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ENTIRE FUNCTION WITH PRESCRIBED INDICATOR

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For an arbitrary $[\kappa_1,\rho_1]$-trigonometrically convex function an entire function is found having it as the indicator with respect to $r^{l(r)}$, where $l(r)$ is a strong oscillate order, $0<\kappa_1<\kappa\le l(r)\le\rho<\rho_1<\infty$.

Key words: indicator of entire function.

In this communication we present an example which shows nontriviality of the class of entire functions of completely regular growth by Kondratyuk (see [1] - [4]). That adds to Kondratyuk's results [4, P. 84-88, 144-150]. Our construction is new and simpler than latter. The notions used in this work can be found in [4] - [6].

Theorem 1. *Let a function $V(r)=r^{l(r)}$, where a function $l\in C^2[1,\infty)$, $0<\kappa_1<\kappa\le\le l(r)\le\rho<\rho_1<\infty$, $l'(r)r\log r\to 0$ as $r\to\infty$, $l''(r)r^2\log r\to 0$ as $r\to\infty$, a 2π-periodic function h be $[\kappa_1,\rho_1]$-trigonometrically convex, a real number $\alpha>\rho$. Then there exists an entire function f, such that*

$$\log|f(re^{i\varphi})|=h(\varphi)V(r)+O(\log r),\ \ E\not\ni re^{i\varphi}\to\infty,$$

where an exceptional set $E\subset\cup_j\{z:|z-z_j|<r_j\}$, and $\sum_{|z_j|\ge r}r_j=o(r^{\rho-\alpha})$, as $r\to\infty$.

Proof. Let us consider the function $u(re^{i\varphi}):=h(\varphi)V(r)$. Its maximum $B(r,u):=:=\max\{u(re^{i\varphi}):\varphi\in[0,2\pi]\}=CV(r)$ is a convex function of $\log r$ since

$$\frac{d^2V(r)}{(d\log r)^2}=r(rV'(r))'=V(r)((l'(r)r\log(r)+l(r))^2+l''(r)r^2\log r+l'(r)r\log r+2rl'(r))=$$

$$=V(r)(l(r)^2+o(1))>0,\ r>r_0.$$

Next, the function u is subharmonic outside the disk $\{z:|z|>r_0\}$. At first, we prove that in the case $h\in C^2[0,2\pi]$. Then the Laplacian

$$\Delta u=h(\varphi)\frac{1}{r^2}\frac{d^2V(r)}{(d\log r)^2}+\frac{1}{r^2}h''(\varphi)V(r)=\frac{1}{r^2}V(r)((l(r)^2+o(1))h(\varphi)+h''(\varphi))\ge 0,\ r>r_0,$$

because h is an ω-trigonometrically convex function for every $\omega \in [\kappa_1, \rho_1]$, i.e. the inequality $h''(\varphi) + \omega^2 h(\varphi) \geq 0$ holds, and $0 < \kappa_1 < \kappa \leq l(r) \leq \rho < \rho_1$.

Turn to the general case. An arbitrary $[\kappa_1, \rho_1]$-trigonometrically convex function satisfies the condition

$$h(\theta_1) sin(\omega(\theta_2 - \theta_3)) + h(\theta_2) sin(\omega(\theta_3 - \theta_2)) + h(\theta_3) sin(\omega(\theta_1 - \theta_2)) \leq 0$$

for all $\omega \in [\kappa_1, \rho_1]$, $\theta_1 < \theta_2 < \theta_3$, and $\theta_3 - \theta_1 < \pi/\omega$.

Let us consider the smoothed indicator

$$\widetilde{h}(\theta) := \int\limits_{-\infty}^{\infty} h(\theta + x)\delta^{-1} K(x/\delta)\, dx = \int\limits_{-\infty}^{\infty} h(y)\delta^{-1} K\left(\frac{y-\theta}{\delta}\right) dy,$$

where a function $K \in C^\infty(\mathbb{R})$; $K(x) = 0$, when $|x| \geq 1$; $K(x) > 0$, when $|x| < 1$; $\int_{-\infty}^{\infty} K(x)\, dx = 1$. It is well known that $\widetilde{h} \in C^\infty(\mathbb{R})$, and it is easy to see that $\widetilde{h}$ is also $[\kappa_1, \rho_1]$-trigonometrically convex. Indeed,

$$\widetilde{h}(\theta_1) sin(\omega(\theta_2 - \theta_3)) + \widetilde{h}(\theta_2) sin(\omega(\theta_3 - \theta_2)) + \widetilde{h}(\theta_3) sin(\omega(\theta_1 - \theta_2)) =$$

$$= \int\limits_{-\infty}^{\infty} h(\theta_1 + x)\sin(\omega(\theta_2 - \theta_3))\delta^{-1} K(x/\delta)\, dx + \int\limits_{-\infty}^{\infty} h(\theta_2 + x)\sin(\omega(\theta_3 - \theta_1))\delta^{-1} K(x/\delta)\, dx +$$

$$+ \int\limits_{-\infty}^{\infty} h(\theta_3 + x)\sin(\omega(\theta_1 - \theta_2))\delta^{-1} K(x/\delta)\, dx = \int\limits_{-\infty}^{\infty} (h(\theta_1 + x)\sin(\omega(\theta_2 + x - \theta_3 - x)) +$$

$$+ h(\theta_2 + x)\sin(\omega(\theta_3 + x - \theta_1 - x)) + h(\theta_3 + x)\sin(\omega(\theta_1 + x - \theta_2 - x)))\delta^{-1} K(x/\delta)\, dx \leq 0,$$

as the integrand is negative. The subharmonic function $\widetilde{h}(\varphi)V(r)$ converges uniformly on compacts outside the disk $\{z : |z| \leq r_0\}$ to the function $u(re^{i\varphi}) = h(\varphi)V(r)$ if δ tends to $0+$, and because of this the function u is subharmonic outside the disk $\{z : |z| \leq r_0\}$ in the general case too. We extend the function u to a subharmonic function in the plane. Finally, applying to the extended function u the approximation theorem by Yulmukhametov [7, P. 278-282], we obtain the statement of Theorem 1. It should be noted that the function h coincides with the Kondratyuk indicator of the entire function f concerning $r^{l(r)}$.

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ЦІЛА ФУНКЦІЯ З ЗАДАНИМ ІНДИКАТОРОМ

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Для довільної $[\kappa_1, \rho_1]$- тригонометрично опуклої функції знайдено цілу функцію з цим індикатором стосовно $r^{l(r)}$, де $l(r)$ – сильний коливний порядок, $0 < \kappa_1 < \kappa \leq l(r) \leq \rho < \rho_1 < \infty$.

Ключові слова: індикатор цілої функції.

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