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THE MIXED PROBLEM FOR A NONLINEAR COUPLED EVOLUTION SYSTEM IN A BOUNDED DOMAIN

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The purpose of this paper is to establish existence and uniqueness of solution of a nonlinear coupled system with variable coefficients of a nonlinear equation with the second order time derivative and a nonlinear equation with the first time derivative in a bounded domain of $\mathbb{R}^n$ with smooth boundary.

Key words: nonlinear system, mixed problem, Faedo-Galerkin method.

1. Problems for evolution system with the time variable derivatives of different orders is considered by many authors [1], [2]. Specifically, in domains which are bounded by spatial variables, the mixed problems for semilinear evolution coupled system with internal damping are quite well studied in [1] and there are established existence, uniqueness, and asymptotic behavior of solutions. Messaoudi [3] considered a multidimensional semilinear system of thermoelasticity and showed that the energy of any weak solution blows up in finite time if the initial energy is negative. Clark and Lima in [4] studied the existence of weak solutions of the nonlinear unilateral mixed problem.

The non-linearity $|v|^\rho v$ usually appears in relativistic quantum mechanic (see Schiff [5] or Segal [6]), and has been considered by various authors for hyperbolic, parabolic and elliptic equations. Lions [7] studied the wave equation with the same non-linearity, i.e., $|v|^\rho v$, in a smooth-bounded-open domain Ω of $\mathbb{R}^n$ with $n \in \mathbb{N}$ and proved existence and uniqueness of solution using Faedo-Galerkin's and compactness' methods.

Clark at al [2] investigated system (2.1)-(2.2) with equal to zero and feedback-homogeneous conditions over a part of the boundary. They established global existence of strong and weak solutions by Faedo-Galerkin's method using a particular basis of the space $H^1_0(\Omega) \cap H^2(\Omega)$ introduced by Medeiros & Milla Miranda in [9] and the exponential stability of total energy associated to the weak solution using Komornik-Zuazua's method [10].

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In this article we study the initial boundary value problem for nonlinear evolution coupled system. Based in the theory developed in the papers of Clark at al [2] and Lions [7] (Theorems 1.1, 1.2 and 1.3), we will prove that problem (2.1)-(2.4) has a unique solution.

The outline of this article is as follows. In Section 2 the basic notations are laid out and existence of solution of nonlinear evolution coupled system in bounded domain are issued, existence and uniqueness of problem is aired in Sections 3, 4.

2. Problem formulation. Let Ω be any bounded domain in $\in \mathbb{R}^n$ with regular in Calderon's sense [11] boundary $\partial\Omega$ and let T be a positive number, $Q_T = (0,T)\times\Omega$, $0 < < T < +\infty$, $Q_{t_1,t_2} = (t_1,t_2)\times\Omega$, $\{t_1,t_2\}\in[0,T]$, $t_1<t_2$; $Q_\tau = Q_{0,\tau}$; $\Omega_\tau = \{t=\tau\}\cap Q_T$; $S_T = (0,T)\times\partial\Omega$.

We will consider the following problem in the domain Q_T :

$$u_{tt}(t,x) - \sum_{i,j=1}^{n}(a_{ij}u_{x_i}(t,x))_{x_j} + \sum_{i=1}^{n}a_i(t,x)u_{x_i}(t,x) + \sum_{i=1}^{n}b_i(t,x)\theta_{x_i} + \\ + \alpha_0(t,x)u(t,x) + \alpha_1(t,x)\theta(t,x) + \gamma_1(t,x)|u_t|^{p-2}u_t = f_1(t,x), \tag{2.1}$$

$$\theta_t(t,x) - \sum_{i,j=1}^{n}(c_{ij}\theta_{x_i}(t,x))_{x_j} - \sum_{i=1}^{n}(c_i(t,x)u_t)_{x_i} + \sum_{i=1}^{n}d_i(t,x)u_{x_i}(t,x) + \\ + \sum_{i=1}^{n}e_i(t,x)\theta_{x_i} + \beta_0(t,x)u(t,x) + \beta_1(t,x)\theta(t,x) + \gamma_2(t,x)|\theta|^{q-2}\theta = f_2(t,x) \tag{2.2}$$

with boundary

$$u(x,t) = 0, \quad \theta(x,t) = 0 \quad \text{on} \quad S_T \tag{2.3}$$

and initial

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x) \quad \text{on} \quad \Omega \tag{2.4}$$

conditions. Here $p,q\in(2,+\infty)$.

We will make the following assumptions concerning the coefficients, nonhomogeneous terms and initial data of problem (2.1)-(2.4):

(**A**) $\{a_{ij}, a_{ijt}\}\in L^\infty(Q_T)$, $a_{ij}(t,x) = a_{ji}(t,x)$ almost everywhere in Q_T;

$$a_0|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \leqslant a^0|\xi|^2$$

a.e. for $(t,x)\in Q_T$ and for every $\xi\in\mathbb{R}^n$, a_0 and a^0 are positive constants;

(**C**) $\{c_{ij}, c_{ijt}\}\in L^\infty(Q_T)$, $c_{ij}(t,x) = c_{ji}(t,x)$ almost everywhere in Q_T;

$$c_0|\xi|^2 \leqslant \sum_{i,j=1}^{n} c_{ij}(t,x)\xi_i\xi_j \leqslant c^0|\xi|^2$$

a.e. for $(t,x)\in Q_T$ and for every $\xi\in\mathbb{R}^n$, c_0 and c^0 are positive constants;

(D) $\{a_i,\ ,a_{it},\ b_i,\ b_{it},\ c_i,\ c_{it},\ d_i,\ d_{it},\ e_i,\ e_{it}\} \in L^\infty(Q_T)$;

(E) $\alpha_0, \alpha_{0t}, \alpha_1, \beta_0, \beta_1 \in L^\infty(Q_T)$;

(F) $\{f_1, f_2, f_{1t}, f_{2t}\} \in L^2(Q_T)$, $u_0 \in H^2(\Omega_0) \cap H_0^1(\Omega_0)$, $u_1 \in H_0^1(\Omega_0) \cap L^{2(p-1)}(\Omega_0)$, $\theta_0 \in H^2(\Omega_0) \cap H_0^1(\Omega_0) \cap L^{2(q-1)}(\Omega_0)$;

(G) $\gamma_1, \gamma_2 \in L^\infty(Q_T)$; $\gamma_1(t,x) \geqslant \tilde{\gamma}_1, \gamma_2(t,x) \geqslant \tilde{\gamma}_2$ almost everywhere in Q_T, $\tilde{\gamma}_1$ i $\tilde{\gamma}_2$ are positive constants.

Definition 1. *A pair of functions* (u,θ), *which* $u \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap C(0,T;L^2(\Omega))$, $u_t \in L^2(0,T;H_0^1(\Omega)) \cap L^p(Q_T) \cap C(0,T;L^2(\Omega))$, $\theta \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap L^q(Q_T) \cap C(0,T;L^2(\Omega))$, *is said to be a generalized solution of problem (2.1)-(2.4) if* (u,θ) *satisfies*

$$\int\limits_{\Omega_T} u_t v\,dx - \int\limits_{\Omega_0} u_1(x) v\,dx + \int\limits_{Q_T} \Big[-u_t v_t + \sum_{i,j=1}^n a_{ij}(t,x) u_{x_i} v_{x_j} + \sum_{i=1}^n a_i(t,x) u_{x_i} v + \sum_{i=1}^n b_i(t,x)\theta_{x_i} v + \alpha_0(t,x) uv + \alpha_1(t,x)\theta v + \gamma_1(t,x)|u_t|^{p-2} u_t v\Big]\,dx\,dt = \int\limits_{Q_T} f_1(t,x) v\,dx\,dt, \tag{2.5}$$

$$\int\limits_{Q_T} \Big[\theta_t w + \sum_{i,j=1}^n c_{ij}(t,x)\theta_{x_i} w_{x_j} + \sum_{i=1}^n c_i(t,x) u_t w_{x_i} + \sum_{i=1}^n d_i(t,x) u_{x_i} w + \sum_{i=1}^n e_i(t,x)\theta_{x_i} w + \beta_0(t,x) uw + \beta_1(t,x)\theta w + \gamma_2(t,x)|\theta|^{q-2}\theta w\Big]\,dx\,dt = \int\limits_{Q_T} f_2(t,x) w\,dx\,dt \tag{2.6}$$

for all $v, w \in C^\infty(0,T;C_0^\infty(\Omega))$ *and initial conditions* $u(0,x) = u_0(x)$, $\theta(0,x) = \theta_0(x)$.

3. Existence of solution.

Theorem 1. *Let assumptions* **(A)** – **(G)** *hold and* $\{a_{ijtt},\ c_{ijtt},\ \alpha_{1t},\ \beta_{0t},\ \beta_{1t},\ \gamma_{1t},\ \gamma_{1tt},\ \gamma_{2t},\ \gamma_{2tt}\} \in L^\infty(Q_T)$. *Then problem (2.1)-(2.4) has a generalized solution* (u,θ).

Remark 1. Equations (2.1) and (2.2) are given in sense of distributions.

Proof. To show the existence of generalized solution of problem (2.1)-(2.4) we will use the Faedo-Galerkin method. We consider $(w^k)_{k\in\mathbb{N}}$ a complete sequence of linearly independent dense everywhere in $H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2(\max\{p,q\}-1)}(\Omega)$ set of functions ("basis") which are orthonormal in $L^2(\Omega)$. Denote $W_m = [w_1, w_2, \ldots, w_m]$ the subspace of $H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2(\max\{p,q\}-1)}(\Omega)$ spanned by the m first vectors of $(w^k)_{k\in\mathbb{N}}$. Let us consider sequences

$$u^m(t,x)=\sum_{k=1}^{m}c_k^m(t)v^m(x),\qquad \theta^m(t,x)=\sum_{l=1}^{m}z_l^m(t)w^m(x)$$

for $v^k, w^l, k, l = 1,\dots,m$ belonging to W_m. The approximated system associated to system (2.1)-(2.2), where c_k^m and z_l^m are solutions to the Cauchy problem, is given by

$$\int_\Omega \Big[u_{tt}^m v^k+\sum_{i,j=1}^{n}a_{ij}(t,x)u_{x_i}^m v_{x_j}^k+\sum_{i=1}^{n}a_i(t,x)u_{x_i}^m v^k+\sum_{i=1}^{n}b_i(t,x)\theta_{x_i}^m v^k+\alpha_0(t,x)u^m v^k+$$
$$+\alpha_1(t,x)\theta^m v^k+\gamma_1(t,x)|u_t^m|^{p-2}u_t^m v^k\Big]\,dx=\int_\Omega f_1(t,x)v^k\,dx, \tag{3.1}$$

$$\int_\Omega \Big[\theta_t^m w^l+\sum_{i,j=1}^{n}c_{ij}(t,x)\theta_{x_i}^m w_{x_j}^l+\sum_{i=1}^{n}c_i(t,x)u_t^m w_{x_i}^l+\sum_{i=1}^{n}d_i(t,x)u_{x_i}^m w^l+$$
$$+\sum_{i=1}^{n}e_i(t,x)\theta_{x_i}^m w^l+\beta_0(t,x)u^m w^l+\beta_1(t,x)\theta^m w^l+$$
$$+\gamma_2(t,x)|\theta^m|^{q-2}\theta^m w^l\Big]\,dx=\int_\Omega f_2(t,x)w^l\,dx,\quad t\in(0,T), \tag{3.2}$$

$$u^m(0)=u_0^m,\ u_t^m(0)=u_1^m,\ \theta^m(0)=\theta_0^m,\ 1\leqslant k,l\leqslant m, \tag{3.3}$$

where v^k and w^l belong to W_m.

From (3.3) we see that u_0^m, u_1^m, θ_0^m belong to W_m and satisfy

$$u_0^m \longrightarrow u_0 \text{ strongly in } H^2(\Omega_0)\cap H_0^1(\Omega_0), \tag{3.4}$$
$$\theta_0^m \longrightarrow \theta_0 \text{ strongly in } H^2(\Omega_0)\cap H_0^1(\Omega_0)\cap L^{2(q-1)}(\Omega_0), \tag{3.5}$$
$$u_1^m \longrightarrow u_1 \text{ strongly in } H_0^1(\Omega_0)\cap L^{2(p-1)}(\Omega_0). \tag{3.6}$$

Thus, from the conditions of the theorem we can conclude that the functions (u,θ) satisfy the conditions of Caratheodory Theorem [12, p.54]. Then there exists continuously differentiable solution of problem (3.1)-(3.3) which is determined in some interval $[0,t^m]$, $t^m\leqslant T$ and has absolutely continuous derivative. This interval will be extended to any interval $(0,T)$ thanks to the first estimate below.

Estimate I. We will multiply equation (3.1) by the functions $c_{kt}^m(t)$ and equation (3.2) by the functions $z_l^m(t)$ respectively, summing over k and l from 1 to m respectively and integrating over t from 0 to τ, $0<\tau<t^m$. We get

$$\int\limits_{Q_\tau}\Big[u_{tt}^m u_t^m+\sum_{i,j=1}^n a_{ij}(t,x)u_{x_i}^m u_{tx_j}^m+\sum_{i=1}^n a_i(t,x)u_{x_i}^m u_t^m+\sum_{i=1}^n b_i(t,x)\theta_{x_i}^m u_t^m+$$
$$+\alpha_0(t,x)u^m u_t^m+\alpha_1(t,x)\theta^m u_t^m+\gamma_1(t,x)|u_t^m|^p\Big]\,dx\,dt=\int\limits_{Q_\tau} f_1 u_t^m\,dx\,dt, \qquad (3.7)$$

$$\int\limits_{Q_\tau}\Big[\theta_t^m\theta^m+\sum_{i,j=1}^n c_{ij}(t,x)\theta_{x_i}^m\theta_{x_j}^m+\sum_{i=1}^n c_i(t,x)u_t^m\theta_{x_i}^m+\sum_{i=1}^n d_i(t,x)u_{x_i}^m\theta^m+$$
$$+\sum_{i=1}^n e_i(t,x)\theta_{x_i}^m\theta^m+\beta_0(t,x)u^m\theta^m+\beta_1(t,x)\theta^m\theta^m+$$
$$+\gamma_2(t,x)|\theta^m|^q\Big]\,dx\,dt=\int\limits_{Q_\tau} f_2\theta^m\,dx\,dt. \qquad (3.8)$$

Let us transform and establish estimates for every term in (3.7) using assumptions of the Theorem 1. It is easy to show that

$$I_1^a=\int\limits_{Q_\tau} u_{tt}^m u_t^m\,dx\,dt=\frac{1}{2}\int\limits_{\Omega_\tau}|u_t^m|^2\,dx-\frac{1}{2}\int\limits_{\Omega_0}|u_1^m|^2\,dx.$$

By assumption (**A**), we have

$$I_2^a=\int\limits_{Q_\tau}\sum_{i,j=1}^n a_{ij}(t,x)u_{x_i}^m u_{x_jt}^m\,dx\,dt\geqslant\frac{a_0}{2}\int\limits_{\Omega_\tau}|\nabla u^m|^2\,dx-\frac{a^0}{2}\int\limits_{\Omega_0}|\nabla u_0^m|^2\,dx-$$
$$-\frac{a^1}{2}\int\limits_{Q_\tau}|\nabla u^m|^2\,dx\,dt,\qquad \text{where } a^1=\operatorname*{ess\,sup}_{Q_T}\sum_{i,j=1}^n|a_{ijt}(t,x)|^2.$$

From (**D**) we obtain

$$I_3^a=\int\limits_{Q_\tau}\sum_{i=1}^n a_i(t,x)u_{x_i}^m u_t^m\,dx\,dt\leqslant\frac{1}{2}\int\limits_{Q_\tau}\left[\nu_a\delta_0^a|\nabla u^m|^2+\frac{1}{\delta_0^a}|u_t^m|^2\right]\,dx\,dt,$$
$$I_4^a=\int\limits_{Q_\tau}\sum_{i=1}^n b_i(t,x)\theta_{x_i}^m u_t^m\,dx\,dt\leqslant\frac{1}{2}\int\limits_{Q_\tau}\left[\nu_b\delta_1^a|\nabla\theta^m|^2+\frac{1}{\delta_1^a}|u_t^m|^2\right]\,dx\,dt,$$

where $\nu_a=\max\limits_i \operatorname{ess\,sup}_{Q_T}|a_i(t,x)|^2$, $\nu_b=\max\limits_i \operatorname{ess\,sup}_{Q_T}|b_i(t,x)|^2$, $\delta_0^a>0$, $\delta_1^a>0$ — any constants.

By assumption (**E**) we find

$$I_5^a = \int\limits_{Q_\tau} \alpha_0(t,x) u^m u_t^m \, dx\, dt \geqslant \frac{1}{2} \int\limits_{Q_\tau} \left(2\nu_{\alpha_0} \delta_2^a T^2 + \frac{1}{\delta_2^a} \right) |u_t^m|^2 \, dx\, dt +$$
$$+ \nu_{\alpha_0} \delta_2^a T \int\limits_{\Omega_0} |u_0^m| \, dx, \quad \text{where } \nu_{\alpha_0} = \operatorname*{ess\,sup}_{Q_T} |\alpha_0(t,x)|^2, \ \delta_2^a > 0,$$

since

$$\int\limits_{Q_\tau} u^2(t,x)\, dx\, dt \leqslant 2T \int\limits_{\Omega_0} u_0^2(x)\, dx + 2T^2 \int\limits_{Q_\tau} u_t^2(t,x)\, dx\, dt. \tag{3.9}$$

Next

$$I_6^a = \int\limits_{Q_\tau} \alpha_1(t,x) \theta^m u_t^m \, dx\, dt \leqslant \frac{1}{2} \int\limits_{Q_\tau} \left[\nu_{\alpha_1} \delta_3^a |u_t^m|^2 + \frac{1}{\delta_3^a} |\theta^m|^2 \right] dx\, dt,$$

where $\nu_{\alpha_1} = \operatorname{ess\,sup}_{Q_T} |\alpha_1(t,x)|^2$, $\delta_3^a > 0$.

From (**F**), (**G**) we get

$$I_7^a = \int\limits_{Q_\tau} \gamma_1(t,x) |u_t^m|^p \, dx\, dt \geqslant \tilde{\gamma}_1 \int\limits_{Q_\tau} |u_t^m|^p \, dx\, dt,$$
$$I_8^a = \int\limits_{Q_\tau} f_1 u_t^m \, dx\, dt \leqslant \frac{1}{2} \int\limits_{Q_\tau} \left[\delta_4^a |f_1|^2 + \frac{1}{\delta_4^a} |u_t^m|^2 \right] dx\, dt, \delta_4^a > 0.$$

Constants $\delta_k^a > 0,\ k = 0, \ldots, 4$.

Estimates of terms $1, 3-5, 7, 8$ from (3.8) can be obtained similarly to the estimates $I_1^a, I_3^a - I_5^a, I_7^a, I_8^a$. Making use of (**C**) we conclude that

$$I_2^c = \int\limits_{Q_\tau} \sum_{i,j=1}^{n} c_{ij}(t,x) \theta_{x_i}^m \theta_{x_j}^m \, dx\, dt \geqslant c_0 \int\limits_{Q_\tau} |\nabla \theta^m|^2 \, dx\, dt.$$

From assumption (**E**) and inequality (3.9) we get

$$I_6^c = \int\limits_{Q_\tau} \beta_0(t,x) u^m \theta^m \, dx\, dt \leqslant \nu_{\beta_0} \delta_3^c T^2 \int\limits_{Q_\tau} |u_t^m|^2 \, dx\, dt +$$
$$+ \frac{1}{2\delta_3^c} \int\limits_{Q_\tau} |\theta^m|^2 \, dx\, dt + \nu_{\beta_0} \delta_3^c T \int\limits_{\Omega_0} |u_0^m| \, dx,$$

where $\delta_3^c > 0$, $\nu_{\beta_0} = \operatorname{ess\,sup}_{Q_T} |\beta_0(t,x)|^2$.

Putting the estimates of all terms into (3.7)-(3.8) we obtain inequality

$$\int\limits_{\Omega_\tau}\left[|u^m|^2+|u_t^m|^2+a_0|\nabla u^m|^2+|\theta^m|^2\right]\,dx+$$
$$+(2c_0-\nu_b\delta_1^a-\nu_c\delta_0^c-\nu_e\delta_2^c)\int\limits_{Q_\tau}|\nabla\theta^m|^2\,dx\,dt+\tilde\gamma_1\int\limits_{Q_\tau}|u_t^m|^p\,dx\,dt+$$
$$+\tilde\gamma_2\int\limits_{Q_\tau}|\theta^m|^q\,dx\,dt\leqslant\int\limits_{\Omega_0}\Big[(2\nu_{\alpha_0}\delta_2^aT+2\nu_{\beta_0}\delta_3^cT)|u_0^m|^2+|u_1^m|^2+$$
$$+a^0|\nabla u_0|^2+|\theta_0^m|^2\Big]\,dx+\int\limits_{Q_\tau}\left[\delta_4^a|f_1|^2+\delta_4^c|f_2|^2\right]\,dx\,dt+$$
$$+\int\limits_{Q_\tau}\Bigg[2T|u^m|^2+\left(\nu_{\alpha_0}\delta_2^aT^2+\nu_{\beta_0}\delta_3^cT^2+\nu_{\alpha_1}\delta_3^a+\sum_{k=0}^{3}\frac{1}{\delta_k^a}+\frac{1}{\delta_0^c}\right)|u_t^m|^2+$$
$$+\left(a^1+\nu_a\delta_0^a+\nu_d\delta_1^c\right)|\nabla u^m|^2+\left(\frac{1}{\delta_3^a}+\sum_{k=1}^{4}\delta_k^c\right)|\theta^m|^2\Bigg]\,dx\,dt, \qquad (3.10)$$

where $\delta_0^a-\delta_4^a$ and $\delta_0^c-\delta_4^c$ are any positive constants. Now choose constants δ_1^a, δ_0^c, δ_2^c in such a way that the following condition holds: $2c_0-\nu_b\delta_1^a-\nu_c\delta_0^c-\nu_e\delta_2^c>0$.

Thus applying Gronwall's lemma from (3.10) we obtain

$$\int\limits_{\Omega_\tau}\left[|u^m|^2+|u_t^m|^2+|\nabla u^m|^2+|\theta^m|^2\right]\,dx+\int\limits_{Q_\tau}|\nabla\theta^m|^2\,dx\,dt+$$
$$+\tilde\gamma_1\int\limits_{Q_\tau}|u_t^m|^p\,dx\,dt+\tilde\gamma_2\int\limits_{Q_\tau}|\theta^m|^q\,dx\,dt\leqslant C_1\Bigg[\int\limits_{Q_\tau}\left[|f_1|^2+|f_2|^2\right]\,dx\,dt+$$
$$+\int\limits_{\Omega_0}\left[|u_0^m|^2+|u_1^m|^2+|\nabla u_0^m|^2+|\theta_0^m|\right]\,dx\Bigg]\leqslant C,\quad \tau\in(0,T), \qquad (3.11)$$

where C_1, C are positive constants independent of m.

Next, in system (3.1)-(3.2) we will multiply the first equation (3.1) by $c_{ktt}^m(0)$, and the second equation (3.2) by $z_{lt}^m(0)$ as $t=0$. Because of the choice of "basis" we have

$$\int\limits_{\Omega_0}|u_{tt}^m(0,x)|^2\,dx\leqslant C_2\int\limits_{\Omega_0}\Bigg(a^0\sum_{i=1}^{n}|u_{0x_ix_i}^m|^2+\nu_a\sum_{i=1}^{n}|u_{0x_ix_i}^m|^2+\nu_b|\nabla\theta_0^m|^2+$$
$$+\nu_{\alpha_0}|u_0^m|^2+\nu_{\alpha_1}|\theta_0^m|^2+\tilde\gamma_1|u_1^m|^{2(p-1)}+|f_1(0,x)|^2\Bigg)\,dx\leqslant C_3, \qquad (3.12)$$

$$\int\limits_{\Omega_0} |\theta_t^m(0,x)|^2\,dx \leqslant C_4 \int\limits_{\Omega_0} \Bigg(c^0 \sum_{i=1}^n |\theta^m_{0x_ix_i}|^2 + \mu_c |\nabla u_1^m|^2 + \nu_d |\nabla u_0^m|^2 + \nu_e |\nabla \theta_0^m|^2 + $$

$$+ \nu_{\beta_0} |u_0^m|^2 + \nu_{\beta_1} |\theta_0^m|^2 + \tilde{\gamma}_2 |\theta_0^m|^{2(q-1)} + |f_2(0,x)|^2 \Bigg)\,dx \leqslant C_5, \tag{3.13}$$

where $C_2,\ C_4$ are positive constants independent of m, and from (3.4)-(3.6) and system (3.1)-(3.2) there are positive constants $C_3,\ C_5$ independent of m such that right-hand-side integrals of (3.12), (3.13) are bounded.

Estimate II. Differentiating equations (3.1) and (3.2) with respect to t, multiplying the first equations (3.1) by $c^m_{ktt}(t)$ and the second equations (3.2) by $z^m_{lt}(t)$ respectively, summing over k and l respectively, integrating over t from 0 to τ, $0<\tau<t^m\leqslant T$, we get

$$\int\limits_{Q_\tau} \Bigg[u^m_{ttt}u^m_{tt} + \sum_{i,j=1}^n a_{ij}(t,x) u^m_{tx_i} u^m_{ttx_j} + \sum_{i,j=1}^n a_{ijt}(t,x) u^m_{x_i} u^m_{ttx_j} + \sum_{i=1}^n a_i(t,x) u^m_{tx_i} u^m_{tt} + $$

$$+ \sum_{i=1}^n a_{it}(t,x) u^m_{x_i} u^m_{tt} + \sum_{i=1}^n b_i(t,x)\theta^m_{tx_i} u^m_{tt} + \sum_{i=1}^n b_{it}(t,x)\theta^m_{x_i} u^m_{tt} +$$

$$+ \alpha_0(t,x) u^m_t u^m_{tt} + \alpha_{0t}(t,x) u^m u^m_{tt} + \alpha_1(t,x)\theta^m_t u^m_{tt} + \alpha_{1t}(t,x)\theta^m u^m_{tt} +$$

$$+ (p-1)\gamma_1(t,x)|u^m_t|^{p-2}(u^m_{tt})^2 + \gamma_{1t}(t,x)|u^m_t|^{p-1}u^m_t u^m_{tt} -$$

$$- f_{1t}(t,x) u^m_{tt} \Bigg]\,dx\,dt = 0, \tag{3.14}$$

$$\int\limits_{Q_\tau} \Bigg[\theta^m_{tt}\theta^m_t + \sum_{i,j=1}^n c_{ij}(t,x)\theta^m_{tx_i}\theta^m_{tx_j} + \sum_{i,j=1}^n c_{ijt}(t,x)\theta^m_{x_i}\theta^m_{tx_j} + \sum_{i=1}^n c_i(t,x) u^m_{tt}\theta^m_{tx_i} +$$

$$+ \sum_{i=1}^n c_i(t,x) u^m_t \theta^m_{tx_i} + \sum_{i=1}^n d_i(t,x) u^m_{tx_i}\theta^m_t + \sum_{i=1}^n d_{it}(t,x) u^m_{x_i}\theta^m_t +$$

$$+ \sum_{i=1}^n e_i(t,x)\theta^m_{tx_i}\theta^m_t + \sum_{i=1}^n e_{it}(t,x)\theta^m_{x_i}\theta^m_t + \beta_0(t,x) u^m_t\theta^m_t + \beta_{0t}(t,x) u^m\theta^m_t +$$

$$+ \beta_1(t,x)|\theta^m_t|^2 + \beta_{1t}(t,x)\theta^m\theta^m_t + (q-1)\gamma_2(t,x)|\theta^m|^{q-2}(\theta^m_t)^2 +$$

$$+ \gamma_{2t}(t,x)|\theta^m|^{q-1}\theta^m\theta^m_t - f_{2t}(t,x)\theta^m_t \Bigg]\,dx\,dt = 0. \tag{3.15}$$

Now, we are going to make transform on the all terms of equations (3.14) -(3.15) using assumptions $(\mathbf{A})-(\mathbf{G})$ and conditions of Theorem 1. Owing to the fact that (3.11), (3.12) estimates of terms $2, 4-11, 14$ for equation (3.14) can be obtained similarly to the estimates from equation (3.7). Estimates of other terms of system (3.14) are

$$J_1^a = \int\limits_{Q_\tau} u_{ttt}^m u_{tt}^m \, dx\, dt = \frac{1}{2}\int\limits_{\Omega_\tau} |u_{tt}^m|^2 \, dx - \frac{1}{2}\int\limits_{\Omega_0} |u_{tt}^m(0,x)|^2 \, dx,$$

$$J_3^a = \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ijt}(t,x) u_{x_i}^m u_{ttx_j}^m \, dx\, dt \geqslant \frac{1}{2}\int\limits_{\Omega_0} \left[\nu_a^1 |\nabla u_1^m|^2 + a^1 |\nabla u_0^m|\right] dx -$$

$$- a^1 \int\limits_{Q_\tau} |\nabla u_t^m|^2 \, dx\, dt - \frac{1}{2}\int\limits_{Q_\tau} \left[a^2 \delta_5^a |\nabla u_t^m|^2 + \frac{1}{\delta_5^a}|\nabla u^m|^2\right] dx\, dt,$$

$$\nu_a^1 = \operatorname*{ess\,sup}_{\Omega_0} \sum_{i,j=1}^{n} |a_{ijt}(0,x)|^2, \quad a^2 = \operatorname*{ess\,sup}_{Q_T} \sum_{i,j=1}^{n} |a_{ijtt}(t,x)|^2, \quad \delta_5^a > 0,$$

$$J_{12}^a = (p-1)\int\limits_{Q_\tau} \gamma_1(t,x) |u_t^m|^{p-2} (u_{tt}^m)^2 \, dx\, dt \geqslant$$

$$\geqslant \tilde{\gamma}_1 \frac{4(p-1)}{p^2} \int\limits_{Q_\tau} \left[\left(\frac{d}{dt}\left(|u_t^m|^{\frac{p-2}{2}} u_t^m\right)\right)^2\right] dx\, dt,$$

$$J_{13}^a = \int\limits_{Q_\tau} \gamma_{1t}(t,x) |u_t^m|^{p-2} u_t^m u_{tt}^m \, dx\, dt \geqslant \frac{\mu_{\gamma_1}}{p} \int\limits_{\Omega_\tau} |u_t^m|^p \, dx - \frac{\nu_{\gamma_1}^1}{p}\int\limits_{\Omega_0} |u_1^m|^p \, dx -$$

$$- \frac{\mu_{\gamma_1}^1}{p}\int\limits_{Q_\tau} |u_t^m|^p \, dx\, dt, \quad \nu_{\gamma_1}^1 = \operatorname*{ess\,sup}_{\Omega_0} |\gamma_{1t}(0,x)|^2, \ \mu_{\gamma_1}^1 = \operatorname*{ess\,sup}_{Q_T} |\gamma_{1tt}(t,x)|^2.$$

In view of the fact that (3.11) and (3.13), estimates of terms $4-13, 16$ from (3.15) can be obtained similarly to the estimates from equality (3.8). In fact, we can easily see that estimates of other terms from equality (3.15)

$$J_1^c = \int\limits_{Q_\tau} \theta_{tt}^m \theta_t^m \, dx\, dt = \frac{1}{2}\int\limits_{\Omega_\tau} |\theta_t^m|^2 \, dx - \frac{1}{2}\int\limits_{\Omega_0} |\theta_t^m(0,x)|^2 \, dx,$$

$$J_2^c = \int\limits_{Q_\tau} \sum_{i,j=1}^{n} c_{ij}(t,x) \theta_{tx_i}^m \theta_{tx_j}^m \, dx\, dt \geqslant c_0 \int\limits_{Q_\tau} |\nabla \theta_t^m|^2 \, dx\, dt,$$

$$J_3^c = \int\limits_{Q_\tau} \sum_{i,j=1}^{n} c_{ijt}(t,x)\theta^m_{x_i}\theta^m_{tx_j}\, dx\, dt \geqslant \frac{c^1}{2}\int\limits_{\Omega_\tau} |\nabla\theta^m|^2\, dx - \frac{\nu_c^1}{2}\int\limits_{\Omega_0} |\nabla\theta_0^m|^2\, dx\, dt -$$
$$- \frac{c^2}{2}\int\limits_{Q_\tau} |\nabla\theta^m|^2\, dx\, dt, \quad \nu_c^1 = \operatorname*{ess\,sup}_{\Omega_0} \sum_{i,j=1}^{n} |c_{ijt}(0,x)|^2,$$
$$c^1 = \operatorname*{ess\,sup}_{Q_T} \sum_{i,j=1}^{n} |c_{ijt}(t,x)|^2, \quad c^2 = \operatorname*{ess\,sup}_{Q_T} \sum_{i,j=1}^{n} |c_{ijtt}(t,x)|^2.$$

Also observe that

$$J_{14}^c = (q-1)\int\limits_{Q_\tau} \gamma_2(t,x)|\theta^m|^{q-2}(\theta_t^m)^2\, dx\, dt \geqslant$$
$$\geqslant \tilde{\gamma}_2 \frac{4(q-1)}{q^2}\int\limits_{Q_\tau} \left[\left(\frac{d}{dt}\left(|\theta^m|^{\frac{q-2}{2}}\theta^m\right)\right)^2\right] dx\, dt,$$
$$J_{15}^c = \int\limits_{Q_\tau} \gamma_{2t}(t,x)|\theta^m|^{q-2}\theta^m\theta_t^m\, dx\, dt \geqslant \frac{\mu_{\gamma_2}}{q}\int\limits_{\Omega_\tau} |\theta^m|^q\, dx - \frac{\nu^1_{\gamma_2}}{q}\int\limits_{\Omega_0} |\theta_0^m|^q\, dx -$$
$$- \frac{\mu^1_{\gamma_2}}{q}\int\limits_{Q_\tau} |\theta^m|^q\, dx\, dt, \quad \nu^1_{\gamma_2} = \operatorname*{ess\,sup}_{\Omega_0} |\gamma_{2t}(0,x)|^2,\ \mu^1_{\gamma_2} = \operatorname*{ess\,sup}_{Q_T} |\gamma_{2tt}(t,x)|^2.$$

Putting the above estimations into (3.14)-(3.15), we obtain inequality

$$\int\limits_{\Omega_\tau} \Big[|u_{tt}^m|^2 + a_0|\nabla u_t^m|^2 + \frac{2\mu_{\gamma_1}}{p}|u_t^m|^p + c^1|\nabla\theta^m|^2 + |\theta_t^m|^2 + \frac{2\mu_{\gamma_2}}{q}|\theta^m|^q\Big]\, dx +$$
$$+ (2c_0 - \nu_b\delta_6^a - \nu_c\delta_5^c - \mu_c\delta_6^c - \nu_e\delta_7^c)\int\limits_{Q_\tau} |\nabla\theta_t^m|^2\, dx\, dt +$$
$$+ \tilde{\gamma}_1 \frac{8(p-1)}{p^2}\int\limits_{Q_\tau} \left(\frac{d}{dt}\left(|u_t^m|^{\frac{p-2}{2}}u_t^m\right)\right)^2 dx\, dt +$$
$$+ \tilde{\gamma}_2 \frac{8(q-1)}{q^2}\int\limits_{Q_\tau} \left(\frac{d}{dt}\left(|\theta^m|^{\frac{q-2}{2}}\theta^m\right)\right)^2 dx\, dt \leqslant C_6 +$$
$$+ \int\limits_{Q_\tau} \left[\delta_{10}^a|f_{1t}|^2 + \delta_{10}^c|f_{2t}|^2\right] dx\, dt + \int\limits_{\Omega_0} \Big(|u_{tt}^m(0,x)|^2 + a^0|\nabla u_1^m|^2 + \nu_{\alpha_0}\delta_8^a|u_1^m|^2 +$$

$$+ a^1|\nabla u_0^m|^2 + \frac{2\nu_{\gamma_1}^1}{p}|u_1^m|^p + |\theta_t^m(0,x)|^2 + \nu_{\beta_1}^1|\theta_0^m|^2 + \frac{2\nu_{\gamma_2}^1}{q}|\theta_0^m|^q\Big)\,dx+$$
$$+\int\limits_{Q_\tau}\Bigg[\left(\delta^a + \nu_{\alpha_0}\delta_8^a + \frac{1}{\delta_4^c}\right)|u_{tt}^m|^2 + (a^2\delta_3^a + 2a^1 + \nu_a\delta_4^a + \nu_d\delta_6^c)|\nabla u_t^m|^2+$$
$$+ (\nu_{\beta_0}\delta_9^c + \frac{1}{\delta_5^c})|u_t^m|^2 + \frac{2\mu_{\gamma_1}^1}{p}|u_t^m|^p + +(\mu_b\delta_7^a + c^2 + \mu_c\delta_8^c)|\nabla\theta^m|^2+$$
$$+ (\delta^c + 2\nu_{\beta_1} + \nu_{\alpha_1}\delta_9^a)|\theta_t^m|^2 + \frac{2\mu_{\gamma_2}^1}{q}|\theta^m|^q\Bigg]\,dx\,dt, \tag{3.16}$$

where $C_6,\ \delta_3^a > 0,\ \delta_4^a > 0,\ \delta_k^a > 0\,(k = 6,\dots,10),\ \delta_l^c > 0,\ (l = 4,\dots,10),\ \delta^a > 0,\ \delta^c > 0$ and constants $\nu_{\beta_1},\ \nu_d,\ \mu_b,\ \mu_c$ dependent on functions $\beta_1,\ d_i,\ ,b_{it},\ c_{it}$ respectively and defined as

$$\nu_{\beta_1} = \operatorname*{ess\,sup}_{Q_T}|\beta_1(t,x)|^2, \quad \mu_b = \max_i \operatorname*{ess\,sup}_{Q_T}|b_{it}(t,x)|^2,$$
$$\mu_c = \max_i \operatorname*{ess\,sup}_{Q_T}|c_{it}(t,x)|^2, \quad \mu_d = \max_i \operatorname*{ess\,sup}_{Q_T}|d_{it}(t,x)|^2.$$

Choose any positive constants $\delta_6^a,\ \delta_5^c,\ \delta_6^c,\ \delta_8^c$ in such a way that the following condition hold: $(2c_0 - \nu_b\delta_6^a - \nu_c\delta_5^c - \mu_c\delta_6^c - \nu_e\delta_7^c) > 0$. From (3.12), (3.13), conditions of Theorem 1, Gronwall's lemma and from (3.16) implies that

$$\int\limits_{\Omega_\tau}\Big[|u_{tt}^m|^2 + |\nabla u_t^m|^2 + |u_t^m|^p + |\nabla\theta^m|^2 + |\theta_t^m|^2 + |\theta^m|^q\Big]\,dx+$$
$$+\int\limits_{Q_\tau}|\nabla\theta_t^m|^2\,dx\,dt + \int\limits_{Q_\tau}\left(\frac{d}{dt}\left(|u_t^m|^{\frac{p-2}{2}}u_t^m\right)\right)^2\,dx\,dt+$$
$$+\int\limits_{Q_\tau}\left(\frac{d}{dt}\left(|\theta^m|^{\frac{q-2}{2}}\theta^m\right)\right)^2\,dx\,dt \leqslant C_7, \quad \tau\in(0,T), \tag{3.17}$$

where C_7 is a positive constant independent of m.

We still can obtain from (3.11), (3.17) the following subsequences u^m, θ^m (still denoted as the original sequences):

$$\begin{aligned}
&u^m \to u && *\text{-weak in} && L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)),\\
&u_t^m \to u_t && *\text{-weak in} && L^\infty(0,T;H_0^1(\Omega)) \text{ and weakly in } L^p(Q_T),\\
&u_{tt}^m \to u_{tt} && *\text{-weak in} && L^\infty(0,T;L^2(\Omega)),\\
&\theta^m \to \theta && *\text{-weak in} && L^\infty(0,T;H_0^1(\Omega)) \text{ and weakly in } L^q(Q_T),\\
&\theta_t^m \to \theta_t && *\text{-weak in} && L^\infty(0,T;L^2(\Omega)),\\
&|u_t^m|^{p-2}u_t^m \to \psi_1 && \text{weakly in} && L^{p/(p-1)}(Q_T),\\
&|\theta^m|^{q-2}\theta^m \to \psi_2 && \text{weakly in} && L^{q/(q-1)}(Q_T).
\end{aligned} \tag{3.18}$$

By Lions' compactness theorem (cf. Lions [7, p.70, theorem 5.1])

$$
\begin{aligned}
&u_t^m \to u_t &&\text{strongly in} && L^2(Q_T) \text{ and a.e. in } Q_T,\\
&\theta^m \to \theta &&\text{strongly in} && L^2(Q_T) \text{ and a.e. in } Q_T
\end{aligned}
\tag{3.19}
$$

as $m \to \infty$. From (3.18), (3.19) and Lions' lemma 1.3 (cf. Lions [7, p.25, lemma 1.3]) $\psi_1 \equiv |u_t|^{p-2}u_t$ and $\psi_2 \equiv |\theta|^{p-2}\theta$.

Remark 2. Moreover,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(|u_t^m|^{\frac{p-2}{2}}u_t^m\right) \text{ bounded in } L^2(Q_T),\\
&\frac{\partial}{\partial t}\left(|\theta^m|^{\frac{q-2}{2}}\theta^m\right) \text{ bounded in } L^2(Q_T).
\end{aligned}
$$

To complete the proof of the Theorem 1, we shall show that the pair of functions (u,θ) is a generalized solution of problem (2.1)-(2.4). From (3.1), (3.2) we obtain the following system of equations

$$
\begin{aligned}
&\int\limits_{\Omega_T} u_t^m v^{m_0}\,dx - \int\limits_{\Omega_0} u_1^m(x)v^{m_0}\,dx + \int\limits_{Q_T}\Big[-u_t^m v_t^{m_0} + \sum_{i,j=1}^n a_{ij}(t,x)u_{x_i}^m v_{x_j}^{m_0} +\\
&\quad + \sum_{i=1}^n a_i(t,x)u_{x_i}^m v^{m_0} + \sum_{i=1}^n b_i(t,x)\theta_{x_i}^m v^{m_0} + \alpha_0(t,x)u^m v^{m_0} +\\
&\quad + \alpha_1(t,x)\theta^m v^{m_0} + \gamma_1(t,x)|u_t^m|^{p-2}u_t^m v^{m_0}\Big]\,dx\,dt = \int\limits_{Q_T} f_1(t,x)v^{m_0}\,dx\,dt,
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
&\int\limits_{Q_T}\Big[\theta^m w^{m_0} + \sum_{i,j=1}^n c_{ij}(t,x)\theta_{x_i}^m w_{x_j}^{m_0} + \sum_{i=1}^n c_i(t,x)u_t^m w_{x_i}^{m_0} + \sum_{i=1}^n d_i(t,x)u_{x_i}^m w^{m_0} +\\
&\quad + \sum_{i=1}^n e_i(t,x)\theta_{x_i}^m w^{m_0} + \beta_0(t,x)u^m w^{m_0} + \beta_1(t,x)\theta^m w^{m_0} +\\
&\quad + \gamma_2(t,x)|\theta^m|^{q-2}\theta^m w^{m_0}\Big]\,dx\,dt = \int\limits_{Q_T} f_2(t,x)w^{m_0}\,dx\,dt, \quad v^{m_0}, w^{m_0} \in W_{m_0}.
\end{aligned}
\tag{3.21}
$$

Since v^{m_0}, w^{m_0} is a "basis" $H_0^1(\Omega)\cap H^2(\Omega)\cap L^{2(\max\{p,q\})-1}(\Omega)$, by denseness it follows that the last two equations are true for all v^{m_0}, $w^{m_0} \in W_{m_0}$. Taking limits in (3.20)-(3.21) as $m \to \infty$, using the fact that $H_0^1(\Omega)\cap H^2(\Omega)\cap L^{2(\max\{p,q\})-1}(\Omega)$ is dense in $H_0^1(\Omega)\cap L^{2(\max\{p,q\})-1}(\Omega)$, we obtain that (u,θ) satisfy (2.1), (2.2). By Lemma 1.2 [7, p.20] $\{u,u_t\} \in C(0,T;L^2(\Omega))$, whenever $\{u,u_t,u_{tt}\} \in L^2(Q_T)$. Also by Lemma 1.2 [7, p.20] $\theta \in C(0,T;L^2(\Omega))$, whenever $\{\theta,\theta_t\} \in L^2(Q_T)$.

It is easy to show, that the initial conditions hold. Using (3.18) and Lemma 1.2 [7, p.20] we see that $u^m(0,x) \to u(0,x)$, $\theta^m(0,x) \to \theta(0,x)$ weakly in $L^2(\Omega)$. As a result of the fact that $u^m(0,x) = u_0^m \to u_0$ in $H^2(\Omega_0)\cap H_0^1(\Omega_0)$ and $\theta_0^m \to \theta_0$ in $H^2(\Omega_0)\cap H_0^1(\Omega_0)\cap L^{2(q-1)}(\Omega_0)$ implyies the first and the third terms of (2.4).

Next, due to (3.18) $\int\limits_\Omega u_{tt}^m w^m\,dx \to \int\limits_\Omega u_{tt} w^m\,dx$ $*$-weakly in $L^\infty(0,T)$, consequently (cf. lemma 1.2, [7, c.20] with $X=\mathbb{R}$) we have

$$\int\limits_{\Omega_0} u_t^m(0,x)w^k\,dx \to \left(\int\limits_\Omega u_t w^k\,dx\right)\Bigg|_{t=0} = \int\limits_\Omega u_t(0,x)w^k\,dx,$$

and, inasmuch as $\int_{\Omega_0} u_t^m(0,x)w^k\,dx \to \int_\Omega u_1 w^k\,dx$, because $u_1^m \to u_1$ in $H_0^1(\Omega_0)\cap$ $\cap L^{2(p-1)}(\Omega_0)$, we obtain

$$\int\limits_{\Omega_0} u_t(0,x)w^k\,dx = \int\limits_{\Omega_0} u_1 w^k\,dx \text{ for any } k\in\mathbb{N}.$$

This implies the second term of (2.4).

Hence, (u,θ) is a generalized solution of problem (2.1)-(2.4). The proof of the Theorem is complete.

Remark 3. First and third parts of (2.3) follows from belonging u and θ to $L^\infty(0,T;H_0^1(\Omega))$.

Corollary 1. (smoothness of solution). *Suppose that conditions of Theorem 1 are fulfilled. Moreover,* $\{a_{ijx_k},\ c_{ijx_k},\ a_{ix_k},\ b_{ix_k},\ c_{ix_k},\ ,d_{ix_k},\ e_{ix_k},\ \alpha_{0x_k},\ \alpha_{1x_k},\ \beta_{0x_k},\ \beta_{1x_k},\ \gamma_{1x_k},\ \gamma_{2x_k}\}\in L^\infty(Q_T)$, $k=1,\dots,n$ *and*

$$p,q \leqslant \frac{2n}{n-2},\ n>2;\ p,q \text{ is an arbitrary when } n=1,2. \tag{3.22}$$

Under these assumptions, the generalized solution of problem (2.1)-(2.4) is the solution almost everywhere on Q_T.

Proof. In similar a way to Theorem 1 we will use the Faedo-Galerkin method with choosing a special "basis". Let w^k are eigenfunctions of Dirichlet problem for operator $-\Delta$:

$$-\Delta w^k = \lambda w^k,\ \ w^k = 0 \text{ on } \partial\Omega.$$

Suppose that the boundary of Ω is sufficiently smooth in such way that $w^k\in H^2(\Omega)$, $w^k\in H_0^1(\Omega)$ and $w^k\in L^{2(\max\{p,q\}-1)}$, especially $\partial\Omega\subset C^2$. Choose u_0^m, , u_1^m, $\theta_0^m\in[w^1,\dots,w^m]$ in such a way, that $u_0^m\to u_0$ strongly in $H^2(\Omega_0)\cap H_0^1(\Omega_0)$, $\theta_0^m\to\theta_0$ strongly in $H^2(\Omega_0)\cap$ $\cap H_0^1(\Omega_0)\cap L^{2(q-1)}$, $u_1^m\to u_1$ strongly in $H_0^1(\Omega_0)\cap L^{2(p-1)}$.

Therefore by chosen "basis", in system (2.5)-(2.6), which is local soluble in some interval $[0,t^m]$, we will multiply the first equation by the function $-\Delta u_t^m(t,x)$ and the second equation by the function $-\Delta\theta^m(t,x)$, integrating over t from 0 to τ, $0<\tau<$ $<t^m\leqslant T$. Taking into account (3.22) it follow that $H^2(\Omega)\cap H_0^1(\Omega)\cap L^{2(\max\{p,q\}-1)}$ is dense in $H_0^1(\Omega)$, hence this operation is true.

In the same manner as Theorem 1 and using Gronwall's lemma, we conclude that

$$\int\limits_{\Omega_\tau} \left[|\nabla u_t^m|^2 + |\Delta u^m|^2 + \alpha_0 |\nabla u_t|^2 + |\nabla \theta^m|^2\right] \, dx + \int\limits_{Q_\tau} |\Delta \theta^m| \, dx \, dt +$$
$$+ \tilde{\gamma_1} \int\limits_{Q_\tau} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|u_t^m|^{p-2} u_t^m\right) \frac{\partial u_t^m}{\partial x_i} \, dx \, dt +$$
$$+ \tilde{\gamma_2} \int\limits_{Q_\tau} \frac{\partial}{\partial x_i} \sum_{i=1}^{n} \left(|\theta^m|^{q-2} \theta^m\right) \frac{\partial \theta^m}{\partial x_i} \, dx \, dt \leqslant C_8, \tag{3.23}$$

where C_8 is a positive constant independent of m.

From transformations, we have

$$\tilde{\gamma_1} \int\limits_{Q_\tau} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|u_t^m|^{p-2} u_t^m\right) \frac{\partial u_t^m}{\partial x_i} \, dx \, dt \geqslant (p-1)\tilde{\gamma_1} \int\limits_{Q_\tau} \sum_{i=1}^{n} \left(|u_t^m|^{\frac{p-2}{2}} \frac{\partial u_t^m}{\partial x_i}\right) \, dx \, dt =$$
$$= \frac{4(p-1)}{p^2} \tilde{\gamma_1} \int\limits_{Q_\tau} \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} (|u_t^m|^{\frac{p-2}{2}} u_t^m)\right) \, dx \, dt,$$
$$\tilde{\gamma_2} \int\limits_{Q_\tau} \frac{\partial}{\partial x_i} \sum_{i=1}^{n} \left(|\theta^m|^{q-2} \theta^m\right) \frac{\partial \theta^m}{\partial x_i} \, dx \, dt \geqslant (p-1)\tilde{\gamma_2} \int\limits_{Q_\tau} \sum_{i=1}^{n} \left(|\theta^m|^{\frac{q-2}{2}} \frac{\partial \theta^m}{\partial x_i}\right) \, dx \, dt =$$
$$= \frac{4(q-1)}{q^2} \tilde{\gamma_2} \int\limits_{Q_\tau} \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} (|\theta^m|^{\frac{q-2}{2}} \theta^m)\right) \, dx \, dt. \tag{3.24}$$

Comparing (3.23) and (3.24), it follows that

$$\int\limits_{\Omega_\tau} \left[|\nabla u_t^m|^2 + |\Delta u^m|^2 + \alpha_0 |\nabla u_t|^2 + |\nabla \theta^m|^2\right] \, dx + \int\limits_{Q_\tau} |\Delta \theta^m| \, dx \, dt +$$
$$+ \frac{4(p-1)}{p^2} \tilde{\gamma_1} \int\limits_{Q_\tau} \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} (|u_t^m|^{\frac{p-2}{2}} u_t^m)\right) \, dx \, dt +$$
$$+ \frac{4(q-1)}{q^2} \tilde{\gamma_2} \int\limits_{Q_\tau} \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} (|\theta^m|^{\frac{q-2}{2}} \theta^m)\right) \, dx \, dt \leqslant C_8. \tag{3.25}$$

From last inequality, we obtain that

$$\begin{aligned}
&(u_t^m)_{m\in\mathbb{N}} \ \text{ is bounded in } \ L^\infty(0,T;H_0^1(\Omega)),\\
&(u^m)_{m\in\mathbb{N}} \ \text{ is bounded in } \ L^\infty(0,T;H^2(\Omega)),\\
&(\theta^m)_{m\in\mathbb{N}} \ \text{ is bounded in } \ L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)),\\
&\left(\frac{\partial}{\partial x_i}\left(|u_t^m|^{\frac{p-2}{2}}u_t^m\right)\right)_{m\in\mathbb{N}} \ \text{ is bounded in } \ L^2(\Omega) \quad (\ i=1,\dots,n),\\
&\left(\frac{\partial}{\partial x_i}\left(|\theta^m|^{\frac{q-2}{2}}\theta^m\right)\right)_{m\in\mathbb{N}} \ \text{ is bounded in } \ L^2(\Omega) \quad (\ i=1,\dots,n), \qquad (3.26)
\end{aligned}$$

The second result in (3.26) follows from the statement: $\Delta\phi \geqslant C\|\phi\|_{H^2(\Omega)}$, where $\phi\in H_0^1(\Omega)$, $\Delta\phi\in L^2(\Omega)$, and true for a regular boundary $\partial\Omega$.

Hence, from Theorem 1 and (3.26) the generalized solution of problem (2.1)-(2.4) is the solution almost everywhere on Q_T

Remark 4. In view of (3.26) and Remark 2. we have $|u_t|^{(p-2)/2}u_t\in H^1(Q_T)$, $|\theta|^{(q-2)/2}\theta\in H^1(Q_T)$.

4. Uniqueness of solution.

Theorem 2. *Suppose that conditions* (**A**)$-$(**E**), (**G**) *hold. Then the generalized solution of initial-boundary value problem* $(2.1)-(2.4)$ *is unique.*

Proof. If (u^1,θ^1), (u^2,θ^2) are solutions of (2.1)-(2.4), then the pair of functions $(u,\theta)=(u^1-u^2,\theta^1-\theta^2)$ satisfies

$$\begin{aligned}
&\int\limits_{\Omega_T}|u_t|^2\,dx+\int\limits_{Q_T}\Big[-|u_t|^2+\sum_{i,j=1}^n a_{ij}(t,x)u_{x_i}u_{tx_j}+\sum_{i=1}^n a_i(t,x)u_{x_i}u_t+\\
&\quad+\sum_{i=1}^n b_i(t,x)\theta_{x_i}u_t+\alpha_0(t,x)uu_t+\alpha_1(t,x)\theta u_t+\\
&\quad+\gamma_1(t,x)\left[(|u_t^1|^{p-2}u_t^1-|u_t^2|^{p-2}u_t^2)(u_t^1-u_t^2)\right]\Big]\,dx\,dt=0, \qquad (4.1)\\
&\int\limits_{\Omega_T}|\theta|^2\,dx+\int\limits_{Q_T}\Big[-|\theta|^2+\sum_{i,j=1}^n c_{ij}(t,x)\theta_{x_i}\theta_{x_j}+\sum_{i=1}^n c_i(t,x)u_t\theta_{x_i}+\\
&\quad+\sum_{i=1}^n d_i(t,x)u_{x_i}\theta+\sum_{i=1}^n e_i(t,x)\theta_{x_i}\theta+\beta_0(t,x)u\theta+\\
&\quad+\beta_1(t,x)|\theta|^2+\gamma_2(t,x)\left[|\theta^1|^{q-2}\theta^1-|\theta^2|^{q-2}\theta^2(\theta^1-\theta^2)\right]\Big]\,dx\,dt=0. \qquad (4.2)
\end{aligned}$$

Using the estimates from Theorem 1, it is easy to find estimates of terms 1-6 for the first equation and terms 1-7 for the second equation. Moreover, from monotonicity, we get

$$\int\limits_{Q_\tau} \gamma_1(t,x)\left[(|u_t^1|^{p-2}u_t^1 - |u_t^2|^{p-2}u_t^2)(u_t^1 - u_t^2)\right]\,dx\,dt \geqslant 0,$$

$$\int\limits_{Q_\tau} \gamma_2(t,x)\left[|\theta^1|^{q-2}\theta^1 - |\theta^2|^{q-2}\theta^2(\theta^1 - \theta^2)\right]\,dx\,dt \geqslant 0.$$

Thus, from (4.1), (4.2) in similar way to (3.11) we obtain estimate

$$\int\limits_{\Omega_\tau}\left[|u|^2 + |u_t| + |\nabla u| + |\theta|^2\right]\,dx + \int\limits_{Q_\tau}|\nabla\theta|^2\,dx\,dt +$$

$$+\,\tilde{\gamma}_1\int\limits_{Q_\tau}|u_t|^p\,dx\,dt + \tilde{\gamma}_2\int\limits_{Q_\tau}|\theta|^q\,dx\,dt \leqslant 0$$

for any $\tau \in (0,T)$.

Finally, according to Gronwall's inequality it follows that $u = 0$, $\theta = 0$ on Q_T. Hence, Theorem 2 is established.

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МІШАНА ЗАДАЧА ДЛЯ НЕЛІНІЙНОЇ ЗВ'ЯЗНОЇ ЕВОЛЮЦІЙНОЇ СИСТЕМИ РІВНЯНЬ В ОБМЕЖЕНІЙ ОБЛАСТІ

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Визначено умови існування та єдиності розв'язку нелінійної зв'язної еволюційної системи зі змінними коефіцієнтами для нелінійного рівняння з другою похідною за часовою змінною та нелінійного рівняння з першою похідною за часовою змінною в обмеженій області $\mathbb{R}^n$ з гладкою межею.

Ключові слова: нелінійна система, мішана задача, метод Фаедо-Гальоркіна.

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