ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2015. Випуск 80. С. 129–133
Visnyk of the Lviv Univ. Series Mech. Math. 2015. Issue 80. P. 129–133

УДК 517.53+519.213

REMARK TO LOWER ESTIMATES FOR CHARACTERISTIC FUNCTIONS OF PROBABILITY LAWS

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Let α be a slowly increasing function and φ be the characteristic function of probability law F that is analytic in $\mathbb{D}_R = \{z : |z| < R\}, 0 < R \leqslant +\infty$, $M(r, \varphi) = \max\{|\varphi(z)| : |z| = r\}$ and $W_F(x) = 1 - F(x) + F(-x)$, $x \geqslant 0$. Conditions on W_F and α, under which $\alpha(\ln M(r, \varphi)) \geqslant (1+o(1))\varrho\alpha(1/(R-r))$ as $r \uparrow R$, are investigated.

Key words: analytic function, characteristic function, probability law.

A non-decreasing, left-continuous function F defined on $(-\infty, +\infty)$ is said [1, p. 10] to be a probability law if $\lim\limits_{x\to+\infty} F(x) = 1$ and $\lim\limits_{x\to-\infty} F(x) = 0$. Given real z, the function $\varphi(z) = \int\limits_{-\infty}^{+\infty} e^{izx} dF(x)$ is called [1, p. 12] the characteristic function of this law. If φ has an analytic continuation on the disk $\mathbb{D}_R = \{z : |z| < R\}$, $0 < R \leqslant +\infty$, then we call φ an analytic in $\mathbb{D}_R$ characteristic function of the law F. In the sequel, we always assume that $\mathbb{D}_R$ is the maximal disk of analyticity of φ. It is known [1, p. 37-38], that φ is an analytic in $\mathbb{D}_R$ characteristic function of the law F if and only if $W_F(x) = 1 - F(x) + F(-x) = O(e^{-rx})$ as $x \to +\infty$ for every $r \in [0, R)$. Hence it follows that $\varliminf\limits_{x\to+\infty} \frac{1}{x} \ln \frac{1}{W_F(x)} = R$. If we put $M(r, \varphi) = \max\{|\varphi(z)| : |z| = r\}$ and $\mu(r, \varphi) = \sup\{W_F(x)e^{rx} : x \geqslant 0\}$ for $0 \leqslant r < R$, then [1, p. 54-55] $\mu(r, \varphi) \leqslant 2M(r, \varphi)$. Therefore, the lower estimates for $\ln \mu(r, \varphi)$ imply the corresponding estimates for $\ln M(r, \varphi)$. Further, we assume that $\ln \mu(r, \varphi) \uparrow +\infty$ as $r \uparrow R$. Hence

$$\varlimsup_{x\to+\infty} W_F(x)e^{Rx} = +\infty. \tag{1}$$

By L_{si} we denote the class of positive, continuous functions α, defined on $(-\infty, +\infty)$, such that $\alpha(x) = \alpha(x_0)$ for $x \leqslant x_0$, $\alpha(x) \uparrow +\infty$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x_0 \leqslant x \uparrow +\infty$ for every $c \in (0, +\infty)$. In [2] the following statements are proved.

Proposition 1. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$, $\frac{d \ln \beta^{-1}(\alpha(x))}{d \ln x} \leqslant q < 1$ for all x large enough and $\alpha\left(\frac{x}{\beta^{-1}(\alpha(x))}\right) = (1+o(1))\alpha(x)$ as $x \to +\infty$, and φ be an analytic in $\mathbb{D}_R$, $0 < R < +\infty$, characteristic function of probability law F, for which $\beta\left(\frac{x_k}{\ln\left(W_F(x_k)e^{Rx_k}\right)}\right) \leqslant \alpha(x_k)$ for some sequence of positive numbers (x_k) increasing to $+\infty$ such that $\beta^{-1}(\alpha(x_{k+1})) = = O(\beta^{-1}(\alpha(x_k)))$ as $k \to \infty$. Then*

$$\alpha(\ln \mu(r,\varphi)) \geqslant (1+o(1))\beta(1/(R-r)), \quad r \uparrow R. \tag{2}$$

Proposition 2. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$, $\frac{d \ln \alpha^{-1}(\beta(x))}{d \ln x} \leqslant q < 1$ for all x large enough $\frac{d\alpha^{-1}(\beta(x))}{dx} = \frac{1}{f(x)} \downarrow 0$ and $\alpha^{-1}(\beta(f(x))) = O(\alpha^{-1}(\beta(x)))$ as $x \to +\infty$, and φ be an analytic in $\mathbb{D}_R$, $0 < R < +\infty$, characteristic function of probability law F, for which $\alpha\left(\ln\left(W_F(x_k)e^{Rx_k}\right)\right) \geqslant \beta(x_k)$ for some sequence of positive numbers (x_k), increasing to $+\infty$, such that $\varlimsup\limits_{k\to\infty}(f(x_{k+1})/f(x_k)) < 2$. Then asymptotic inequality (2) holds.*

The condition on α and β in Proposition 1 assume that the function α increases slower than the function β. In Proposition 2, α increases quicker than β.

Here we consider the case when $\beta(x) = \varrho\alpha(x)$ for all $x \geqslant x_0$, where $0 < \varrho < +\infty$, that is the functions β and α have the same growth. We use a result from [2].

Let $\Omega(R)$ be a class of positive, unbounded functions Φ, defined on $(0,R)$, such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(0,R)$. For $\Phi \in \Omega(R)$ we denote by ϕ the function inverse to Φ', and let $\Psi(r) = r - \frac{\Phi(r)}{\Phi'(r)}$ be the function associated with Φ in the sense of Newton.

Lemma 1. *Let $\Phi \in \Omega(R)$, $0 < R < +\infty$, and φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F for which (1) holds and*

$$\ln W_F(x_k) \geqslant -x_k\Psi(\phi(x_k)) \tag{3}$$

for some sequence of positive numbers (x_k) increasing to $+\infty$ such that $\phi(x_{k+1}) - \phi(x_k) \leqslant h(x_{k+1})$, where h is a positive continuous and non-increasing function on $[x_0, +\infty)$ and $R > \phi(x) - h(x) \to R$ as $x \to +\infty$. Then

$$\ln \mu(r,f)) \geqslant \Phi(r - h(\Phi'(r))), \quad r_0 \leqslant r < R. \tag{4}$$

Using this lemma we prove the following theorem.

Theorem 1. *Let $\alpha \in L_{si}$ be a continuously differentiable function and φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F. Suppose that one of the following conditions is fulfilled:*

1) $\varrho > 1$, $\varlimsup\limits_{x\to+\infty} \frac{d \ln \alpha^{-1}(x)}{d \ln \alpha^{-1}(\varrho x)} = q(\varrho) < 1$, $\alpha\left(\frac{x}{\alpha(x)}\right) = (1+o(1))\alpha(x)$ as $x \to +\infty$ and

$$\alpha\left(\frac{x_k}{\ln\left(W_F(x_k)e^{Rx_k}\right)}\right) \leqslant \frac{\alpha(x_k)}{\varrho} \tag{5}$$

for some sequence of positive numbers (x_k) increasing to $+\infty$ such that $\alpha^{-1}(\alpha(x_{k+1})/\varrho) = = O(\alpha^{-1}(\alpha(x_k)/\varrho))$ as $k \to \infty$;

2) $0<\varrho<1$, $\varlimsup\limits_{x\to+\infty}\frac{d\ln\alpha^{-1}(\varrho x)}{d\ln\alpha^{-1}(x)}=q(\varrho)<1$, $\frac{d\alpha^{-1}(\varrho\alpha(x))}{dx}=\frac{1}{f(x)}\downarrow 0$, $\alpha^{-1}(\varrho\alpha(f(x)))=$ $=O(\alpha^{-1}(\varrho\alpha(x)))$ *as* $x\to+\infty$ *and*

$$\alpha\left(\ln\left(W_F(x_k)e^{Rx_k}\right)\right)\geqslant\varrho\alpha(x_k) \tag{6}$$

for some sequence of positive numbers (x_k) *increasing to* $+\infty$ *such that* $\varlimsup\limits_{k\to\infty}\frac{f(x_{k+1}}{f(x_k)}<2$.

Then

$$\alpha(\ln\mu(r,\varphi))\geqslant(1+o(1))\varrho\alpha\left(\frac{1}{R-r}\right),\quad r\uparrow R. \tag{7}$$

Proof. At first let $\varrho>1$. Then (5) implies the inequality $\ln W_F(x_k)\geqslant -Rx_k+\frac{x_k}{\alpha^{-1}(\alpha(x_k)/\varrho)}$. Since $\varlimsup\limits_{x\to+\infty}\frac{d\ln\alpha^{-1}(x)}{d\ln\alpha^{-1}(\varrho x)}=q(\varrho)<1$, we have $\frac{d\ln\alpha^{-1}(\alpha(x)/\varrho)}{d\ln x}\leqslant$ $\leqslant(1+o(1))q(\varrho)$ and $\frac{x}{\alpha^{-1}(\alpha(x)/\varrho)}\uparrow+\infty$ as $x_0\leqslant x\to+\infty$. Therefore, using l'Hospital's rule we obtain

$$\frac{x}{\alpha^{-1}(\alpha(x)/\varrho)}\geqslant(1+o(1))(1-q(\varrho))\int\limits_{x_0}^{x}\frac{dt}{\alpha^{-1}(\alpha(t)/\varrho)},\quad x\to+\infty,$$

and, thus,

$$\ln W_F(x_k)\geqslant -Rx_k+(1-q_1)\int\limits_{x_0}^{x_k}\frac{dt}{\alpha^{-1}(\alpha(t)/\varrho)} \tag{8}$$

for every $q_1\in(q(\varrho),1)$ and all k large enough. We put

$$\Phi(r)=\int\limits_{r_0}^{r}\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{R-x}\right)\right)dx,\quad q_1<q_2<1. \tag{9}$$

Then $\Phi'(r)=\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{R-x}\right)\right)$, $\phi(x)=R-\frac{1-q_2}{\alpha^{-1}(\alpha(x)/\varrho)}$ and

$$x\Psi(\phi(x))=\int\limits_{x_0}^{x}\phi(t)dt+\text{const}=Rx-(1-q_2)\int\limits_{x_0}^{x}\frac{dt}{\alpha^{-1}(\alpha(t)/\varrho)}+\text{const},$$

i. e. in view of (8) and of the inequality $q_1<q_2$ we obtain (3).

Since $\alpha^{-1}(\alpha(x_{k+1})/\varrho)\leqslant K\alpha^{-1}(\alpha(x_k)/\varrho)$, $K>1$, for all $k\geqslant k_0$, we have

$$\frac{1}{\alpha^{-1}(\alpha(x_k)/\varrho)}-\frac{1}{\alpha^{-1}(\alpha(x_{k+1})/\varrho)}\leqslant\frac{K-1}{\alpha^{-1}(\alpha(x_{k+1})/\varrho)}.$$

Therefore, putting $h(x)=\frac{(K-1)(1-q_2)}{\alpha^{-1}(\alpha(x)/\varrho)}$, we obtain $\phi(x)-h(x)=R-\frac{K(1-q_2)}{\alpha^{-1}(\alpha(x)/\varrho)}\to R$ as $x\to+\infty$, $h(\Phi'(r))=(K-1)(R-r)$ and $\phi(x_{k+1})-\phi(x_k)\leqslant h(x_{k+1})$ for $k\geqslant k_0$.

Finally, for every $\eta>0$ and all $r\in[r_0(\eta),R)$ from (9) it follows that

$$\Phi(r)=\int\limits_{r-\eta(R-r)}^{r}\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{R-x}\right)\right)dx\geqslant\eta(R-r)\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{(1+\eta)(R-r)}\right)\right).$$

Therefore, by Lemma 1

$$\ln\mu(r,\varphi)\geqslant \eta(R-r+h(\Phi'(r)))\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{(1+\eta)(R-r+h(\Phi'(r)))}\right)\right)=$$

$$=\eta K(R-r)\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{(1+\eta)K(R-r)}\right)\right),$$

whence in view of the condition $\alpha(x/\alpha(x))=(1+o(1))\alpha(x)$ as $x\to+\infty$ we have

$$\alpha(\ln\mu(r,\varphi))\geqslant\alpha\left(\eta K(R-r)\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{(1+\eta)K(R-r)}\right)\right)\right)=$$

$$=(1+o(1))\alpha\left(\frac{\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{(1+\eta)K(R-r)}\right)\right)}{\frac{1-q_2}{(1+\eta)K(R-r)}}\right)=$$

$$=(1+o(1))\varrho\alpha\left(\frac{1-q_2}{(1+\eta)K(R-r)}\right)=(1+o(1))\varrho\alpha\left(\frac{1}{R-r}\right),\ r\uparrow R,$$

thus we obtain (7).

Now let $0<\varrho<1$. If we put $x\Psi(\phi(x))=Rx-\alpha^{-1}(\varrho\alpha(x))$ then (6) implies (3), $\phi(x)=(x\Psi(\phi(x)))'=R-1/f(x)$, $\Phi'(r)=f^{-1}(1/(R-r))$ and since $\frac{d\ln\alpha^{-1}(\rho\alpha(x))}{d\ln x}\leqslant q(\varrho)(1+o(1))$ as $x\to+\infty$ we have

$$\Phi(r)=\int\limits_{r_0}^{r}f^{-1}\left(\frac{1}{R-x}\right)dx=\int\limits_{f^{-1}(1/(R-r_0))}^{f^{-1}(1/(R-r))}td\left(\frac{-1}{f(t)}\right)=$$

$$=-(R-r)f^{-1}(1/(R-r))+\alpha^{-1}(\varrho\alpha(f^{-1}(1/(R-r))))+\text{const}=$$

$$=\alpha^{-1}(\varrho\alpha(f^{-1}(1/(R-r))))\left\{1-\frac{(R-r)f^{-1}(1/(R-r))}{\alpha^{-1}(\varrho\alpha(f^{-1}(1/(R-r))))}\right\}+\text{const}\geqslant$$

$$\geqslant(1-q)\alpha^{-1}(\varrho\alpha(f^{-1}(1/(R-r))))$$

for every $q\in(q(\varrho),1)$ and all $r\in[r_0(q),R)$. But the condition $\alpha^{-1}(\varrho\alpha(f(x)))=O(\alpha^{-1}(\varrho\alpha(x)))$ as $x\to+\infty$ implies that $\alpha^{-1}(\varrho\alpha(1/(R-r)))\leqslant K\alpha^{-1}(\varrho\alpha(f^{-1}(1/(R-r))))$, $K=\text{const}>0$. Therefore, $\Phi(r)\geqslant K_1\alpha^{-1}(\varrho\alpha(1/(R-r)))$, $K_1=\text{const}>0$, and if we put $h(x)=a(R-\phi(x)), 0<a<1$, then

$$\Phi(r-h(\Phi'(r))\geqslant K_1\alpha^{-1}\left(\varrho\alpha\left(\frac{1}{(1+a)(R-r)}\right)\right).\qquad(10)$$

It is clear that, in view of the relation $\phi(x)=R-1/f(x)$, the condition $\phi(x_{k+1})-\phi(x_k)\leqslant h(x_{k+1})$ is equivalent to the condition $f(x_{k+1})\leqslant(1+a)f(x_k)$ and the last one follows from the condition $\varlimsup\limits_{k\to\infty}\frac{f(x_{k+1}}{f(x_k)}<2$. Therefore, by Lemma 1 we see that (4) and (10) implies (7). The proof of Theorem 1 is complete.

Since $\ln M(r,\varphi)\geqslant\ln\mu(r,\varphi)-\ln 2$, choosing $\alpha(x)=\ln x$ from Theorem 1 we obtain the following assertion.

Corollary 1. *Let φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F. Suppose that one of the following conditions is fulfilled:*

1) $\varrho > 1$ *and* $\ln\big(W_F(x_k)e^{Rx_k}\big) \geqslant x_k^{(\varrho-1)/\varrho}$ *for some sequence of positive numbers* (x_k) *increasing to* $+\infty$ *such that* $x_{k+1} = O(x_k)$ *as* $k \to \infty$*;*

2) $0 < \varrho < 1$ *and* $\ln\big(W_F(x_k)e^{Rx_k}\big) \geqslant x^{\varrho}$ *for some sequence of positive numbers* (x_k) *increasing to* $+\infty$ *such that* $\varlimsup\limits_{k\to\infty}\left(\frac{x_{k+1}}{x_k}\right)^{1-\varrho} < 2$.

Then $\ln\ln M(r,\varphi)) \geqslant (1+o(1))\varrho\ln(1/(R-r))$ *as* $r \uparrow R$.

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Стаття: надійшла до редколегії 12.12.2014.
доопрацьована 02.11.2015.
прийнята до друку 11.11.2015.

ЗАУВАЖЕННЯ ЩОДО ОЦІНОК ЗНИЗУ ДЛЯ ХАРАКТЕРИСТИЧНИХ ФУНКЦІЙ ЙМОВІРНІСНИХ ЗАКОНІВ

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Нехай α — повільно зростаюча функція, а φ — аналітична в $\mathbb{D}_R = \{z : |z| < R\}$, $0 < R \leqslant +\infty$, характеристична функція ймовірнісного закону F, $M(r,\varphi) = \max\{|\varphi(z)| : |z| = r < R\}$ і $W_F(x) = 1 - F(x) + F(-x)$, $x \geqslant 0$. Досліджено умови на функції W_F і α, за яких правильна нерівність $\alpha(\ln M(r,\varphi)) \geqslant (1+o(1))\varrho\alpha(1/(R-r))$, $r \uparrow R$.

Ключові слова: аналітична функція, характеристична функція, ймовірнісний закон.