)^\triangle$, $F \neq S'^\top$, does not contain a bottom element $0' \in S'$, therefore the mapping $(f^\top)^\triangle : (S'^\top)^\triangle \to (S^\top)^\triangle$ takes such F to the open filter $(f^\top)^{-1}(F)$ which does not contain a bottom element $0 \in S$, hence $(f^\top)^\triangle(F) \neq S^\top$. On the other hand, $(f^\top)^\triangle(S'^\top) = S^\top$. Thus we define the mapping $f^\wedge : S'^\wedge \to S^\wedge$ as a restriction of $(f^\top)^\triangle$. By the above the assignment $s \mapsto \{F \in S^\wedge \mid s \in F\}$ is an isomorphism $u_S : S \to S^{\wedge\wedge}$ which is a component of a natural transformation $u : \mathbf{1}_{\mathcal{CS}\text{em}_0} \to (-)^{\wedge\wedge}$. Thus $\mathcal{CS}\text{em}_0$ is self-dual under the contravariant functor $(-)^\wedge$. By the above we consider this self-duality as a restriction of the self-duality for $\mathcal{CS}\text{em}$ via $(-)^\triangle$ to the subcategory $\mathcal{CS}\text{em}_0 \to \mathcal{CS}\text{em}$.

Definition 3. *A binary relation $P \subset S \times S'$ is called a* separating polarity *if:*

(1) for all $x_1, x_2 \in S$, $x_1 \not\leq x_2$ iff there is $y \in S'$ such that $\neg x_1 P y$, $x_2 P y$;

(1') for all $y_1, y_2 \in S'$, $y_1 \not\leq y_2$ iff there is $x \in S$ such that $\neg x P y_1$, $x P y_2$;

(2) for all $x_1, x_2 \in S$ and $y \in S'$, $(x_1 \wedge x_2) P y$ iff $x_1 P y$ or $x_2 P y$;

(2') for all $x \in S$ and $y_1, y_2 \in S'$, $x P (y_1 \wedge y_2)$ iff $x P y_1$ or $x P y_2$;

(3) for all $x \in S$ and $y \in S'$, the sets $xP \subset S'$ and $Py \subset S$ are non-empty and closed under directed suprema.

It is easy to see that an equivalent definition can be given using characteristic functions.

Definition 4. *A binary relation $P \subset S \times S'$ is called a* separating polarity *if the characteristic mapping $\bar{P} : S \times S' \to \{0, 1\}$ of its complement satisfies the following:*

(1) $\bar{P}$ is distributive w.r.t. $\wedge$ in the both variables, and $\bar{P}(0, y) = \bar{P}(x, 0') = 0$ for all $x \in S$, $y \in S'$;

(2) $\bar{P}$ separates elements of S and of S', i.e.:

(2a) for each $x_1, x_2 \in S$, if $\bar{P}(x_1, y) = \bar{P}(x_2, y)$ for all $y \in S'$, then $x_1 = x_2$;

(2b) for each $y_1, y_2 \in S'$, if $\bar{P}(x, y_1) = \bar{P}(x, y_2)$ for all $x \in S$, then $y_1 = y_2$;

(3) $\bar{P}$ is Scott continuous.

Proposition 2. *Let S, S' be continuous meet semilattices with bottom elements $0, 0'$ resp. If $P \subset S \times S'$ is a separating polarity, then the mapping i that takes each $x \in S$ to $x\bar{P} \cup \{\top\}$ is an isomorphism $S \to S'^\wedge$. Conversely, each isomorphism $i : S \to S'^\wedge$ is determined by the above formula for a unique separating polarity $P \subset S \times S'$.*

Proof. ($\Longrightarrow$) Recall that by Lemma II.2-8 [1] the joint Scott continuity of $\bar{P}$ is equivalent to its Scott continuity in each variable separately. Due to (1),(3) $x\bar{P}$ and $\bar{P}y$ are Scott open filters (although not necessarily non-empty), which are distinct from S' and S resp. Then $i(x)=x\bar{P}\cup\{\top\}\in S'^{\wedge}$ for all $x\in S$. We similarly define i' as follows: $y\mapsto \bar{P}y\cup\{\top\}$ for all $y\in S'$. Then i and i' are meet-preserving injective mappings $S\to S'^{\wedge}$ and $S'\to S^{\wedge}$, respectively. From (1),(3) we also infer that i preserves directed suprema and a bottom element, hence is Scott continuous, and therefore is a morphism in the category $\mathcal{CS}\mathrm{em}_0$. We can apply to i the contravariant functor $(-)^{\wedge}$. The mapping $i^{\wedge}:S'^{\wedge\wedge}\to S^{\wedge}$ takes each non-empty Scott open filter $\mathrm{F}\subsetneq (S'^{\top})^{\triangle}$ to $i^{-1}(\mathrm{F})\cup\{\top\}$. Then $i^{\wedge}\circ u_{S'}:S'\to S^{\wedge}$ sends all y to

$$\{x\in S^{\top}\mid x=\top \text{ or } i(x)\ni y\}=\{x\in S^{\top}\mid x=\top \text{ or } y\in x\bar{P}\}=\bar{P}y\cup\{\top\}=i'(y).$$

Since $u_{S'}$ is an isomorphism and i' is injective by (2), the mapping $i^{\wedge}$ is injective as well, hence i is surjective. Taking into account that i is meet-preserving, we arrive at conclusion that i is an order isomorphism.

Observe that similarly $i'^{\wedge}\circ u_S:S\to S'^{\wedge}$ coincides with i, hence i' is an isomorphism as well.

($\Longleftarrow$) Let $i:S\to S'^{\wedge}$ be an isomorphism. It is straightforward to verify that the relation $P=\{(x,y)\in S\times S'\mid y\notin i(x)\}$ satisfies the properties (1)–(3) and determines i in the above manner.

By the above in the sequel, if such S' and P exist, we may assume that S' is equal to $S^{\wedge}$, and $P=\{(s,\mathrm{F})\in S\times S^{\wedge}\mid s\notin \mathrm{F}\}$. It is important that, for $S=\exp_{\supset}X$, we may put $S'=S$. A required $P\subset \exp_{\supset}X\times\exp_{\supset}X$ is the following: $(F,G)\in P$ iff $F\cap G\neq\varnothing$.

We are interested in cases when both S and $S^{\wedge}$ are compact Hausdorff w.r.t. the Lawson topologies. A continuous semilattice S is called *stably continuous* if $x\ll y,z$ implies $x\ll y\wedge z$ for any $x,y,z\in S$. By Exercise IV-2.23 [1] a continuous semilattice L is a continuous lattice if and only if $L^{\triangle}$ is stably continuous with a top element isolated from below. Therefore:

Lemma 1. *For a continuous semilattice S with a bottom element, the poset $(S^{\top})^{\triangle}$ is a continuous lattice (i.e. $S^{\wedge}$ is a complete semilattice) iff S is stably continuous.*

Corollary 1. *For a continuous semilattice S with a bottom element, the Lawson topologies on S and $S^{\wedge}$ are compact and Hausdorff if and only if S is complete and stably continuous.*

Hence the full subcategory $\mathcal{CSCS}\mathrm{em}_0$ of $\mathcal{CS}\mathrm{em}_0$ with complete stably continuous semilattices with 0 as objects is self-dual under a restriction of the functor $(-)^{\wedge}$.

Note that, for a compactum X, the semilattice $\exp_{\supset}X$ is complete, stably continuous, and contains a bottom element.

Lemma II.2-9 [1] implies that, for a morphism $f:S_1\to S_2$ in $\mathcal{CS}\mathrm{em}_0$, a mapping $[f\to L]:[S_2\to L]\to[S_1\to L]$, $[f\to L](c)=c\circ f$ for $c\in[S_2,L]$, is a Scott continuous lattice morphism. Thus a contravariant functor $[-\to L]$ from $\mathcal{CS}\mathrm{em}_0$ to the category $\mathcal{S}\mathrm{up}$ of complete lattices and mappings that preserve all suprema is obtained. For each object S of $\mathcal{CS}\mathrm{em}_0$, the subset $[S\to L]_0=\{c\in[S\to L]\mid c(0)=0\}$ is closed under arbitrary meets and joins, hence is an object of $\mathcal{S}\mathrm{up}$ as well. The inclusion $[S\to L]_0\hookrightarrow[S\to L]$ is a morphism in $\mathcal{S}\mathrm{up}$, and $[f\to L]\big([S_2\to L]_0\big)\subset[S_1\to L]$ for $f:S_1\to S_2$ above. Therefore we define a mapping $[f\to L]_0:[S_2\to L]_0\to[S_1\to L]_0$ as a restriction of $[f\to L]$ and obtain a contravariant functor $[-\to L]_0:\mathcal{CS}\mathrm{em}_0\to\mathcal{S}\mathrm{up}$. Recall that

$[S \to L]$ is identified with the opposite to the poset $\underline{M}_{[L^{op}]}S^{op}$ of L^{op}-capacities on S^{op}. It is easy to see that $[S \to L]_0$ is opposite to the poset $M_{[L^{op}]}S^{op}$ of *normalized* L^{op}-capacities on S^{op}.

Let continuous semilattices with bottom elements S, S' and a separating polarity $P \subset S \times S'$ be fixed. For a Scott continuous function $c : S \to L$ (i.e. for an L^{op}-capacity $c : S^{op} \to L^{op}$), we define a function $\tilde{c} : S' \to \tilde{L}$ by the equality

$$\tilde{c}(s') = \inf\{c(s) \mid s \in S, (s, s') \notin P\} \text{ in } L,\ s' \in S'.$$

For a particular "canonical" case $S' = S^\wedge$, $P = \{(s, \mathrm{F}) \in S \times S^\wedge \mid s \in \mathrm{F}\}$, the function $\tilde{c} \in [S' \to \tilde{L})$ is of the form

$$\tilde{c}(\mathrm{F}) = \inf\{c(s) \mid s \in F, s \neq \top\} \text{ in } L,\ F \in S^\wedge.$$

Proposition 3. *The function $\tilde{c}$ is Scott continuous, takes a bottom element to a bottom element, and its epigraph is equal to*

$$\widetilde{\operatorname{epi}}\,\tilde{c} = \{(s', \alpha') \in S' \times \tilde{L} \mid (s, s') \notin P \text{ or } \alpha' \leqslant \alpha \text{ for all } (s, \alpha) \in \operatorname{epi} c\}.$$

If for S' the same S, P are fixed in the above sense, then

$$\tilde{\tilde{c}}(s) = \begin{cases} c(s), s \neq 0, \\ 0, s = 0, \end{cases} \qquad s \in S.$$

Remark 1. This statement can be equivalently formulated in terms of capacities: under the conditions of the latter proposition, if $c : S^{op} \to L$ is an L-capacity, then the function $\tilde{c} : S'^{op} \to \tilde{L}$ defined as

$$\tilde{c}(s') = \sup\{c(s) \mid s \in S, (s, s') \in P\} \text{ in } L,\ s' \in S',$$

is a normalized $\tilde{L}$-capacity with a subgraph determined by the formula

$$\widetilde{\operatorname{sub}}\,\tilde{c} = \{(s', \alpha') \in S' \times \tilde{L} \mid (s, s') \notin P \text{ or } \alpha \leqslant \alpha' \text{ for all } (s, \alpha) \in \operatorname{sub} c\},$$

and $\tilde{\tilde{c}} = c \vee c_0$, where c_0 is the least normalized capacity taking $0 \in S$ (i.e. a top element of S^{op}) to $1 \in L$ and all other elements of S to 0.

To prove the latter statement and to clarify the nature of the relationship between c and $\tilde{c}$, we need a notion of Galois connection.

Definition 5. [1] *If S, S' are posets and $p : S \to S'$ and $q : S' \to S$ are functions such that for all $s \in S$ and $s' \in S'$*

$$s \leqslant_S q(s') \text{ iff } p(s) \leqslant_{S'} s',$$

then p is called a lower adjoint *to q, q is an* upper adjoint *to p, and the quadruple (S, p, q, S') is called a* Galois connection *(or a* Galois correspondence*).*

Clearly such p, q are isotone mappings, and each mapping of the adjoint pair (p, q) is uniquely determined by the other one.

If $p_1 : S \to S'^{op}$ and $p_2 : S'^{op} \to S$ are such that (S, p_1, p_2, S'^{op}) is a Galois connection, then the quadruple (S, p_1, p_2, S') is called a *contravariant Galois connection*. An equivalent definition is the following:

Definition 6. *If S, S' are posets and $p : S \to S'$ and $q : S' \to S$ are functions such that for all $s \in S$ and $s' \in S'$*

$$s \leqslant_S q(s') \text{ iff } s' \leqslant_{S'} q(s),$$

then the quadruple (S, p, q, S') is called a contravariant Galois connection.

Such p, q are antitone, and the latter definition is symmetric, i.e. (S', q, p, S) is a contravariant Galois connection as well.

For a fixed separating polarity $P \subset S \times S'$ and a mapping $f \in [S \to L]$, we define a mapping $p(f) : S' \to \tilde{L}$ by the formula

$$p(f)(s') = \widetilde{\sup}\{f(s) \tilde{\wedge} P(s, s') \mid s \in S\}, s' \in S'.$$

For $f' \in [S' \to \tilde{L}]$, a mapping $q(f') : S \to L$ is defined similarly:

$$q(f')(s) = \sup\{f'(s') \wedge \bar{P}(s, s') \mid s \in S\}, s \in S.$$

For all $s \in S$, the function $S' \to \tilde{L}$ that takes each s' to $f(s) \tilde{\wedge} P(s, s')$ is Scott continuous, therefore the pointwise supremum of all such functions is Scott continuous as well. It is obvious that $p(f)(0) = 1 = \tilde{0}$, hence $p(f) \in [S' \to \tilde{L}]_0$, and similarly $q(f')$ is in $[S \to L]_0$. Observe that, for all $f \in [S \to L]$ and $f' \in [S' \to \tilde{L}]$, the both inequalities $p(f) \tilde{\leqslant} f'$ and $q(f') \leqslant f$ are equivalent to

$$f(s) \geqslant f'(s') \text{ for all } s \in S, s' \in S' \text{ such that } s\bar{P}s'.$$

Thus $([S \to L]^{op}, p, q, [S' \to \tilde{L}]^{op})$ is a contravariant Galois connection. Separation of points of S and S' by $\bar{P}$ implies that the restriction $p_0 : [S \to L]_0 \to [S' \to \tilde{L}]_0$ of p is injective, as well as the restriction $q_0 : [S' \to \tilde{L}]_0 \to [S \to L]_0$ of q, therefore p_0 and q_0 are mutually inverse order antiisomorphisms.

Now by observing that $p(f) = \tilde{f}$, $q(f') = \tilde{f}'$, the Proposition 3 is at hand. If $S' = S^\wedge$, and $P = \{(s, \mathrm{F}) \in S \times S^\wedge \mid s \notin \mathrm{F}\}$, then the constructed antiisomorphism $p_0 : [S \to L]_0 \to [S^\wedge \to \tilde{L}]_0$ is denoted by $\varkappa_{[L]}S$.

If $S = S' = \exp_\supset X$, $P = \{(F, G) \in (\exp_\supset X)^2 \mid F \cap G = \varnothing\}$, $c \in \underline{M}_{[L]}S$, then

$$\tilde{c}(F) = \sup\{c(G) \mid F \cap G = \varnothing\}, F \in \exp_\supset X,$$

which is precisely a definition of a *conjugate $\tilde{L}$-capacity* to an L-capacity c on a compactum X, cf. [4]. Hence we also call the just defined $\tilde{c} : S' \to \tilde{L}$ a *conjugate mapping* to $c : S \to L$, and $\tilde{c} : S'^{op} \to \tilde{L}$ is called a *conjugate capacity* to $c : S^{op} \to L$.

From the above we obtain also:

Proposition 4. *For a compact Hausdorff Lawson lattice L and a continuous semilattice S with a bottom element, the posets $[S \to L]_0$ and $[S^\wedge \to \tilde{L}]_0$ are order antiisomorphic.*

The posets $M_{[L]}S^{op}$ and $M_{[\tilde{L}]}(S^{\wedge op})$ are also order antiisomorphic.

Let $P_1 \subset S_1 \times S_1^\wedge$, $P_2 \subset S_2 \times S_2^\wedge$ be "standard" separating polarities, i.e.
$P_i = \{(s, \mathrm{F}) \in S_1 \times S_i^\wedge \mid s \notin \mathrm{F}\}$, $i = 1, 2$, and contravariant Galois connections
$([S_i \to L]_0^{op}, p_i, q_i, [S_i^\wedge \to \tilde{L}]_0^{op})$, $i = 1, 2$, are as described above.

Consider the following diagram:

$$\begin{array}{ccc} [S_1 \to L]_0 & \underset{q_1}{\overset{p_1}{\rightleftarrows}} & [S_1^\wedge \to \tilde{L}]_0 \\ {\scriptstyle [f\to L]_0}\uparrow & & \downarrow{\scriptstyle [f^\wedge \to \tilde{L}]_0} \\ [S_2 \to L]_0 & \underset{q_2}{\overset{p_2}{\rightleftarrows}} & [S_2^\wedge \to \tilde{L}]_0 \end{array}$$

Lemma 2. *The quadruple* $([S_1 \to L]_0^{op}, [f^\wedge \to \tilde{L}]_0 \circ p_1, [f \to L]_0 \circ q_2, [S_2^\wedge \to \tilde{L}]_0^{op})$ *is a contravariant Galois connection.*

Proof. Observe that, for all $\varphi \in [S_1 \to L]_0$, $\psi \in [S_2^\wedge \to \tilde{L}]_0$, the inequality $\tilde{\varphi} \circ f^\wedge \mathrel{\tilde{\leqslant}} \psi$ is equivalent to

$$\psi(y) \leqslant \varphi(x) \text{ for all } x \in S_1, y \in S_2^\wedge \text{ such that } (x, f^\wedge(y)) \notin P_1,$$

and the inequality $\tilde{\psi} \circ f \leqslant \varphi$ is equivalent to

$$\psi(y) \leqslant \varphi(x) \text{ for all } x \in S_1, y \in S_2^\wedge \text{ such that } (f(x), y) \notin P_2.$$

Since $(x, f^\wedge(y)) \notin P_1$ if and only if $(f(x), y) \notin P_2$, we arrive at the required conclusion.

Corollary 2. *The mapping* $q_2 \circ [f^\wedge \to \tilde{L}]_0 \circ p_1 : [S_1 \to L]_0 \to [S_2 \to L]_0$ *is an upper adjoint to the mapping* $[f \to L]_0 : [S_2 \to L]_0 \to [S_1 \to L]_0$.

For a mapping of complete lattices $\varphi : L \to L'$, we denote by φ^* its *lower inverse*, i.e. the mapping $L' \to L$ that is defined by the equality

$$\varphi^*(y) = \sup\{x \in L \mid \varphi(x) \leqslant y\},\ y \in L'.$$

Recall that by the adjoint theorem for order structures [2] any mapping of complete lattices $\varphi : L \to L'$ which preserves arbitrary suprema is a lower adjoint of a unique Galois connection between these lattices, and the upper adjoint is equal to φ^*.

Theorem IV.1-3 [1] implies that the category $\mathcal{S}$up of complete lattices and mappings that preserve all suprema is self-dual under the contravariant functor $\tilde{D}$ that sends each lattice L to L^{op} and each morphism $f : L_1 \to L_2$ to $(f^*)^{op} : L_2^{op} \to L_1^{op}$.

The diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{up} & \xrightarrow{\tilde{D}} & \mathcal{S}\mathrm{up} \\ {\scriptstyle [-\to L]_0}\uparrow & & \uparrow{\scriptstyle [-\to \tilde{L}]_0} \\ \mathcal{CS}\mathrm{em}_0 & \xrightarrow[(-)^\wedge]{} & \mathcal{CS}\mathrm{em}_0 \end{array}$$

generally is not commutative, but the two pairwise compositions are isomorphic [3] (observe that all functors are contravariant).

Putting the previous propositions together, we obtain the main result of this work.

Theorem 1. *The collection $\varkappa_{[L]}$ of all isomorphisms $\varkappa_{[L]}S : [S \to L]_0^{op} \to [S^\wedge \to \tilde{L}]_0$, for all objects of $\mathcal{CS}\mathrm{em}_0$, is an isomorphism of functors $\tilde{D} \circ [- \to L]_0 \to [- \to \tilde{L}]_0 \circ (-)^\wedge$.*

Thus the constructions of lattices of normalized L-capacities and of normalized $\tilde{L}$-capacities on continuous semigroups with zero are functorial and linked via two classic dualities and a functor isomorphism. The components of the latter send each capacity to its conjugate.

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СПРЯЖЕНІ МІРИ НА НАПІВҐРАТКАХ

Олег НИКИФОРЧИН, Оксана МИКИЦЕЙ

Прикарпатський національний університет імені Василя Стефаника,
76025 Івано-Франківськ, вул. Шевченка, 57
e-mail: oleh.nyk@gmail.com

Як узагальнення спряжених ґраткозначних ємностей на компакті, запроваджено відношення спряженості неперервних за Скоттом відображень з неперервних напівґраток з нулем у повні неперервні ґратки.

Ключові слова: спряжена ємність, неперервна напівґратка, двоїстість категорій.

СОПРЯЖЕННЫЕ МЕРЫ НА ПОЛУРЕШЁТКАХ

Олег НИКИФОРЧИН, Оксана МИКИЦЕЙ

Прикарпатский национальный университет имени Василия Стефаника,
76025 Ивано-Франковск, ул. Шевченко, 57
e-mail: oleh.nyk@gmail.com

Как обобщение сопряженных решёткозначных ёмкостей на компакте, определено отношение сопряженности непрерывных за Скоттом отображений с непрерывных полурешёток с нулём в полные непрерывные решётки.

Ключевые слова: сопряженная ёмкость, непрерывные полурешётки, двоичность категорий.

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