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ON THE BASIS OF SYSTEM OF EXPONENTIAL FUNCTIONS WITH UNIT IN WEIGHT SPACE

Zaur KASIMOV

Institute of Mathematics and Mechanics of NAS of Azerbaijan
F.Ahaeva Str. A31141 Baku, Azerbaijanian Republic

The system of exponents with unity with complex coefficients is examined in this article. The necessary and sufficient terms of basis of this system are found in weight space of Lebesgue.

Key words: system of exponential function, basis.

We will consider the next system of exponent with unit

$$1 \cup \left\{ e^{i(nt - \alpha(t) \, sign \, n)} \right\}_{n \neq 0}, \tag{1}$$

where $\alpha(t)$ – is an actual function on a segment $[-\pi, \pi]$. Let suppose that

$$\rho(t) \equiv \prod_{i=1}^{l} |t - \tau_i|^{\beta_i},$$

where $\{\tau_i\} \subset (-\pi, \pi)$, $\{\beta_i\} \subset R$ is some great numbers.

It is easy to see that the system (1) differs from the system of exponent,

$$\left\{ e^{i(nt - \alpha(t) \, sign \, n)} \right\}_{n \in Z}, \tag{2}$$

by one member. The similar idea of replacement belongs to N. Levinson [1], which found that if the full exponent in L_p system $\{e^{i\lambda_k t}\}_{k \in Z}$ (Z - is a set of integers) replace an eventual number $\{\lambda_k\}_{k \in M}$, with other $\{\mu_k\}_{k \in M}$, where $\{\mu_k\}_{k \in M} \cap \{\lambda_k\}_{k \in Z \setminus M} = \{\varnothing\}$, $\mu_i \neq \mu_j$, with $i \neq j$, then obtained after replacement system of exponent will remain complete in L_p, where $M \subset Z$ - is some finite subset of Z. The presence of variable coefficients in the system (2) does not allow to apply the method of Levinson to study the basic properties of the system (1). It appears under certain conditions that the basic properties of the systems (1) and (2) are identical in the weight space $L_{p,\rho}$:

$$L_{p,\rho} \equiv \left\{ f : \int_{-\pi}^{\pi} |f(t)|^p \rho(t) dt < +\infty \right\},$$

with a norm

$$\|f\|_{p,\rho} = \left(\int_{-\pi}^{\pi} |f|^p \rho dt < +\infty \right)^{1/p}.$$

The previous questions of the basis of system (1), (2) in L_p, $1 < p < +\infty$, in case when $\alpha(t) \equiv \alpha t$, is fully studied in the works $[2-5]$, where $\alpha \in C$ - is a complex parameter. Interest to the similar systems has originated from Paley-Wiener's basic results [6]. These systems in model cases are the own functions of maximal differential operators of first-order. Therefore the study of basic properties of these systems is of particular interest from in terms of spectral theory of differential operators. On the other hand, the study of basic properties of the systems of type of sines and cosines

$$\{\sin(nt + \gamma(t))\}_{n \geq 1},$$

in $L_p(0,\pi)$, as it is shown in the work [7], substantially uses analogical properties in relation to the systems (1), (2) in $L_p(-\pi,\pi)$.

In this article basic properties of the system is studied (1) in $L_{p,\rho}$. The basic properties of the system (2) in L_p have been previously studied by the author [9].

We will do next basic suppositions:

1) $\alpha(t)$ is a piecewise-Hölder, odd function on $[-\pi,\pi]$: $\{s_i : -\pi < s_1 < ... < s_r < \pi\}$ - is a great number of irremovable points of breaks of function $\alpha(t)$, where $h_k = \alpha(s_k + 0) - \alpha(s_k - 0)$ - is gallops of function $\alpha(t)$ in points s_k, $k = \overline{1,r}$, $h_\pi = -2\alpha(\pi)$;

2) Great Numbers $T \equiv \{\tau_i\}_1^l$ и $S = \{s_i\}_1^r$ do not intersect: T I $S = \{\varnothing\}$.

The following main theorem is proved in the work

Theorem 1. Let executed terms 1); 2) and inequalities:

$$-1 < \beta_i < \frac{p}{q}, \quad i = \overline{1,l}; \quad -\frac{1}{q} < \frac{h_k}{\pi} < \frac{1}{p}, \ k = \overline{1,r};$$
$$-\frac{\pi}{2p} < \alpha(\pi) < \frac{\pi}{2q}$$

Then the system (1) forms a base in $L_{p,\rho}$.

Before passing to the proof of the theorem, we will consider the following homogeneous regional task of Riemann in the classes of Hardy $H_p^{\pm}$:

$$Z_0^+\left(e^{it}\right)+e^{-2i\alpha(t)}Z_0^-\left(e^{it}\right)=0 \quad \text{п.в. on } [-\pi,\pi]. \tag{3}$$

By a solution of task (3) it is meant the following thing: it is needed to find the pair of analytical functions $\left\{Z_0^+(z);Z_0^-(z)\right\}$, which belong to the classes of Hardy H_p^+, H_p^- – , inside and outside of the unit circle, accordingly, non-tangential boundary values which almost everywhere on a single circumference satisfy to correlation (3). The theory of such tasks is well worked out and sanctified in a monograph [8]. Functions $Z_0^\pm(z)$ are determined by next expressions [8]:

$$Z_0(z)=\begin{cases}X(z), & |z|<1,\\ -Y\ (z), & |z|>1,\end{cases}$$

where

$$X(z)=\exp\left\{\frac{i}{2\pi}\int_{-\pi}^{\pi}\alpha(s)\frac{e^{is}+z}{e^{is}-z}ds\right\},\ |z|<1,$$

$$Y(z)=\exp\left\{-\frac{i}{2\pi}\int_{-\pi}^{\pi}\alpha(s)\frac{e^{is}+z}{e^{is}-z}ds\right\},\ |z|>1,$$

We have previously [9] proved that under the conditions of theorem 1 system (2) forms a base in $L_{p,\rho}(-\pi,\pi)$, the biorthogonal to her system $\left\{e_n^+(t);e_{n+1}^-(t)\right\}_{n\geq 0}$ looks like

$$e_n^\pm(t)\equiv\frac{h_n^\pm(t)}{\rho(t)},$$

where

$$h_n^+(t)=\frac{e^{i\alpha(t)}\sum_{k=0}^{n}b_{n-k}^+e^{-ikt}}{2\pi Z_0^+\left(e^{it}\right)},\ n\geq 0;\ h_n^-(t)=\frac{e^{i\alpha(t)}\sum_{k=1}^{n}b_{n-k}^-e^{ikt}}{2\pi Z_0^+\left(e^{it}\right)},\ n\geq 1;$$

$\left\{b_n^\pm\right\}_{n\geq 0}$ are coefficients at z^n function $Z_0^\pm(z)$ in decompositions of Taylor on degrees z in zero and in the vinicity of infinitely-remote point, accordingly.

Formulas of Sokhotski-Plemel [8] gives

$$\ln Z_0^+\left(e^{it}\right)=\ln G(t)+\ln\left[-Z_0^-\left(e^{it}\right)\right]=\ln G(t)-\ln Y\left(e^{it}\right),$$

where $G(t)\equiv e^{2i\alpha(t)}$.

Once again applying the formulas of Sokhotski-Plemel to expression $Y(z)$ we have

$$\ln Z_0^+\left(e^{it}\right)=\frac{1}{2}\ln G(t)+\frac{i}{4\pi}\int_{-\pi}^{\pi}\ln G(s)ctg\frac{t-s}{2}ds .$$

If to pay attention to identity

$$ctg\frac{t-s}{2}=i+i\frac{2e^{is}}{e^{it}-e^{is}} ,$$

we will get

$$\ln Z_0^+\left(e^{it}\right)=\frac{1}{2}\ln G(t)-\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln G(s)ds+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\ln G(s)e^{is}}{e^{is}-e^{it}}ds .$$

It is clear, that , $\forall t\in[0,\pi]$. Therefore it is $\int_{-\pi}^{\pi}\ln G(s)ds=0$.

Let

$$I(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\ln G(s)e^{is}}{e^{is}-e^{it}}ds .$$

Thus

$$I(t)=\frac{1}{2\pi}\int_{0}^{\pi}\ln G(s)\left[\frac{1}{1-e^{i(t-s)}}-\frac{1}{1-e^{i(t+s)}}\right]ds .$$

It is easy to establish the validity of correlation

$$\frac{1}{1-e^{i(t-s)}}-\frac{1}{1-e^{i(t+s)}}=-\frac{i}{2}\frac{\cos\frac{s}{2}}{\cos\frac{t}{2}}\left[\frac{1}{\sin\frac{s-t}{2}}+\frac{1}{\sin\frac{s+t}{2}}\right].$$

Taking into consideration this correlation in the expression for $I(t)$ we will get

$$I(t)=-\frac{i}{4\pi}\int_{0}^{\pi}\ln G(s)\frac{\cos\frac{s}{2}}{\cos\frac{t}{2}}\left[\frac{1}{\sin\frac{s-t}{2}}+\frac{1}{\sin\frac{s+t}{2}}\right]ds .$$

Thus $I(t)$ even function on $[-\pi,\pi]$ and finally for $Z_0^+\left(e^{it}\right)$ we have

$$Z_0^+\left(e^{it}\right)=G^{\frac{1}{2}}(t)e^{I(t)}.$$

Quite obviously, that it is

$$b_0^+ = Z_0^+(0) = X(0)Y(0) = \exp\left\{\frac{i}{2\pi}\int_{-\pi}^{\pi}\alpha(s)ds\right\} \neq 0 .$$

Then for $h_0^+(t)$ we get expression

$$h_0^+(t) = \frac{b_0^+}{2\pi}\left[Z_0^+\left(e^{it}\right)A(t)\right]^{-1} = \frac{b_0^+}{2\pi}e^{-I(t)}\left|A(t)B(t)\right|^{-\frac{1}{2}} .$$

Consequently

$$c_0^+ = \int_{-\pi}^{\pi} e_0^+(t)dt = \int_{-\pi}^{\pi}\frac{h_0^+(t)}{\rho(t)}dt = \frac{b_0^+}{2\pi}\int_{-\pi}^{\pi}e^{-I(t)}\left|A(t)B(t)\right|^{-\frac{1}{2}}\rho^{-1}(t)dt \neq 0 .$$

Let's assume

$$c_n^{\pm} = \int_{-\pi}^{\pi}\overline{e_n^{\pm}(t)}dt , \quad n \geq 1 .$$

We will consider the system $\left\{H_n^+(t); H_k^-\right\}_{n\geq 0, k\geq 1}$, which is defined by formulas

$$H_0^+(t) = \frac{1}{c_0^+}e_0^+(t); H_n^{\pm}(t) = e_n^{\pm}(t) - b_n^{\pm}e_0^+(t), n \geq 1,$$

where $b_n^{\pm} = \frac{c_n^{\pm}}{c_0^+}$..

It is easy to check, that system

$$H_0^+ \cup \left\{H_n^+; H_n^-\right\}_{n\geq 1},$$

is biorthogonal attended to the system (1). Further we examine a partial sum ($\varphi_n^+(t) \equiv e^{-i\alpha(t)}e^{\mathrm{int}}$; $\varphi_n^-(t) \equiv e^{i\alpha(t)}e^{-\mathrm{int}}$)

$$S_{N^+,N^-} = \sum_{n=0}^{N^+}\left(H_n^+, \psi\right)_\rho \varphi_n^+(t) + \sum_{n=0}^{N^-}\left(H_n^-, \psi\right)_\rho \varphi_n^-(t) ,$$

where

$$(f,g)_\rho = \int_{-\pi}^{\pi} g(t)\overline{f(t)}\rho(t)dt .$$

After simple transformations it can be easily obtained

$$\left\| S_{N^+,N^-} - \psi \right\|_{p,\rho} \le \left\| \sum_{n=0}^{N^+} \left(e_n^+, \psi\right)_\rho \varphi_n^+ + \sum_{n=0}^{N^-} \left(e_n^-, \psi\right)_\rho \varphi_n^- - \psi \right\|_{p,\rho} +$$

$$+ \left\| \frac{1}{c_0^+} \left(e_0^+, \psi\right)_\rho - \left(e_0^+, \psi\right)_\rho \left[\sum_{n=1}^{N^+} b_n^+ \varphi_n^+ + \sum_{n=1}^{N^+} b_n^- \varphi_n^- \right] - \left(e_0^+, \psi\right)_\rho \varphi_0^+ \right\|_{p,\rho} \to 0$$

when $N^{\pm} \to \infty$.

Thus, any function can be decomposed into a series of system (2.1). From $\left\{H_n^+; H_{n+1}^-\right\} \subset L_{q,\rho} \left(\frac{1}{p} + \frac{1}{q} = 1 \right)$ it is obvious that it is minimal in $L_{p,\rho}$ and consequently, any function ψ from $L_{p,\rho}$ decomposes in a row and thus by only character on the system (1) in $L_{p,\rho}$. By it basis of systems (1) is proved

A theorem is well-proved.

Now, we will suppose that a condition is executed

3) $\left\{ \frac{h_k}{\pi} - \frac{1}{p} : k = \overline{1,r};\ h_\pi \right\} \mathbf{I}\ Z = \{\varnothing\}$.

We will define $\{n_k\}_0^r \subset Z$ from correlations

$$-\frac{1}{q} < \frac{h_k}{\pi} + n_{k-1} - n_k < \frac{1}{p},\ \ k = \overline{1,r};\ \ n_0 = 0.$$

We will accept $\omega_r = \frac{-2\alpha(\pi)}{\pi} + n_r$.

We will enter a function in consideration

$$C(t) \equiv e^{i2n_k\pi},\ t \in \left(s_{k-1}, s_k\right),\ k = \overline{1, r+1},\quad s_0 = -\pi,\ \ s_{r+1} = \pi.$$

Along with (1) we will consider the system

$$1\,\mathbf{U}\left\{ \tilde{A}(t) e^{\text{int}};\ e^{i\alpha(t)} e^{-\text{int}} \right\}_{n\ge 1}, \tag{4}$$

where $\tilde{A}(t) \equiv e^{i\alpha(t)}\ C(t)$. It is quite obviously, that base properties of the systems (1) and (4) in $L_{p,\rho}$ are identical. If $-\frac{1}{q} < \omega_r < \frac{1}{p}$, then for the system (4) all terms of theorem 1 are executed , and as a result it forms a base in $L_{p,\rho}$. We will consider a case $\omega_r < -\frac{1}{q}$,, let e.g., $-\frac{1}{q} - 1 < \omega_r < \frac{1}{q}$. . In this case we examine the system

$$1\,\mathbf{U}\left\{ \tilde{A}(t) e^{\text{int}};\ \tilde{B}(t) e^{-\text{int}} \right\}_{n\ge 1}, \tag{5}$$

where $\tilde{B}(t) \equiv e^{-it}$. we will denote $\tilde{G}(t) \equiv \frac{\tilde{B}(t)}{\tilde{A}(t)}$. It is easy to notice, that in this case it is executed $\tilde{G}(-t) \equiv \tilde{G}^{-1}(t)$ on $[-\pi,\pi]$. Then quite similar to the proof of theorem 1 we set that the system (5) forms a base in $L_{p,\rho}$ and as result the system (1) is full, but not minimal in $L_{p,\rho}$. Extending this argument as a result we get that system (1) at $\omega_r < -\frac{1}{q}$ is full, but not minimal in $L_{p,\rho}$.

Similarly, we prove that at $\omega_r > \frac{1}{p}$ the system (1) is unfull, but minimal in $\omega_r > \frac{1}{p}$. Thus we have the following theorem.

Theorem 2. Let executed terms 1) - 3) have inequalities $-1 < \beta_i < \frac{p}{q}$, $i = \overline{1,l}$. System (1) forms a base in $L_{p,\rho}$ $1 < p < +\infty$, only when $-\frac{1}{q} < \omega_r < \frac{1}{p}$. Thus, at $\omega_r < -\frac{1}{q}$ it is full, but unminimal, and at $\omega_r > \frac{1}{p}$ it is unfull, but minimal in $L_{p,\rho}$.

We will consider the separate special cases.

$$1 \cup \left\{ e^{i(n-\alpha \, sign \, n)t} \right\}_{n\neq 0} \tag{6}$$

Directly from a theorem we get

Investigation 1. Let $\alpha \in C$ is complex parameter and inequalities take place

$$-1 < \beta_i < \frac{p}{q}, \qquad i = \overline{1,l};$$

$$-\frac{1}{2q} < \operatorname{Re}\alpha < \frac{1}{2p}, \ \frac{1}{p} + \frac{1}{q} = 1.$$

Then the system (6) forms a base in $L_{p,\rho}$, $1 < p < +\infty$.

Actually, in this case it is $\alpha(t) \equiv \alpha t$ and it is easily to notice that all terms are executed 1), 2).

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ПРО БАЗИСНІСТЬ СИСТЕМИ ЕКСПОНЕНТ З ОДИНИЦЕЮ У ВАГОВОМУ ПРОСТОРІ

Заур КАСІМОВ

Інститут математики і механіки НАН Азербайджану, вул.Ф.Агаева Аз1141 Баку, Республіка Азербайджан,

Розглянуто систему експонент з одиницею з комплексними коефіцієнтами. Знайдено необхідні і достатні умови базисності цієї системи у ваговому просторі Лебега.

Ключові слова: система експонент, базисність.

О БАЗИСНОСТИ СИСТЕМЫ ЭКСПОНЕНТ С ЕДИНИЦЕЙ В ВЕСОВОМ ПРОСТРАНСТВЕ

Заур КАСИМОВ

Институт математики и механики НАН Азербайджана, вул.Ф. Агаева Аз1141 Баку, Республика Азербайджан,

Рассмотрено систему экспонент с единицей с комплексными коэффициентами. Найдены необходимые и достаточные условия базисности этой системы в весовом пространстве Лебега.

Ключевъые слова: система экспонент, базисность.

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