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EILENBERG-MACLANE SPACES OF COMPACT CONVEX BODIES OF CONSTANT WIDTH

Lidiya BAZYLEVYCH

L'viv Polytechnic National University,
79013 L'viv, St. Bandera Str., 12
e-mail: izar@litech.lviv.ua

The Lie group S^1 acts on the hyperspace $\mathrm{cw}(I^2)$ of convex bodies of constant width inscribed into the unit square I^2 by rotation. It is proved that the orbit space $\mathrm{cw}(I^2)/S^1$ of this action contains a Q-manifold which is a $K(\mathbb{Z},2)$-space.

Key words: max-min convex set, hyperspace, Hilbert cube.

1. Introduction. Let X be a compact convex body in the euclidean space $\mathbb{R}^n$ (to avoid trivialities, we always assume that $n \geq 2$). By S^{n-1} we denote the unit sphere in $\mathbb{R}^n$. For any $u \in S^{n-1}$, let $h_X(u) = \max\{\langle x, u\rangle \mid x \in X\}$. We call the function $h_X \colon S^{n-1} \to \mathbb{R}$ the *support function* of X. A compact convex body X is said to be of *constant width* $d > 0$ if $h_X(u) - h_X(-u) = d$, for every $u \in S^{n-1}$.

Let $I = [0,1]$. By $\mathrm{cw}(I^2)$ of convex bodies of constant width inscribed into the unit square I^2. We endow $\mathrm{cw}(I^2)$ with the Hausdorff metric d_H:

$$d_H(A,B) = \inf\{r > 0 \quad | A \subset O_r(B), \quad B \subset O_r(A)\},$$

where $O_t(C)$ stands for the t-neighborhood of C in $\mathbb{R}^2$ (actually, this formula also works for the hyperspace $\exp \mathbb{R}^2$ of nonempty compact subsets in $\mathbb{R}^2$).

The Minkowski combination in $\mathrm{cw}(I^2)$ is defined as follows:

$$tA + (1-t)B = \{ta + (1-t)b \quad | \quad a \in A, \quad b \in B\}, \quad A, B \in \mathrm{cw}(I^2), \quad t \in I.$$

With respect to this combination, the space $\mathrm{cw}(I^2)$ is a convex subset of the space $\mathrm{cw}(\mathbb{R}^n)$. Since the space $\mathrm{cw}(I^2)$ is infinite-dimensional, it follows from Keller's theorem [5] that $\mathrm{cw}(I^2)$ is homeomorphic to the Hilbert cube.

The Lie group S^1 acts on the hyperspace $\mathrm{cw}(I^2)$ by rotations. The simplest way to describe this action is to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Then for every $A \in \mathrm{cw}(I^2)$ and every $\alpha \in S^1$ there exists $x_A(\alpha) \in \mathbb{C}$ such that the set $\alpha A + x_A(\alpha)$ is inscribed in I^2, i.e. belongs to the space $\mathrm{cw}(I^2)$. We define $\alpha \cdot A = A + x_A(\alpha)$. The circle B of radius $1/2$ centered at $(1/2, 1/2)$ is the unique fixed point of this action.

The natural action of the Lie group S^1 on the hyperspace $\exp S^1$ is investigated in [7]. It is proved therein that the orbit space of this action contains Q-manifolds which are Eilenberg-MacLane spaces $K(G,2)$, for some groups G. Recall that X is a space $K(\mathbb{Z},2)$, if $\pi_2(X)=G$ and $\pi_i(X)=0$ for $i\neq 2$.

The aim of this note is to prove that the orbit space $\mathrm{cw}(I^2)/S^1$ contains a Q-manifold which is an Eilenberg-MacLane space $K(\mathbb{Z},2)$. Moreover, this Q-manifold is homeomorphic to its product by $[0,1)$. We therefore can conclude that this Q-manifold is homeomorphic to $T(\mathbb{C}P^\infty)\times Q\times[0,1)$, where T stands for the infinite telescope construction.

2. Notation and auxiliary results. If a group G acts on a set X, for any $x\in X$, we denote by x_G the stabilizer of x, i.e. $x_G=\{g\in G\mid gx=x\}$.

We denote by $q\colon \mathrm{cw}(I^2)\to \mathrm{cw}(I^2)/S^1$ the quotient map.

For every finite subgroup G of S^1, we let $Z(G)=\{A\in \mathrm{cw}(I^2)\mid S^1_A\subset G\}$.

We will need the following construction due to Sallee [11]. Let K be a convex body of constant width h in $\mathbb{R}^2$ and let $p\in\mathbb{R}^2\setminus K$ be a point sufficiently close to K. Let $q,r\in\partial K$ be points of the intersection of ∂K with the circumference of radius h centered at p. Let $q',r'\in\partial K$ be the points opposite to q and r respectively. Following [11] we call the Λ-modification of K the convex set bounded by the union of the following pieces: 1) the smaller arc of the circumference connecting q and r of radius h centered at p; 2) the part of ∂K that connects q and r'; 3) the part of ∂K that connects r and q'; 4) the arc of the circumference centered at q of radius h that connects q' and p; 5) the arc of the circumference centered at r of radius h that connects r' and p.

Note that this construction works only in the case when p is close enough to ∂K; in the sequel we speak about the Λ-modification only in the situation where it is well-defined, i.e. leads to a body of constant width. We denote this modification by $\Lambda(K,p)$. In [11], it is remarked that $\Lambda(K,p)$ is also a convex set of constant width h. Note that the boundary of $\Lambda(K,p)$ is not a smooth curve.

In the sequel, we apply the Λ-modification to the sets from $\mathrm{cw}(I^2)$; the result, $\Lambda(K,p)$, is a convex set of constant width 1 but does not belong to $\mathrm{cw}(I^2)$. We denote by $\Lambda'(K,p)$ the (unique) shifted copy of $\Lambda(K,p)$ that belongs to $\mathrm{cw}(I^2)$. We will call it the Λ'-modification.

Proposition 1. *The Λ'-modification map $(K,p)\mapsto\Lambda'(K,p)$ is continuous as a map of two variables.*

By an AR-space (ANR-space) we mean an absolute (neighborhood) retract for the class of metric spaces.

Recall that a Q-manifold is a separable metrizable space which is locally homeomorphic to the Hilbert cube.

A metric space (X,d) is said to satisfy the Disjoint Approximate Property if, for every $\alpha\colon X\to(0,\infty)$ there exist maps $f,g\colon X\to X$ such that $d(f(x),g(x))<\alpha(f(x))$, for every $x\in X$, and $f(X)\cap g(X)=\emptyset$.

The following characterization theorem for Q-manifolds is proved by H. Toruńczyk [6].

Theorem 1 (Characterization theorem for Q-manifolds)**.** *A separable locally compact metrizable ANR-space is a Q-manifold if it satisfies the Disjoint Approximation Property.*

Recall that a map is called *proper* if the preimage of every compact subset is compact.

An important consequence of the Slice theorem is that if X is a G-space with the orbits all of the same type, then the orbit map $X \to X/G$ is a locally trivial fibration [12, Chapter II, Theorem 5.8].

3. Main result.

Lemma 1. *There exists an equivariant deformation retraction (f_t) of the hyperspace $\mathrm{cw}(I^2)$ to the set $\{B\}$ such that $f_t(Z(S^1)) \subset Z(S^1)$.*

Proof. Let $h\colon [0,1] \to [1,\infty]$ be any nondecreasing homeomorphism (we endow $[1,\infty]$ with the order topology). For any $\tau \in [1,\infty]$ and any $A \in \mathrm{cw}(I^2)$, the closed τ-neighborhood $\bar{O}_\tau(A)$ is a convex body of constant width and there is a unique homothetic copy of it (we denote it by $\varphi(\bar{O}_\tau(A))$) which belongs to $\mathrm{cw}(I^2)$. We finally let $f_t(A) = \varphi(\bar{O}_{h(t)}(A))$. It is easy to to deduce from known geometric properties of bodies of constant width that (f_t) is a required retraction.

Theorem 2. *The space $M(S^1) = Z(S^1)/\alpha$ is a Q-manifold which is $K(\mathbb{Z}, 2)$.*

Proof. First, note that M is an open subset in $\mathrm{cw}(I^2)$; this follows immediately from the Slice Lemma. We are going to prove that the space M satisfies the Disjoint Approximate Property. By a consequence of the Slice Theorem, the quotient map $q\colon Z(S^1) \to M$ is a fibration. Given $x \in M$, find a neighborhood U of x such that q is trivial over U. Let $\alpha\colon U \to (0,\infty)$ be a continuous function. We will assume that $B_{\alpha(y)}(y) \subset U$, for every $y \in U$.

There exists a section $s\colon U \to q^{-1}(U)$ of the fibration $q|q^{-1}(U)$. There exists a continuous function $\beta\colon U \to (0,\infty)$ such that $d_H(f_{\beta(q(A))}(A), A) < \alpha(q(A))/2$, for every $A \in q^{-1}(U)$.

There exists a small enough continuous function $\gamma\colon U \to (0,\infty)$ such that, for every $y \in U$, we have $d_H(\Lambda'(f_{\beta(q(s(y)))}(s(y)), p(y)), s(y)) < \alpha(y)$, where $p(y)$ is the point defined as follows. Let $r(y) = (r_1(y), r_2(y))$ denote the (unique) point of intersection of $s(y)$ and the upper side of the square I^2; then $p(y) = (r_1(y), r_2(y) + \gamma(y))$. Note that $\Lambda'(f_{\beta(q(s(y)))}(s(y)), p(y)) \in Z(S^1)$, because the boundary of $\Lambda'(f_{\beta(q(s(y)))}(s(y)), p(y))$ has exactly three non-differentiability points. If γ is small enough, then also two of these points are close enough so that three of them cannot form the vertices of an equilateral triangle.

Now, define the maps $F, G\colon U \to U$ by the following formulas:

$$F(x) = f_{\beta(y)}(s(y)),\ \ G(y) = \Lambda'(f_{\beta(q(s(y)))}(s(y)), p(y)).$$

Since $d(F(y), y) < \alpha(y)$, $d(G(y), y) < \alpha(y)$ and $F(U) \cap G(U) = \emptyset$, we see that U satisfies the Discrete Approximation Property.

We are going to show that the orbit space is an ANR. This follows from the fact that the action of S^1 on $\mathrm{cw}(I^2)$ is linear and results of [].

We conclude that U is a Q-manifold. Thus, M is also a Q-manifold.

In order to prove that M is homeomorphic to $M \times [0,1)$ it is sufficient to prove that there exists a proper homotopy $H\colon M \times [0,1) \to M$ such that $H(x,0) = x$, for every $x \in M$ (see [14]). This homotopy is defined by the formula $H(x,t) = q(f_t s(x))$, $x \in M$.

It follows from the homotopy exact sequence of the bundle $Z \to M$ (with fiber S^1) and from contractibility of Z that M is a $K(\mathbb{Z}, 2)$-space. Consider the space

$$T(\mathbb{C}P^\infty) = \bigcup_{i=1}^{\infty} \mathbb{C}P^i \times [i, \infty) \subset \mathbb{C}P^\infty \times [1, \infty).$$

This is a locally compact $K(\mathbb{Z}, 2)$-space and therefore $T(\mathbb{C}P^\infty) \times Q \times [0,1)$ is a $[0,1)$-stable Q-manifold. Since the $[0,1)$-stable Q-manifolds are classified by their homotopy types, we conclude that $Z(S^1) \simeq T(\mathbb{C}P^\infty) \times Q \times [0,1)$.

4. Remarks. One can ask whether there are counterparts of the main result of this note for the hyperspace of rotors. Recall that a *rotor* in a regular polygon P is a convex set X which admits a complete rotation in P such that X touches every side of P in the process of rotation. It is proved in [4] that the hyperspace $\mathcal{R}(P_n)$ of the rotors inscribed in the regular n-gon is homeomorphic to the Hilbert cube.

Let (X, Y) be a pair of compact convex subsets in $\mathbb{R}^n$, $n \geq 2$. Following [13] we say that (X, Y) is of constant relative width $r > 0$ if

$$w_{(X,Y)}(u) = h_X(u) + h_Y(-u) = r \quad \text{for every} \quad u \in S^{n-1}.$$

Let Oxy be the standard rectangular coordinate system in $\mathbb{R}^2$. Denote by $\mathrm{crw}_r(\mathbb{R}^2)$ the set of all pairs (X, Y) of compact convex subsets in $\mathbb{R}^2$ of constant relative width r such that X lies in the first quadrant of the system and touches both the x-axis and y-axis. The topology on the set $\mathrm{crw}_r(\mathbb{R}^2)$ is inherited from the product $\exp \mathbb{R}^2 \times \exp \mathbb{R}^2$, where the Hausdorff metric is considered on each factor. The Lie group S^1 acts on $\mathrm{crw}_r(\mathbb{R}^2)$ by rotations. Are there counterparts of the main result of this note for the hyperspace $\mathrm{crw}_r(\mathbb{R}^2)$?

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ПРОСТОРИ ЕЙЛЕНБЕРГА-МАКЛЕЙНА КОМПАКТНИХ ОПУКЛИХ ТІЛ СТАЛОЇ ШИРИНИ

Лідія БАЗИЛЕВИЧ

Національний університет "Львівська політехніка",
79013 Львів, вул. Степана Бандери, 12
e-mail: izar@litech.lviv.ua

Група Лі S^1 діє поворотами на гіперпросторі $\mathrm{cw}(I^2)$ опуклих тіл сталої ширини, вписаних в одиничний квадрат I^2. Доведено, що простір орбіт цієї дії містить Q-многовид, який є $K(\mathbb{Z}, 2)$-простором.

Ключові слова: min-max опукла множина, гіперпростір, гільбертів куб.

ПРОСТРАНСТВА ЭЙЛЕНБЕРГА-МАКЛЕЙНА КОМПАКТНЫХ ВЫПУКЛЫХ ТЕЛ ПОСТОЯННОЙ ШИРИНЫ

Лидия БАЗИЛЕВИЧ

Национальный университет "Львовская политехника",
79013 Львов, ул. Степана Бандеры, 12
e-mail: izar@litech.lviv.ua

Группа Ли S^1 действует поворотами на гиперпросторанстве $\mathrm{cw}(I^2)$ выпуклых тел постоянной ширины, вписанных в единичный квадрат I^2. Доказано, що простоpанство орбит этого действия содержит Q-многообразие, являющееся $K(\mathbb{Z}, 2)$-пространством.

Ключевые слова: max-min выпуклое множество, гипперпространство, гильбертов куб.

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