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# ON THE ISOMORPHISMS OF FREE PARATOPOLOGICAL GROUPS AND FREE HOMOGENEOUS SPACES II

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In the paper we prove that a free paratopological group on a $T_0$-topological space is a $T_0$-topological space. We consider the functors that preserve the isomorphisms of the free (abelian) paratopological groups and free homogeneous spaces.

*Key words:* free paratopological group, free homogeneous space, isomorphism of paratopological groups, isomorphism of homogeneous spaces.

**1. Preliminaries.** The paper is a continuation of the paper [11]. All the notations and definition are taken from [11].

In the second section of the paper we prove that a free paratopological group on a $T_0$-space is a $T_0$-space. The third section is devoted to functors preserving isomorphisms of free (abelian) paratopological groups and free homogeneous spaces. The fourth section contains a method of the reducing of the isomorphic classification of free (abelian) paratopological groups to the isomorphic classification of free (abelian) paratopological groups on $T_0$-spaces.

Some results of the paper were announced in [10].

**2. Free paratopological groups on $T_0$-spaces.** For every $n \geq 1$, by $D_n$ we denote the set $\{1, 2, \ldots, n\}$ with the topology $\{\emptyset, U_1, U_2, ...U_n\}$, where $U_k = \{1, 2, \ldots, k\}$.

It was proved in [13, Pr. 3.4] that a Markov free abelian paratopological group on $T_0$-space is a $T_0$-space.

**Theorem 1.** *A Markov free paratopological group over a $T_0$-space is a $T_0$-space.*

To prove the theorem we need the following lemmas.

**Lemma 1.** *Let $X$ be a $T_0$-space, $Y$ a finite non-empty subset of $X$ and $n = |Y|$. Then there exists a continuous mapping $f\colon X \to D_n$ such that $f|Y$ is injective.*

*Proof.* Let $Y = \{x_1, x_2, ..., x_n\}$, $G = (\mathbb{R}, +)$ and $\tau$ be the topology on $G$ with the base $\{[x; +\infty) : x \in \mathbb{R}\}$. Then $(G, \tau)$ is a paratopological group [14, Ex. 2.14]. We shall denote this group by $\mathbb{R}^*$. Since $X$ is a $T_0$-space, for each pair $\{i, j\}$ such that $i \neq j$ there exists an open set $U_{ij}$ containing exactly one of the points $x_i$ and $x_j$. Consider the mapping $f_{ij}\colon X \to \mathbb{R}^*$ defined by $f_{ij}(U_{ij}) = 2^{ni+j}$ and $f_{ij}(X \setminus U_{ij}) = 0$. The mapping $f_{ij}$ is continuous [13, Lem. 2.3]. Since $\mathbb{R}^*$ is a paratopological group, the mapping $g\colon X \to \mathbb{R}^*$ such that $g(x) = \sum f_{ij}(x)$ is continuous. Then $f_{ij}(x) = 2^{ni+j}([g(x)/2^{ni+j}] \mod 2)$ for every $x \in X$ and $i \neq j$. Since $f_{ij}(x_i) \neq f_{ij}(x_j)$ provided $i \neq j$, we see that $g|Y$ is an injection. Let $g(Y) = \{a_1, a_2, ..., a_n\}$ where $a_1 > a_2 > \cdots > a_n$. Consider the mapping $h\colon \mathbb{R}^* \to D_n$ such that $h(x) = i$, where $i = n$ if $x < a_n$ and $i$ is the smallest number such that $x \geq a_i$ otherwise. It is easy to check that $h$ is continuous. Now we put $f = hg\colon X \to D_n$. Since $g|Y$ is an injection and $h(a_i) = i$ for each $i$, the map $f|Y$ is an injection too.

**Lemma 2.** *(T.O. Banakh) A Markov free paratopological group $F_p(D_n)$ is a $T_0$-space for every positive integer $n$.*

*Proof.* It was proved in [13, Pr. 3.4] that a Markov free abelian paratopological group on a $T_0$-space is a $T_0$-space. Let $\varphi : F_p(D_n) \to A_p(D_n)$ be a continuous homomorphism such that $\varphi(x) = x$ for each $x \in D_n$, $K$ be the commutant of $F_p(D_n)$. Since $A_p(D_n)$ is abelian, $K \subset \ker\varphi$. Let $\pi : F_p(D_n) \to F_p(D_n)/K$ be the quotient homomorphism. Since the group $F_p(D_n)/K$ is abelian, there exists a continuous homomorphism $\psi : A_p(D_n) \to F_p(D_n)/K$ such that $\psi(x) = \pi(x)$ for every $x \in D_n$. Since the group $F_p(D_n)$ is generated by the set $D_n$, we obtaine $\pi = \psi\varphi$. Then $K = \ker\pi \supset \ker\varphi$, thus $K = \ker\varphi$.

Therefore, in order to prove that $F_p(D_n)$ is a $T_0$-space it suffices to construct a topology $\tau$ on $F(D_n)$ which separates every point from $K \setminus \{e\}$ and the identity $\{e\}$ of $F_p(D_n)$ and $D_n$ is a subspace of $(F_p(D_n), \tau)$. Using results from [15] it is easy to prove that the group $F_p(D_n)$ is algebraically free over the set $D_n$. For every word $A \in F_p(D_n)$ let $\varphi_i(A)$ be the sum of degrees of the letters "$i$" in the word $A$. Consider the subsemigroup $S$ of $F_p(D_n)$ generated by $\{e\}$ and the set of all the words $A \in F_p(D_n)$ over the alphabet $D_n$ such that the last nonzero element in the sequence $(\varphi_1(A), \varphi_2(A), \dots, \varphi_n(A))$ is positive. For every $s \in S$ and for every $g \in F_p(D_n)$ we see that $g^{-1}xg \in S$, thus the semigroup $S$ defines a semigroup topology $\tau$ on $F_p(D_n)$ [14, 2] such that $S \subset \tau$. Then $D_n$ is a subspace of $(F_p(D_n), \tau)$ and the topology $\tau$ induces the discrete topology on $K$.

*Proof of the theorem.* Using results from [15] it is easy to prove that the group $F_p(X)$ is algebraically free over the set $X$. Since the space of paratopological group is homogeneous, it is sufficient to prove that for each word $A \in F_p(X)$ over the alphabet $X$ there exists an open set $U$ separating $A$ and the identity of $F_p(X)$. Let $A = x_1^{\epsilon_1} x_2^{\epsilon_2} ... x_n^{\epsilon_n}$ be a word in the irreducible form and $a_1, a_2, ..., a_k$, $k \leqslant n$, be its letters. Then by Lemmma 1 there exists a continuous mapping $f\colon X \to D_k$ such that $f(a_i) \neq f(a_j)$ provided $i \neq j$. We may extend the mapping $f$ to a continuous homomorphism $f^*\colon F_p(X) \to F_p(D_k)$. Then $f^*(A) \neq e_{F_p(D_k)}$. Since $F_p(D_k)$ is a $T_0$-space, there exists an open set $U \subseteq F_p(D_k)$ containing exactly one of the points $f^*(A)$ and $e_{F_p(D_k)}$. The set $(f^*)^{-1}(U)$ is open and contains exactly one of the points $A$ and $e_{F_p(X)}$.

**3. The reflections of spaces and the isomorphisms of free paratopological groups.** A topological space is *totally disconnected* if each its quasicomponent is a singleton.

Let $T$ be a class of spaces satisfying the following property:

Let $X$ be a space such that for every $x, y \in X$ there exists $f\colon X \to Y$, where $Y \in T$ with $f(x) \neq f(y)$, then $X \in T$. $(*)$

Examples of the classes spaces satisfying property $(*)$ are: $T_0$-spaces, $T_1$-spaces, $T_2$-spaces, functionally Hausdorff spaces, totally disconnected spaces.

A class $T$ of spaces is *hereditary* provided that if $X \in T$ then $Y \in T$ for each subspace $Y$ of $X$. The following observation was made by T. O. Banakh.

**Proposition 1.** *A class $T$ of spaces satisfies condition $(*)$ if and only if $T$ is a hereditary class closed under Tychonoff products and strengthening of topology.*

Let $T$ be a class of spaces satisfying condition $(*)$ and let $X$ be a space. Consider the following equivalence relation on $X$. Let $x, y \in X$. Put $x \sim_T y$ if and only if $f(x) = f(y)$ for each continuous mapping $f\colon X \to Y$, where $Y \in T$. The quotient space $X/\sim_T$ is called *the $T$-reflection of $X$* and is denoted by $TX$. If $X \in T$ then the identity homeomorphism $i\colon X \to X$ separates all pairs of different points of $X$, thus $X = TX$.

For some classes $T$ of spaces the equivalence relation $\sim_T$ has an other descriptions. If $T_0$ is the class of $T_0$-spaces and $x, y \in X$ then $x \sim_{T_0} y$ if and only if either $x = y$ or there is no open subset of the space $X$ containing exactly one of the points $x, y$. If $fT_2$ is the class of functionally Hausdorff spaces and $x, y \in X$ then $x \sim_{fT_2} y$ if and only $f(x) = f(y)$ for each continuous mapping $f\colon X \to [0;1]$, where the segment $[0;1]$ has the standard topology. If $TD$ is the class of totally disconnected spaces and $x, y \in X$ then $x \sim_{TD} y$ if and only if the points $x$ and $y$ have the same quasicomponent (see also [5, §46, V.]).

**Proposition 2.** *Any class $T$ satisfying condition $(*)$ determines a covariant functor $T\cdot$ from the category of spaces and continuous mappings to the category of spaces from the class $T$ and their continuous mappings.*

*Proof.* Let us check that $TX \in T$ for each space $X$. Note that for each continuous mapping $f\colon X \to Y \in T$ there exists a continuous mapping $g\colon TX \to Y$ such that $f = g \circ t_X$, where $t_X\colon X \to TX$ is the quotient mapping. Let $x, y \in TX$, $x \neq y$. Choose points $x_1 \in t_X^{-1}(x), y_1 \in t_X^{-1}(y)$. Then there exists a continuous mapping $f\colon X \to Y \in T$ such that $f(x_1) \neq f(y_1)$. Then for the above defined $g$ we have that $g(x) \neq g(y)$, therefore $TX \in T$.

Let $f\colon X \to Y$ be a continuous mapping, $t_X\colon X \to TX$, $t_Y\colon Y \to TY$ be the quotient mappings. Let us prove that there exists a unique continuous mapping $g\colon TX \to TY$ such that $g \circ t_X = t_Y \circ f$. Let $u \in TX$ and $x \in t_X^{-1}(u)$. Put $g(u) = t_Y(f(x))$. Let us check that the mapping $g$ is well-defined. If $z \in t_X^{-1}(u)$ then $h(x) = h(z)$ for all continuous mappings $h\colon X \to Z$, where $Z \in T$. Since $TY \in T$, we obtaine $t_Y(f(x)) = t_Y(f(z))$, and we are done. Since $t_X$ is the quotient mapping and the composition $t_Y \circ f$ is continuous, the mapping $t_X$ is continuous too. Put $Tf = g$.

It is easy to check that the rule which corresponds a space $TX$ to each space $X$ and a mapping $Tf : TX \to TY$ to each continuous mapping $f : X \to Y$ is a covariant functor.

The functor from Proposition 2 is called *the $T$-reflection*.

**Theorem 2.** *Let $T$ be a class of spaces satisfying condition $(*)$ such that $F_p(X') \in T$ for each space $X' \in T$. Let $X$ and $Y$ be spaces such that the Markov free paratopological groups $F_p(X)$ and $F_p(Y)$ are topologically isomorphic. Then the quotient mappings $t_X \colon X \to TX$ and $t_Y \colon Y \to TY$ are $M_p$-equivalent and hence the Markov free paratopological groups $F_p(TX)$ and $F_p(TY)$ are topologically isomorphic.*

*Proof.* Let $i \colon F_p(X) \to F_p(Y)$ be a topological isomorphism, $t_X \colon X \to TX$, $t_Y \colon Y \to TY$ be the quotient mappings, $t_X^* \colon F_p(X) \to F_p(TX)$ and $t_Y^* \colon F_p(Y) \to F_p(TY)$ be their homomorphic extensions.

Let us construct a continuous mapping $h \colon TX \to F_p(TY)$ such that $h \circ t_X = t_Y^* \circ (i|X)$. Let $x' \in TX$. Choose an arbitrary point $x \in X$ such that with $t_X(x) = x'$ and put $h(x') = t_Y^* i(x)$. Let $y \in X$. There is a point $x \in X$ such that $t_X(x) = t_X(y)$ and $ht_X(x) = t_Y^* i(x)$. Thus $ht_X(y) = ht_X(x) = t_Y^* i(x) = t_Y^* i(y)$ since $TY \in T$ and therefore $F_p(TY) \in T$. Thus $h \circ t_X = t_Y^* \circ (i|X)$. The continuity of the mapping $h$ is implied from the continuity of $i$ and $t_Y^*$ and the fact that the mapping $t_X$ is quotient.

Similarly, we can construct a continuous mapping $g \colon TY \to F_p(TX)$ such that $g \circ t_Y = t_X^* \circ (i^{-1}|Y)$. Let us extend the mappings $h, g$ to the continuous homomorphisms $h^* \colon F_p(TX) \to F_p(TY)$ and $g^* \colon F_p(TY) \to F_p(TX)$. Let $x \in X$. Then

$$h^* t_X^*(x) = h^* t_X(x) = ht_X(x) = t_Y^* i(x).$$

Since the group $F_p(X)$ is generated by the set $X$, we have $h^* \circ t_X^* = t_Y^* \circ i$. Similarly we can show that $g^* \circ t_Y^* = t_X^* \circ i^{-1}$. Since

$$g^* \circ h^* \circ t_X^* = g^* \circ t_Y^* \circ i = t_X^* \circ i^{-1} \circ i = t_X^*,$$

we obtain $g^* \circ h^* = 1_{F_p(TX)}$. Similarly, we can prove that $h^* \circ g^* = 1_{F_p(TY)}$. Thus $h^* \colon F_p(TX) \to F_p(TY)$ is a topological isomorphism. Since $h^* \circ t_X^* = t_Y^* \circ i$, the mappings $t_X$ and $t_Y$ are $M_p$-equivalent.

**Corollary 1.** *Let $T$ be one of the following clasess:*

- *$T_0$-spaces,*
- *functionally Hausdorff spaces,*
- *totally disconnected spaces.*

*Let $X$ and $Y$ be spaces such that the Markov free paratopological groups $F_p(X)$ and $F_p(Y)$ are topologically isomorphic. Then the Markov free paratopological groups $F_p(TX)$ and $F_p(TY)$ are topologically isomorphic too.*

*Proof.* If $X'$ is a $T_0$-space then $F_p(X')$ is a $T_0$-space too [12]. If $X'$ is a functionally Hausdorff space then $F_p(X')$ is a functionally Hausdorff space too [13, Pr. 3.8]. If $X'$ is a totally disconnected space then by [13, Pr. 2.15] the quasicomponent of the unit in $F_p(X')$ is a singleton, thus $F_p(X')$ is a totally disconnected space too.

**Corollary 2.** *Let $T$ be a class of spaces satisfying condition $(*)$ such that $F_p(X') \in T$ for each space $X' \in T$. Let $X_1, X_2, Y_1, Y_2$ be spaces, $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$ be $M_p$-equivalent mappings. Then the mappings $Tf_1$ and $Tf_2$ are $M_p$-equivalent.*

*Proof.* Let $i\colon F_p(X_1) \to F_p(X_2)$, $j\colon F_p(Y_1) \to F_p(Y_2)$ be topological isomorphisms such that $f_2^* \circ i = j \circ f_1^*$. Similarly to the proof of Theorem 2 we can build topological isomorphisms $i_T\colon F_p(TX_1) \to F_p(TX_2)$ and $j_T\colon F_p(TY_1) \to F_p(TY_2)$ such that $i_T \circ t_{X_1}^* = t_{X_2}^* \circ i$ and $j_T \circ t_{Y_1}^* = t_{Y_2}^* \circ j$. Proposition 2 implies that $Tf_1 \circ t_{X_1} = t_{Y_1} \circ f_1$ and $Tf_2 \circ t_{X_2} = t_{Y_2} \circ f_2$. If $x \in X_2$ then

$$t_{Y_2}^* f_2^*(x) = t_{Y_2}^* f_2(x) = t_{Y_2} f_2(x) = (Tf_2)t_{X_2}(x) = (Tf_2)^* t_{X_2}(x) = (Tf_2)^* t_{X_2}^*(x).$$

Since the group $F_p(X_2)$ is generated by the set $X_2$, we have $t_{Y_2}^* \circ f_2^* = (Tf_2)^* \circ t_{X_2}^*$. Let $x \in X_1$. Then

$$j_T(Tf_1)^* t_{X_1}^*(x) = j_T(Tf_1)^* t_{X_1}(x) = j_T(Tf_1)t_{X_1}(x) = j_T t_{Y_1} f_1(x) = j_T t_{Y_1}^* f_1(x) =$$
$$= t_{Y_2}^* j f_1(x) = t_{Y_2}^* j f_1^*(x) = t_{Y_2}^* f_2^* i(x) = (Tf_2)^* t_{X_2}^* i(x) = (Tf_2)^* i_T t_{X_1}^*(x).$$

Since the group $F_p(TX_1)$ is generated by the set $t_{X_1}^*(X_1)$, we obtain $(Tf_2)^* \circ i_T = j_T \circ (Tf_1)^*$. Thus, the mappings $Tf_1$ and $Tf_2$ are $M_p$-equivalent.

If we replace the words "free paratopological group" by the words "free abelian paratopological group" in the Definitions 1.8 and 1.9 from the paper [11] then we obtain the definitions of $A_p$-equivalent spaces and $A_p$-equivalent mappings (remark that in the paper [11] the author did misprints in these definitions; there must we written "in Definitions 1.8 and 1.9" instead of "in Definitions 1.10 and 1.11").

Similarly to Theorem 2 we can prove the following

**Theorem 3.** *Let $T$ be a class of spaces satisfying condition $(*)$ such that $A_p(X') \in T$ for each space $X' \in T$. Let $X$ and $Y$ be spaces such that the Markov free abelian paratopological groups $A_p(X)$ and $A_p(Y)$ are topologically isomorphic. Then the quotient mappings $t_X\colon X \to TX$ and $t_Y\colon Y \to TY$ are $A_p$-equivalent and hence the Markov free abelian paratopological groups $A_p(TX)$ and $A_p(TY)$ are topologically isomorphic.*

**Corollary 3.** *Let $T$ be one of the following clasess:*

- *$T_0$-spaces,*
- *$T_1$-spaces,*
- *functionally Hausdorff spaces,*
- *totally disconnected spaces.*

*Let $X$ and $Y$ be spaces such that the Markov free abelian paratopological groups $A_p(X)$ and $A_p(Y)$ are topologically isomorphic. Then Markov free abelian paratopological groups $A_p(TX)$ and $A_p(TY)$ are topologically isomorphic too.*

*Proof.* If $X'$ is a $T_0$-space then $A_p(X')$ is a $T_0$-space too [13, Pr. 3.4]. If $X'$ is a $T_1$-space then $A_p(X')$ is a $T_1$-space too [12, Pr. 3.5]. If $X'$ is a functionally Hausdorff space then $F_p(X')$ is a functionally Hausdorff space too [13, Pr. 3.8].

Now let $X'$ be a totally disconnected space. We are going to show that the quasi-component of the zero in $A_p(X')$ is a singleton. Let $x \in A_p(X')\backslash\{0\}$. Then there exists a finite nonempty subset $F \in X'$ and a set $\{n_y : y \in F\}$ of non-zero integers such that $x = \sum\{n_y y : y \in F\}$. Since the space $X'$ is totally disconnected, for every point $y \in F$ there exists a clopen neighborhood $U_y \subset X'$ of $y$ such that $U_y \cap F = \{y\}$. For every point $y \in F$ put $V_y = U_y \backslash \bigcup\{U_{y'} : y' \in F\backslash\{y\}\}$. Then $\{V_y : y \in F\}$ is a family of pairwise disjoint clopen subsets of $X'$. Let $f : X \to \mathbb{Z}$ be a mapping such that $f(z) = n_y$ if $z \in V_y$ for some $y \in F$ and $F(X'\backslash \bigcup\{V_y : y \in F\}) = \{0\}$. Then $f$ is a continuous mapping.

Let $f^* : A_p(X') \to \mathbb{Z}$ be a continuous homomorphic extension of the mapping $f$. Then $f^*(0) = 0$ but $f^*(x) = \sum\{n_y^2 : y \in F\} > 0$. Therefore $f^{*-1}(0)$ is a clopen neighborhood of the zero of the group $A_p(X')$ not containing $x$. Thus $A_p(X')$ is a totally disconnected space.

**Corollary 4.** *Let $T$ be a class of spaces satisfying condition $(*)$ such that $A_p(X') \in T$ for each space $X' \in T$. Let $X_1, X_2, Y_1, Y_2$ be spaces, $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$ be $A_p$-equivalent mappings. Then the mappings $Tf_1$ and $Tf_2$ are $A_p$-equivalent.*

*Proof.* The proof is similar to the proof of Corollary 2.

Let $X_1, X_2, Y_1, Y_2$ be spaces. A mapping $f_1 : X_1 \to Y_1$ is called *$B$-equivalent* to a mapping $f_2 : X_2 \to Y_2$ if there exist isomorphisms $i : H(X_1) \to H(X_2)$ and $j : H(Y_1) \to H(Y_2)$ such that $j \circ \bar{f}_1 = \bar{f}_2 \circ i$. Recall that here by $H(X) = (H_B(X), G(X), h)$ we denote the free homogeneous space on a space $X$ described in the beginning of [11, Part 2].

We shall need the following

**Lemma 3.** *Let $X, Y$ be spaces and $(i, \varphi) : H(X) \to H(Y)$ be a morphism. Let $n \geq 0$ and $z_1, z_2, \ldots, z_{2n+1} \in H_B(X)$. Then $z = z_1 z_2^{-1} \cdots z_{2n}^{-1} z_{2n+1} \in H_B(X)$ and*

$$i(z) = i(z_1) i(z_2)^{-1} \cdots i(z_{2n})^{-1} i(z_{2n+1}).$$

*Proof.* Let $x, y \in H_B(x)$. Then $xy^{-1} \in G(X)$ and since $(i, \varphi)$ is a morphism,

$$\varphi(xy^{-1}) = i(xy^{-1}y) i(y)^{-1} = i(x) i(y)^{-1}.$$

It is clear that $z \in H_B(X)$. Put $g = z_1 z_2^{-1} \cdots z_{2n}^{-1}$ if $n > 0$ and $g = e$ if $n = 0$. Then $g \in G(X)$ and $i(z) = i(g z_{2n+1}) = \varphi(g) i(z_{2n+1})$. Since $\varphi$ is a homomorphism,

$$\varphi(g) = \varphi(z_1 z_2^{-1}) \cdots \varphi(z_{2n-1} z_{2n}^{-1}) = i(z_1) i(z_2)^{-1} \cdots i(z_{2n-1}) i(z_{2n})^{-1}.$$

**Corollary 5.** *Let $X, Y$ be spaces and $(i, \varphi), (j, \psi) : H(X) \to H(Y)$ be morphisms. If $i|X = j|X$ then $(i, \varphi) = (j, \psi)$.*

**Theorem 4.** *Let $T$ be a class of spaces satisfying condition $(*)$ such that $H_B(X') \in T$ for each space $X' \in T$. Let $X$ and $Y$ be spaces such that the free homogeneous spaces $H(X)$ and $H(Y)$ are isomorphic. Then the quotient mappings $t_X \colon X \to TX$ and $t_Y \colon Y \to TY$ are $B$-equivalent and hence the free homogeneous spaces $H(TX)$ and $H(TY)$ are isomorphic.*

*Proof.* Let $(i, \varphi) \colon H(X) \to H(Y)$ be an isomorphism of the homogeneous spaces, $t_X \colon X \to TX$, $t_Y \colon Y \to TY$ be the quotient mappings and $\bar{t}_X = (t_X^*, \psi_X) \colon H(X) \to H(TX)$, $\bar{t}_Y = (t_Y^*, \psi_Y) \colon H(Y) \to H(TY)$ be the morphisms constructed from the mappings $t_X$ and $t_Y$ (see [11, Part 2]).

Let us construct a continuous mapping $h \colon TX \to H_B(TY)$ such that $h \circ t_X = t_Y^* \circ (i|X)$. Let $x' \in TX$. Choose an arbitrary point $x \in X$ such that with $t_X(x) = x'$ and put $h(x') = t_Y^* i(x)$. Let $y \in X$. There is a point $x \in X$ such that $t_X(x) = t_X(y)$ and $ht_X(x) = t_Y^* i(x)$. Thus $ht_X(y) = ht_X(x) = t_Y^* i(x) = t_Y^* i(y)$ because $TY \in T$ and therefore $H_B(TY) \in T$. So $h \circ t_X = t_Y^* \circ (i|X)$. The continuity of the mapping $h$ follows from the continuity of $i$ and $t_Y^*$ and the fact that the mapping $t_X$ is quotient.

Similarly, we can construct a continuous mapping $g\colon TY \to H_B(TX)$ such that $g \circ t_Y = t^*_X \circ (i^{-1}|Y)$. Let $(h^*, \varphi_X)\colon H(TX) \to H(TY)$, $(g^*, \varphi_Y)\colon H(TY) \to H(TX)$ be the morphisms constructed from the mappings $h$ and $g$. Let $x \in X$. Then

$$h^* t^*_X(x) = h^* t_X(x) = h t_X(x) = t^*_Y i(x).$$

Corollary 5 implies that $h^* \circ t^*_X = t^*_Y \circ i$. Similarly we can show that $g^* \circ t^*_Y = t^*_X \circ i^{-1}$. Since $g^* \circ h^* \circ t^*_X = g^* \circ t^*_Y \circ i = t^*_X \circ i^{-1} \circ i = t^*_X$, $g^* \circ h^* = 1_{H_B(TX)}$. Similarly, we can prove that $h^* \circ g^* = 1_{H_B(TY)}$. Corollary 5 implies that $(h^*, \varphi_X) \circ (g^*, \varphi_Y) = 1_{H(TY)}$ and $(g^*, \varphi_Y) \circ (h^*, \varphi_X) = 1_{H(TX)}$. Hence $(h^*, \varphi_X)$ is an isomorphism. Since $h^* t^*_X = t^*_Y i$, $\bar{t}_Y \circ (i, \varphi) = h^* \circ \bar{t}_X$ by Corollary 5 and the mappings $t_X$ and $t_Y$ are $B$-equivalent.

**Corollary 6.** *Let $T$ be one of the following clasess:*

- *$T_0$-spaces,*
- *$T_1$-spaces,*
- *$T_2$-spaces,*
- *functionally Hausdorff spaces,*
- *totally disconnected spaces.*

*Let $X$ and $Y$ be spaces such that the free homogeneous spaces $H(X)$ and $H(Y)$ are isomorphic. Then the free homogeneous spaces $H(TX)$ and $H(TY)$ are isomorphic too.*

*Proof.* If $T$ is either the class of $T_0$-spaces or the class of totally disconnected spaces or the class of functionally Hausdorff spaces and $X' \in T$ then $F_p(X') \in T$ (see the proof of Corollary 1) and therefore $H_p(X') \in T$ thus $H_B(X') \in T$ by Lemma 1 from [11]. If $X'$ is a $T_1$-space then $H_B(X')$ is a $T_1$-space too [6]. If $X'$ is a $T_2$-space then $H_B(X')$ is a $T_2$-space too [7].

**Corollary 7.** *Let $T$ be a class of spaces satisfying condition $(*)$ such that $H_B(X') \in T$ for each space $X' \in T$. Let $X_1, X_2, Y_1, Y_2$ be spaces, $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ be $B$-equivalent mappings. Then the mappings $Tf_1$ and $Tf_2$ are $B$-equivalent.*

*Proof.* Let $(i, \varphi)\colon H(X_1) \to H(X_2)$, $(j, \psi)\colon H(Y_1) \to H(Y_2)$ be topological isomorphisms such that $\bar{f}_2 \circ (i, \varphi) = (j, \psi) \circ \bar{f}_1$. Similarly to the proof of Theorem 4 we can construct isomorphisms $(i_T, \varphi_T)\colon H(TX_1) \to H(TX_2)$ and $(j_T, \psi_T)\colon H(TY_1) \to H(TY_2)$ such that $(i_T, \varphi_T) \circ \bar{t}_{X_1} = \bar{t}_{X_2} \circ (i, \varphi)$ and $(j_T, \psi_T) \circ \bar{t}_{Y_1} = \bar{t}_{Y_2} \circ (j, \psi)$. Proposition 2 implies that $Tf_1 \circ t_{X_1} = t_{Y_1} \circ f_1$ and $Tf_2 \circ t_{X_2} = t_{Y_2} \circ f_2$. If $x \in X_2$ then

$$t^*_{Y_2} f^*_2(x) = t^*_{Y_2} f_2(x) = t_{Y_2} f_2(x) = (Tf_2) t_{X_2}(x) = (Tf_2)^* t_{X_2}(x) = (Tf_2)^* t^*_{X_2}(x).$$

Corollary 5 implies that $t^*_{Y_2} \circ f^*_2 = (Tf_2)^* \circ t^*_{X_2}$. Let $x \in X_1$. Then

$$j_T (Tf_1)^* t^*_{X_1}(x) = j_T (Tf_1)^* t_{X_1}(x) = j_T (Tf_1) t_{X_1}(x) = j_T t_{Y_1} f_1(x) = j_T t^*_{Y_1} f_1(x) =$$
$$= t^*_{Y_2} j f_1(x) = t^*_{Y_2} j f^*_1(x) = t^*_{Y_2} f^*_2 i(x) = (Tf_2)^* t^*_{X_2} i(x) = (Tf_2)^* i_T t^*_{X_1}(x).$$

Since the set $H_B(TX_1)$ is generated by the set $t^*_{X_1}(X_1)$, we see that $\overline{Tf_2} \circ (i_T, \varphi_T) = (j_T, \psi_T) \circ \overline{Tf_1}$ by Corollary 5. Thus the the mappings $Tf_1$ and $Tf_2$ are $B$-equivalent.

## 4. On $T_0$-reflection.

**Proposition 3.** *For each topological space $X$ the quotient mapping $t_X$ has a continuous right inverse.*

*Proof.* Let $X_1$ be a subset of $X$ such that $X_1 \cap C$ is a singleton for each class $C$ of the relation $\sim_{T_0}$ on $X$. Define the mapping $f\colon T_0X \to X_1$ by putting $f(x) = y$, where $y = t^{-1}(x) \cap X_1$. It is clear that $t_X \circ f$ is the identity mapping on the space $TX$. Let us check that the mapping $f$ is continuous. Let $U$ be an open subset in $X_1$. Let us put $V = \{x \in X : \text{there exists a point } y \in U \text{ such that } x \sim_{T_0} y\}$. Since $U$ is open in $X_1$, there exists an open set $W$ in $X$ such that $U = W \cap X_1$. Let us prove that $V = W$. Suppose that there exists $z \in V \setminus W$. Then there exists $z_1 \in U$ such that $z \sim_{T_0} z_1$. Since the points $z$ and $z_1$ are not separated by open subsets in $X$, we see that $z_1 \notin W$. We get a contradiction with the fact that $U = W \cap X_1$. Let $z \in W$. Then there exists $z_1 \in X_1$ such that $z \sim_{T_0} z_1$. Since the points $z$ and $z_1$ are not separated by open subsets in $X$, we have $z_1 \in W$, therefore $z_1 \in U$ and $z \in V$. Thus $V = W$ and the set $V$ is open in $X$. By the construction, $V = t_X^{-1}(f^{-1}(U))$. Since the mapping $t_X$ is quotient and $V$ is open subset in $X$, we see that $f^{-1}(U)$ is an open subset in $T_0X$.

*Remark* 1. Let $X$ be a topological space. Let $X_1$ be a subset of $X$ such that $X_1 \cap C$ is a singleton for each class $C$ of the relation $\sim_{T_0}$ on $X$. The above lemma imply that the mapping $t_X|X_1$ is a homeomorphism. Since every neighborhood of the set $X_1$ coincides with $X$, the quotient space $X/X_1$ is antidiscrete. It easy to check that the size of the set $X/X_1$ does not depend on the choice of $X_1$. The cardinal of this size with antidiscrete topology is denoted as the space $X/T_0X$.

Let $(X, x_0)$ and $(Y, y_0)$ be pointed spaces such that $X \cap Y = \varnothing$. The quotient space $(X \oplus Y)/\{x, y\}$ is called a bouquet of pointed spaces $(X, x_0)$ and $(Y, y_0)$ and is denoted by $(X, x_0) \vee (Y, y_0)$.

**Lemma 4.** *Let $X, Y$ be disjoint spaces, $x_1, x_2 \in X$, $y_1, y_2 \in Y$. Then spaces $(X, x_1) \vee (Y, y_1)$ and $(X, x_2) \vee (Y, y_2)$ are $B$-equivalent.*

*Proof.* For $i = 1, 2$ put $K_i = \{x_i, y_i\}$ and define maps $r_i : X \oplus Y \to K_i$ such that $r_i(X) = \{x_i\}$ and $r_i(Y) = \{y_i\}$. Then the maps $r_1$ and $r_2$ are parallel retractions. So by [11, Pr. 3] the spaces $(X, x_1) \vee (Y, y_1)$ and $(X, x_2) \vee (Y, y_2)$ are $B$-equivalent.

We shall write sometimes "$X \vee Y$" instead of "$(X, x_0) \vee (Y, y_0)$". We also recall that the $B$-equivalence of spaces implies their $M_p$-equivalence.

**Lemma 5.** *Let $X, Y$ be spaces and $f : X \to Y$ be a continuous map. Let $f^* : F_p(X) \to F_p(Y)$ be the homomorphic extension of the map $f$. Then $\ker f^*$ is the subgroup $N$ of $F_p(X)$ generated by the set $\{g^{-1}xy^{-1}g : x, y \in X, g \in F_p(X),\ f(x) = f(y)\}$.*

*Proof.* It is clear that $N \subset \ker f^*$. Now we prove the opposite inclusion. Using results from [15] it is easy to prove that the group $F_p(X)$ is algebraically free over the set $X$ and the group $F_p(Y)$ is algebraically free over the set $Y$. If $w$ is an arbitrary element of $F_p(X)$ then $w = x_1^{\varepsilon_1} \cdots x_1^{\varepsilon_1}$ where $\{x_1, \ldots, x_n\} \subset X$ and $\{\varepsilon_1, \ldots, \varepsilon_n\} \subset \{-1, 1\}$. Then by easy induction on $n$ we can prove that if $f^*(w) = e$ then $w \in N$.

**Proposition 4.** *Let $X$ be a nonempty topological space. Then $X \overset{M_p}{\sim} (T_0X \times \{1\} \vee (X/T_0X) \times \{2\})$.*

*Proof.* Let $X_1$ be a subset of $X$ such that $X_1 \cap C$ is a singleton for each class $C$ of the relation $\sim_{T_0}$ on $X$. Put $Z = X_1 \times \{1\} \oplus X \times \{2\}$. Choose an arbitrary point $x_0 \in X$ and put $Z' = Z/\{(x_0, 1), (x_0, 2)\}$ and $\pi : Z \to Z'$ be the quotient mapping.

Define a mapping $r : X \to X_1$ as follows. Let $x \in X$. There is a unique point $x_1 \in X_1$ such that $x_1 \sim_{T_0} x$. Put $r(x) = x_1$. The proof of Proposition 3 implies that $r^{-1}(U)$ is open for each open subset $U$ of $X_1$ so $r$ is continuous.

Let $t \in \{1, 2\}$. Define a mapping $r_t : Z \to Z$ putting $r_t(x, s) = (r(x), t)$ for each $x \in X, s \in \{1, 2\}$ such that $(x, s) \in Z$. Since

$$r_t^{-1}(U \times \{t\}) = (r^{-1}(U \cap X_1) \cap X_1) \times \{1\} \cup r^{-1}(U \cap X_1) \times \{2\}$$

for each open set $U \subset X_1$, the mapping $r_t$ is a continuous retraction. Since $r_t((x_0, 1)) = r_t((x_0, 2))$, there exists a mapping $r'_t : Z' \to Z'$ such that $r'_t\pi = \pi r_t$. Since $\pi$ is the quotient mapping and the mapping $r_t\pi$ is continuous then the mapping $r'_t$ is continuous too.

It is easy to check that $r_1$ and $r_2$ are parallel retractions. Let $t, t' \in \{1, 2\}$. Then $r'_t r'_{t'}\pi = r'_t\pi r_{t'} = \pi r_t r_{t'} = \pi r_t = r'_t\pi$. Since the mapping $\pi$ is surjective then $r'_t r'_{t'} = r'_t$ so the mappings $r'_1$ and $r'_2$ are parallel retractions too.

Let $i\colon Z' \to F_p(Z')$ be the mapping such that $i(z') = r'_1(z')z'^{-1}r'_2(z')$ for each $z' \in Z'$. Let us check that the mapping $i$ is continuous. It is sufficient to prove that its restrictions onto $\pi(X_1 \times \{1\})$ and $\pi(X \times \{2\})$ are continuous. If $z \in X_1 \times \{1\}$ then

$$i\pi(z) = r'_1\pi(z) \times \pi(z)^{-1} \times r'_2\pi(z) = \pi r_1(z) \times \pi(z)^{-1} \times \pi r_2(z) =$$

$$= \pi(z) \times \pi(z)^{-1} \times \pi r_2(z) = \pi r_2(z) = r'_2\pi(z).$$

Therefore $i|\pi(X_1 \times \{1\})$ is a continuous map. Now let $z \in X \times \{2\}$. Define a mapping $j : \pi(X \times \{2\}) \to F_p(Z')$ putting $j(z') = z'^{-1}r'_2(z')$ for each $z' \in \pi(X \times \{2\})$. Let us check that the mapping $j$ is continuous. For this purpose we prove that $j\pi(X \times \{2\})$ is an antidiscrete subspace of $F_p(Z')$. It is easy to check that for each point $z' \in \pi(X \times \{2\})$ such that $z' \neq r'_2(z')$ there is no open subset $U$ of $Z'$ such that $U$ contains exactly one of the points $z'$ and $r'_2(z')$. Let $z'$ be an arbitrary point of $\pi(X \times \{2\})$. Let $R_{z'}$ be a subset of $F_p(Z')$ such that $R_{z'} = z'^{-1}\{z', r'_2(z')\} = \{e, j(z')\}$. Thus, by the homogeneity, for each open subset $U$ of $F_p(Z')$ we have the following dichotomy: $R_{z'} \subset U$ or $R_{z'} \subset F_p(Z')\backslash U$. Let $V$ be an open subset of $F_p(Z')$ such that $V \cap j\pi(X \times \{2\}) \neq \varnothing$. Choose a point $z' \in \pi(X \times \{2\})$ such that $j(z') \in V$. Then $R_{z'} \subset V$ so $e \in V$. The dichotomy implies that $R_{u'} \subset V$ for each point $u' \in \pi(X \times \{2\})$ so $j\pi(X \times \{2\}) \subset V$. Since $F_p(Z')$ is a paratopological group and the mappings $j$ and $r'_2$ are continuous and $i(z') = j(z') \times r'_2(z')$ for each $z' \in \pi(X \times \{2\})$, the mapping $i$ is continuous too.

Denote by $i^*\colon F_p(Z') \to F_p(Z')$ the continuous homomorphic extension of the mapping $i$. It was proved in [9] that $i^* \circ i^* = 1_{F_p(Z')}$.

Let $t \in \{1, 2\}$. Let $Y_t$ be the quotient space $Z'/\pi(X_1 \times \{t\})$, $p_t : Z' \to Y_t$ be the quotient mapping and $p_t^*\colon F_p(Z') \to F_p(Y_t)$ be the continuous homomorphic extension of $p_t$. Lemma 5 implies that $\ker p_t^*$ is a smallest normal subgroup of $F_p(Z')$ containing the set $\{xy^{-1} : x, y \in Z', f(x) = f(y)\} = \{xy^{-1} : x, y \in \pi(X_1 \times \{t\})\}$.

Let $x \in X_1$. Then $i\pi((x, 1)) = r'_2\pi((x, 1)) = \pi r_2((x, 1)) = \pi((r(x), 2)) = \pi((x, 2))$. So $i(\pi(X_1 \times \{1\}) = \pi(X_1 \times \{2\})$ and thus $i^*(\ker p_1^*) = \ker p_2^*$. Then Proposition 6 from [11] implies that the spaces $Y_1$ and $Y_2$ are $M_p$-equivalent.

Let $f_1 : Z \to X$ be a mapping such that $f_1(x,1) = x_0$ for each $x \in X_1$ and $f_1(x,2) = x$ for each $x \in X$. Using this mapping we can construct a homeomorphism from $Y_1$ to $X$.

Let $q_1 : X \to X/X_1$ be the quotient mapping, $\widetilde{f}_2 : Z \to X_1 \times \{1\} \oplus (X/X_1) \times \{2\}$ be a mapping such that $\widetilde{f}_1(x_1,1) = (x_1,1)$ for each $x \in X_1$ and $\widetilde{f}_2(x,2) = (q_1(x),2)$ for each $x \in X$. Let

$$Y_2' = X_1 \times \{1\} \oplus (X/X_1) \times \{2\}/\{(x_0,1),(q_1(x_0),2)\}$$

and $q : X_1 \times \{1\} \oplus (X/X_1) \times \{2\} \to Y_2'$ be the quotient mapping. Let $f_2 = q\widetilde{f}_2$. Using this mapping we can construct a homeomorphism from $Y_2$ to $Y_2'$.

Since the space $X_1$ is homeomorphic to the space $T_0X$ and the space $X/X_1$ is homeomorphic to the space $X/T_0X$, we obtain that the space $Y_2'$ is $M_p$-equivalent to the space $T_0X \times \{1\} \vee (X/T_0X) \times \{2\}$. Thus

$$X \overset{M_p}{\sim} Y_1 \overset{M_p}{\sim} Y_2 \overset{M_p}{\sim} Y_2' \overset{M_p}{\sim} T_0X \times \{1\} \vee (X/T_0X) \times \{2\}.$$

Let $X$ be a pseudometrizable space, and $d$ be a pseudometric generating the topology of $X$. Then one can easily check that $T_0X$ is a metrizable space.

**Corollary 8.** *Each pseudometrizable space is $M_p$-equivalent to the bouquet of metrizable and antidiscrete spaces.*

**Proposition 5.** *Let $X_1$ and $X_2$ be spaces with topologically isomorphic Graev free paratopological groups, $Y_1$ and $Y_2$ be spaces with topologically isomorphic Markov free paratopological groups. If $X_i \cap Y_i = \varnothing$ for $i \in \{1,2\}$ then Graev free paratopological groups on spaces $X_1 \oplus Y_1$ and $X_2 \oplus Y_2$ are topologically isomorphic.*

*Proof.* Let $i\colon FG_p(X_1) \to FG_p(X_2)$ be an isomorphism of the Graev free paratopological groups with distinguished points $a_i \in X_i$, $i = 1,2$, $j\colon F_p(Y_1) \to F_p(Y_2)$ be an isomorphism of the Markov free paratopological groups.

Let $t \in \{1,2\}$. Let $i_{Xt} : X_t \to X_t \oplus Y_t$ and $i_{Yt} : Y_t \to X_t \oplus Y_t$ be the identity embeddings, and $i^*_{Xt} : FG_p(X_t) \to FG_p(X_t \oplus Y_t, a_t)$ and $i^*_{Yt} : F_p(Y_t) \to FG_p(X_t \oplus Y_t, a_t)$ be their extensions to the continuous homomorphisms of paratopological groups.

Consider the mapping $k\colon X_1 \oplus Y_1 \to FG_p(X_2 \oplus Y_2)$ defined as $k(z) = i^*_{X2}i(z)$ if $z \in X_1$ and $k(z) = i^*_{Y2}j(z)$, if $z \in Y_1$. Similarly to [4, Pr. 8.8] one can check that the extension of the mapping $k$ to the continuous homomorphism $k^*\colon FG_p(X_1 \oplus Y_1) \to FG_p(X_2 \oplus Y_2)$ is a topological isomorphism of the Graev free paratopological groups $FG(X_1 \oplus Y_1)$ and $FG(X_2 \oplus Y_2)$ with the distinguished points $a_i \in X_i \oplus Y_i$.

**Proposition 6.** *Let $X_1$ and $X_2$ be spaces with topologically isomorphic Graev free abelian paratopological groups, $Y_1$ and $Y_2$ spaces with topologically isomorphic Markov free abelian paratopological groups. If $X_i \cap Y_i = \varnothing$ for $i \in \{1,2\}$ then Graev free abelian paratopological groups on spaces $X_1 \oplus Y_1$ and $X_2 \oplus Y_2$ are topologically isomorphic.*

*Proof.* The proof is similar to the proof of the previous proposition.

**Corollary 9.** *Let $X_1$ and $X_2$ be nonempty topological spaces with topologically isomorphic Markov free paratopological groups, $Y$ be a nonempty topological space such that*

*$Y \cap (X_1 \cup X_2) = \varnothing$. Then Markov free paratopological groups on spaces $X_1 \vee Y$ and $X_2 \vee Y$ are topologically isomorphic.*

*Proof.* By Proposition 5 we have that Graev free paratopological groups on the spaces $X_1 \oplus Y$ and $X_2 \oplus Y$ are topologically isomorphic. Similarly to [3, §5] one can check that Graev free paratopological groups on the spaces $X_i \oplus Y$ and $(X_i \vee Y)^+$ are topologically isomorphic. Since Graev free paratopological group on the space $X^+$ is naturally isomorphic to the Markov free paratopological group on the space $X$,

$$F_p(X_1 \vee Y) \simeq FG_p((X_1 \vee Y)^+) \simeq FG_p(X_1 \oplus Y) \simeq$$
$$\simeq FG_p(X_2 \oplus Y) \simeq FG_p((X_2 \vee Y)^+) \simeq F_p(X_2 \vee Y).$$

**Corollary 10.** *Let $X_1$ and $X_2$ be nonempty topological spaces with topologically isomorphic Markov free abelian paratopological groups, $Y$ be a nonempty topological space such that $Y \cap (X_1 \cup X_2) = \varnothing$. Then Markov free abelian paratopological groups on spaces $X_1 \vee Y$ and $X_2 \vee Y$ are topologically isomorphic.*

*Proof.* The proof is similar to the proof of the previous corollary.

**Theorem 5.** *Topological spaces $X$ and $Y$ are $A_p$-equivalent if and only if $T_0X \overset{A_p}{\sim} T_0Y$ and $X/T_0X = Y/T_0Y$.*

*Proof.* Without loss of the generality it suffices to consider only the case $X \neq \varnothing$ and $Y \neq \varnothing$.

Sufficiency. Since $A_p(T_0X) \simeq A_p(T_0Y)$ and $X/T_0X = Y/T_0Y$, Corollary 10 implies that $A_p(T_0X \times \{1\} \vee (X/T_0X) \times \{2\}) \simeq A_p(T_0Y \times \{1\} \vee (Y/T_0Y) \times \{2\})$. Since the $M_p$-equivalence of two spaces implies the $A_p$-equivalence,

$$A_p(X) \simeq A_p(T_0X \times \{1\} \vee (X/T_0X) \times \{2\})$$

and $A_p(Y) \simeq A_p(T_0Y \times \{1\} \vee (Y/T_0Y) \times \{2\})$ by proposition 4. Thus

$$X \overset{A_p}{\sim} T_0X \times \{1\} \vee (X/T_0X) \times \{2\} \overset{A_p}{\sim} T_0Y \times \{1\} \vee (Y/T_0Y) \times \{2\} \overset{A_p}{\sim} Y.$$

Necessity. Let $X$ and $Y$ be $A_p$-equivalent. Then Corollary 3 implies that $T_0X \overset{A_p}{\sim} T_0Y$. Theorem 3 implies that the quotient mappings $t_X \colon X \to T_0X$ and $t_Y \colon Y \to T_0Y$ be $A_p$-equivalent. Since $\ker t_X^*$ is an algebraically free abelian group on the set of generators with cardinality $X/T_0X$, $X/T_0X = 1 + \operatorname{rank} \ker t_X^* = 1 + \operatorname{rank} \ker t_Y^* = Y/T_0Y$.

**Theorem 6.** *Topological spaces $X$ and $Y$ are $M_p$-equivalent if and only if $T_0X \overset{M_p}{\sim} T_0Y$ and $X/T_0X = Y/T_0Y$.*

*Proof.* The proof of the necessity is similar to the abelian case. Let us prove the sufficiency. Let $X$ and $Y$ be $M_p$-equivalent. Then Corollary 1 implies that $T_0X \overset{M_p}{\sim} T_0Y$. Since the spaces $X$ and $Y$ are $A_p$-equivalent, Theorem 5 implies that $X/T_0X = Y/T_0Y$.

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## ІЗОМОРФІЗМИ ВІЛЬНИХ ПАРАТОПОЛОГІЧНИХ ГРУП І ВІЛЬНИХ ОДНОРІДНИХ ПРОСТОРІВ II


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Доведено, що вільна паратопологічна група $T_0$-простору є $T_0$-простором. Подано приклади функторів, які зберігають ізоморфізми вільних (абелевих) паратопологічних груп і вільних однорідних просторів. Також наведено метод зведення ізоморфної класифікації вільних (абелевих) паратопологічних груп над топологічними просторами до ізоморфної класифікації вільних (абелевих) паратопологічних груп над $T_0$-просторами.



*Ключові слова:* вільна паратопологічна група, вільний однорідний простір, ізоморфізм паратопологічних груп, ізоморфізм однорідних просторів.

# ИЗОМОРФИЗМЫ СВОБОДНЫХ ПАРАТОПОЛОГИЧЕСКИХ ГРУПП И СВОБОДНЫХ ОДНОРОДНЫХ ПРОСТРАНСТВ II

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Доказано, что свободная паратопологическая группа $T_0$-пространства является $T_0$-пространством. Рассмотрено функторы, сохраняющие изоморфизмы свободных (абелевых) паратопологических групп и свободных однородных пространств.

*Ключевые слова:* свободная паратопологическая группа, свободное однородное пространство, изоморфизм паратопологических групп, изоморфизм однородных пространств.

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