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RADICAL FILTERS OF SEMISIMPLE MODULES WITH FINITE NUMBER OF HOMOGENEOUS COMPONENTS

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Radical filters of semisimple modules with finite homogeneous components are described.

Key words: semisimple ring, module, radical filter.

All rings are assumed to be associative with unit 1≠0 and all modules are left unitary.

Let R be a ring. The category of left R-modules will be denoted by $R-Mod$. We shall write $N \leq M$ if N is a submodule of M. The set of all R-endomorphisms of M will be denoted by $End(M)$. Let $soc(M)$ denote the socle of M. Let $N \leq M$ and $f \in End(M)$. Put

$$(N:f)_M = \{x \in M \,|\, f(x) \in N\}\,, End(M)_N = \{f \in End(M) \,|\, f(M) \subseteq N\}\,.$$

Let E be some non-empty collection of submodules of a left R-module M.

Consider the following conditions:

$$L \in E, L \leq N \leq M \Rightarrow N \in E; \tag{1}$$

$$L \in E, f \in End(M) \Rightarrow (L:f)_M \in E; \tag{2}$$

$$N, L \in E \Rightarrow N \cap L \in E; \tag{3}$$

$$N \in E, N \in Gen(M), L \leq N \leq M \wedge \forall g \in End(M)_N : (L:g)_M \in E \Rightarrow L \in E; \tag{4}$$

Definition 1. *A non-empty collection E of submodules of a left R-module M satisfying* (1), (2), (3) *is called a preradical filter of M.*

Definition 2. *A non-empty collection E of submodules of a left R-module M satisfying* (1), (2), (4) *is called a radical filter of M.*

Definition 3. *A preradical (radical) filter E of a left R-module M is said to be trivial if either $E = \{L \,|\, L \leq M\}$ or $E = \{M\}$.*

Proposition 1. *Let M be a semisimple R-module with a unique homogeneous component and let $M = \underset{i\in I}{\oplus} M_i$, where M_i is simple for each $i \in I$. If $Card(I) < \infty$, then every preradical filter of M is trivial.*

Proof. Suppose that $Card(I) < \infty$. Let E be a preradical filter of M such that $E \neq \{M\}$. We claim that $E = \{L \,|L \leq M\}$. Indeed, consider the submodule $L = \bigcap_{H\in E} H$ of M. Since L is a submodule of M, L is semisimple (see Proposition 9.4 [1, p. 117]. Suppose that $L \neq 0$. Hence there exists a simple submodule T of L.

Let $f : T \to M$ be an arbitrary R-homomorphism. Since M is semisimple, there exists a submodule H such that $M = H \oplus T$. Consider the map $g : M \to M$ such that $\forall t \in T \forall h \in H : g(t + h) = f(t)$. It is obvious that g is an R-homomorphism. Then $g(T) = f(T)$. $T \subseteq L$ implies that $f(T) \subseteq g(L)$. It is easy to see that $g(L) \subseteq L$. Therefore $f(T) \subseteq L$. It follows from this that $\sum_{q\in Hom_R(T,M)} q(T) = Tr_M(T) \subseteq L$(see [p. 109, 1]). However, $Tr_M(T) = M$. Hence $M = L$. Now we obtain $E = \{M\}$. However, this contradicts the original assumption that $E \neq \{M\}$. Therefore, we must conclude that $L = 0$. By Proposition 10.6 [p.125,1], since $Card(I) < \infty$, M is a finitely cogenerated module. Taking into consideration this fact and $\bigcap_{H\in E} H = 0$, we see that there exist submodules $H_1, H_2, ..., H_n$ of M belonging to E such that $H_1 \bigcap H_2 \bigcap ... \bigcap H_n = 0$. Thus, by (3), $H_1 \bigcap H_2 \bigcap ... \bigcap H_n \in E$. Now we obtain $0 \in E$. Hence $E = \{L \,|L \leq M\}$.

Lemma 1. *If E is a radical filter of an R-module M, then E satisfies the following condition*

$$N, L \in E, N \in Gen(M) \Rightarrow N \bigcap L \in E. \quad (3')$$

Proof. Let E be a radical filter of an R-module M, $N \in Gen(M)$, and $N, L \in E$.

Consider an arbitrary g belonging to $End(M)_N$. Let x be an arbitrary element of $(L : g)_M$. Then $g(x) \in N$ and $g(x) \in L$. Therefore, $x \in (L \bigcap N : g)_M$. And now we obtain $(L : g)_M \subseteq (L \bigcap N : g)_M$. By (2), since E is a radical filter of an R-module M, $(L : g)_M \in E$. Taking into account $(L : g)_M \subseteq (L \bigcap N : g)_M$, by (1), $(L \bigcap N : g)_M \in E$. However, $N \in E$, $L \bigcap N \subseteq N$, and $N \in Gen(M)$. By (4), $N \bigcap L \in E$.

Corollary 1. *Let M be an R-module. If every submodule of M is generated by M, then every radical filter of M is a preradical filter.*

Example 1. Every radical filter of any ring is a preradical filter.

Proposition 2. *If M is a semisimple R-module, then every radical filter of M is a preradical filter.*

Proof. Let K be any submodule of M. By Lemma 9.2 [1, p. 116], $M = K \oplus H$, where K is a submodule of M. Consider the epimorphism $f : M \to K$ such that $f(k + h) = k$ for every $k \in K$ and $h \in H$. Hence $K \in Gen(M)$. Now apply Corollary 1.

Proposition 3. *Let M be a semisimple R-module with a unique homogeneous component and let $M = \underset{i\in I}{\oplus} M_i$, where M_i is simple for each $i \in I$. If $Card(I) < \infty$, then every radical filter of M is trivial.*

Proof. Apply Proposition 2 and Proposition 1.

Let M be a semisimple left R-module with a unique homogeneous component and let $M = \underset{i\in I}{\oplus} M_i$, where M_i is simple for each $i \in I$. If $N = \underset{i\in J}{\oplus} N_i$, where N_i is simple for each $i \in J$ and $M \cong N$, then $Card(I) = Card(J)$. Put

$$Card_s(M) := Card(I).$$

Proposition 4. *Let M be a semisimple R-module with a unique homogeneous component. If $Card_s(M)$ is infinite, then the collection*

$$E_p(M) := \{L \,|L \le M, Card_s(M/L) < p\}$$

is a non-trivial radical [preradical] filter of M for each infinite cardinal number $p \le Card_s(M)$.

Proof. Let $Card_s(M)$ be infinite and p be an infinite cardinal number such that $p \le Card_s(M)$. (1) Let $L \in E_p(M)$ and $L \le N \le M$.

Hence $Card_s(M/L) < p$. Since M, N are semisimple modules and $N \le M, L \le N$, there exist submodules $K \le M, H \le N$ such that

$$M = N \oplus K, N = L \oplus H.$$

This implies that $M = L \oplus H \oplus K$.

It is easily seen that $Card_s(M) = Card_s(L) + Card_s(H \oplus K)$. Since $H \oplus K \cong M/L$, $Card_s(H \oplus K) = Card_s(M/L) < p$. However, $Card_s(H \oplus K) = Card_s(H) + Card_s(K)$. Therefore $Card_s(H) + Card_s(K) < p$. Since $Card_s(K) \le Card_s(H) + Card_s(K)$, $Card_s(K) < p$. It is easy to see that $K \cong M/N$. Hence $Card_s(M/N) = Card_s(K) < p$. This means that $N \in E_p(M)$.

(2) Let $L \in E_p(M)$ and $f \in End(M)$.

Let $m_1, m_2 \in M$ such that $m_1 - m_2 \in (L : f)_M$. Hence $f(m_1) - f(m_2) = f(m_1 - m_2) \in L$. Therefore, we have a map

$$g : M/(L : f)_M \to M/L,$$

where $\forall m \in M : g(m + (L : f)_M) = f(m) + L$. It is obvious that g is monomorphism. This implies that $M/L = D \oplus U$, where $D \cong M/(L : f)_M, U \le M/L$. Thus $Card_s(D) + Card_s(U) = Card_s(M/L)$. But $Card_s(D) = Card_s(M/(L : f)_M)$. Hence, $Card_s(M/(L : f)_M) + Card_s(U) = Card_s(M/L) < p$. This implies that $Card_s(M/(L : f)_M) < p$. This means that $M/(L : f)_M \in E_p(M)$.

(4) Let $N \in E_p(M), N \in Gen(M), L \le N \le M$ and $(L : g)_M \in E_p(M)$ for every $g \in End(M)_N$.

As M is semisimple and $N \le M$ we see that there exists a submodule T of M such that $M = N \oplus T$. Consider the projection

$$g_N : M \to M, g_N(n + t) = n \text{ for every } n \in N, t \in T.$$

Let $m_1, m_2 \in M$ be such that $m_1 - m_2 \in (L : g_N)_M$. This implies that $g_N(m_1) - g_N(m_2) = g_N(m_1 - m_2) \in L$. Let n be an arbitrary element of N. Hence $g_N(n) = n$. If $g_N(m_1) - g_N(m_2) \in L$, then $m_1 - m_2 \in (L : g_N)_M$. From what has already been proved, we deduce that $q : M/(L : g_N)_M \to N/L$ is a bijection. It is easy to see that

q is an R-homomorphism. Therefore, $Card_s(M/(L:g_N)_M) = Card_s(N/L)$. Since $(L:g_N)_M \in E_p(M)$, $Card_s(M/(L:g_N)_M) < p$. Thus $Card_s(N/L) < p$. Taking into account that M/L is semisimple and $N/L \leq M/L$, we have that there exists a submodule D of M/L such that $M/L = N/L \oplus D$. Therefore $D \cong (M/L)/(N/L) \cong M/N$. Since $M/L = N/L \oplus D$, $Card_s(M/L) = Card_s(N/L) + Card_s(D) = Card_s(N/L) + Card_s(M/N)$. We have $Card_s(M/N) < p$, because $N \in E_p(M)$. Consider the following cases:

(i) $Card_s(N/L) < \infty$ and $Card_s(M/N) < \infty$;

(ii) $Card_s(N/L) = \infty$ or $Card_s(M/N) = \infty$.

(i) Assume $Card_s(N/L) < \infty$ and $Card_s(M/N) < \infty$. Hence

$$Card_s(M/L) = Card_s(N/L) + Card_s(M/N) < \infty.$$

Therefore $Card_s(M/L) = Card_s(N/L) + Card_s(M/N) < p$, because p is infinite.

(ii) Assume $Card_s(N/L) = \infty$ or $Card_s(M/N) = \infty$. Taking into account $Card_s(M/L) = Card_s(N/L) + Card_s(M/N)$, by (2.1) [2, p. 417],

$$Card_s(M/L) = \max\{Card_s(N/L), Card_s(M/N)\}.$$

But $Card_s(N/L) < p$, $Card_s(M/N) < p$. Thus we have $Card_s(M/L) < p$.

In both cases we obtain $Card_s(M/L) < p$. It means that $L \in E_p(M)$. Therefore $E_p(M)$ is a non-empty set satisfying (1), (2), (4). Now apply Proposition 2.

Since M is semisimple and $Card_s(M) \neq 0$, there exists a minimal submodule T of M. Hence $M = T \oplus W$for some submodule $W \neq M$ of M. Therefore $M/W \cong T$. Hence, $Card_s(M/W) = Card_s(T) = 1$. Thus, $W \in E_p(M)$ for each infinite cardinal number $p \leq Card_s(M)$. We obtain $E_p(M) \neq \{M\}$ for each infinite cardinal number $p \leq Card_s(M)$.

Since $Card_s(M/0) = Card_s(M)$, $0 \notin E_p(M)$ for each infinite cardinal number $p \leq Card_s(M)$.

Proposition 5. *(Theorem 1* [5]*). Let M be a semisimple R-module with a unique homogeneous component. If $Card_s(M)$ is infinite, then every non-trivial radical [preradical] filter of M is of the form $E_p(M)$ for some infinite cardinal number $p \leq Card_s(M)$.*

Corollary 2. *If M is a semisimple R-module with a unique homogeneous component, then:*

(i) The set of all radical filters of M and the set of all preradical filters of M are equal.

(ii) If $Card_s(M)$ is finite, then all radical [preradical] filters are trivial.

(iii) If $Card_s(M)$ is infinite, then $\{E_p(M)\,|p = \infty, p \leq Card_s(M)\}$ is the set of all non-trivial radical [preradical] filters of M.

Proof. Apply Propositions 1, 3, 4, 5.

Proposition 6. *If M is a left R-module such that $M = M_1 \oplus M_2 \oplus ... \oplus M_n$, where $M_i = Tr_M(M_i)$ for each $i \in \{1,2,...,n\}$ and $S \leq M \Rightarrow S \in Gen(M)$ for every S, then:*

(i) Every radical [preradical] filter E of M is of the form

$$E = \{J_1 + J_2 + ... + J_n\,|J_i \in E_i(i \in \{1,2,...,n\})\},$$

where E_i is a radical [preradical] filter of M_i for each $i \in \{1, 2, ..., n\}$.

(ii) If E_i is a radical [preradical] filter of M_i for each $i \in \{1, 2, ..., n\}$, then $E = \{J_1 + J_2 + ... + J_n \,|J_i \in E_i(i \in \{1, 2, ..., n\})\}$ is a radical [preradical] filter of M.

Proof. (i) By Theorem 2 [5]. (ii) By Theorem 1 [6].

Theorem 1. *If M is a semisimple R-module with a finite number of homogeneous components $M_1, M_2, ..., M_n$, then:*

(i) The set of all radical filters of M and the set of all preradical filters of M are equal.

(ii) $\{\{J_1 + J_2 + ... + J_n \,|J_i \in E_i(i \in \{1, 2, ..., n\})\} \,|E_i \in \{E_{p_i}(M_i) \,|p_i = \infty, p_i \leq \leq Card_s(M_i)\} \bigcup\cdot \{\{M_i\}, \{L_i \,|L_i \leq M_i\}\}(i \in \{1, 2, ..., n\})\}$ is the set of all radical filters of M.

Proof. By Propositions 1, 6 and Corollary 2.

Corollary 3. *If M is a finitely generated semisimple R-module, then the set of all radical [preradical] filters of M is a 2^n-element set, where n is a number of homogeneous components of M.*

Remark 1. Let $M = \underset{\alpha<\xi}{\oplus} M_\alpha$ be a semisimple R-module, where $M_\alpha \neq 0$ is a homogeneous component of M for any ordinal number $\alpha < \xi$, ξ is a limit ordinal number, and σ is a limit ordinal number such that $\sigma \leq \xi$. Consider

$$F_\sigma := \{K \leq M \,\Big|\, \underset{\chi\leq\alpha<\xi}{\oplus} M_\alpha \subseteq K, \chi < \sigma\}.$$

We see that $\bigcap F_\sigma = \underset{\sigma\leq\alpha<\xi}{\oplus} M_\alpha \notin F_\sigma$ and F_σ is a radical filter of M.

Indeed, let $\sigma \leq \eta < \xi$. If $\chi < \sigma$, then $\chi < \eta < \xi$ and we have $M_\eta \subseteq \underset{\chi\leq\alpha<\xi}{\oplus} M_\alpha$ for every $\chi < \sigma$. Thus $M_\eta \subseteq \bigcap_{\chi<\sigma} \underset{\chi\leq\alpha<\xi}{\oplus} M_\alpha = \bigcap F_\sigma$ for $\sigma \leq \eta < \xi$.

Let $\eta < \sigma$. Since σ is a limit ordinal number, $\eta+1 < \sigma$. Since $(\underset{\eta+1\leq\alpha<\xi}{\oplus} M_\alpha) \bigcap M_\eta = 0$, $\bigcap F_\sigma \bigcap M_\eta = 0$ for $\eta < \sigma$.

Put $D := \bigcap F_\sigma$. Let K_α be a minimal submodule of M_α for any $\alpha < \xi$. Taking into account Proposition 9.4 [1], we obtain $D \in Gen(\{K_\alpha \,|\alpha < \xi\})$. Hence $D = \underset{\alpha<\xi}{\oplus} tr_D(K_\alpha)$. Since $W \mapsto tr_W(K_\alpha), W \in R - Mod$ is a hereditary preradical [4], $tr_D(K_\alpha) = tr_M(K_\alpha) \bigcap D = M_\alpha \bigcap D$. Thus $\bigcap F_\sigma = D = \underset{\alpha<\xi}{\oplus}(M_\alpha \bigcap D) = \underset{\alpha<\sigma}{\oplus}(M_\alpha \bigcap D) \oplus \underset{\sigma\leq\alpha<\xi}{\oplus}(M_\alpha \bigcap D) = 0 \oplus \underset{\sigma\leq\alpha<\xi}{\oplus} M_\alpha = \underset{\sigma\leq\alpha<\xi}{\oplus} M_\alpha$.

Let $\chi < \sigma$. Hence $M_\chi \bigcap \bigcap F_\sigma = M_\chi \bigcap \underset{\sigma\leq\alpha<\xi}{\oplus} M_\alpha = 0$. Thus $\underset{\chi\leq\alpha<\xi}{\oplus} M_\alpha$ is not contained in $\bigcap F_\sigma$ for any $\chi < \sigma$. Hence $\bigcap F_\sigma \notin F_\sigma$.

Consider conditions 1, 2, 4 for radical filters.

(1) This is clear.

(2) Let $\chi < \sigma$, $K \leq M$, $\underset{\chi\leq\alpha<\xi}{\oplus} M_\alpha \subseteq K$, and $f \in End(M)$. Since M_α is a fully invariant submodule of M for any $\alpha < \xi$,$f(\underset{\chi\leq\alpha<\xi}{\oplus} M_\alpha) \subseteq \underset{\chi\leq\alpha<\xi}{\oplus} M_\alpha$. Hence $\underset{\chi\leq\alpha<\xi}{\oplus} M_\alpha \subseteq (K : f)_M$. Therefore $(K : f)_M \in F_\sigma$.

(4) Let $N \in F_\sigma, L \leq N \leq M$ and $\forall g \in End(M)_N : (L : g)_M \in F_\sigma$. Hence there exists an ordinal number $\chi < \sigma$ such that $\underset{\chi\leq\alpha<\xi}{\oplus} M_\alpha \subseteq N$. Consider

$$g : M \to M, g(m_1 + m_2) = m_1, (m_1 \in \underset{\chi\leq\alpha<\xi}{\oplus} M_\alpha, m_2 \in \underset{\alpha<\chi}{\oplus} M_\alpha).$$

It is easily seen that $g \in End(M)_N$. Thus $(L : g)_M \in F_\sigma$. Hence there exists $\beta < \sigma$ such that $\underset{\beta\leq\alpha<\xi}{\oplus} M_\alpha \subseteq (L : g)_M$. Put $\gamma := \max(\chi, \beta)$. Hence

$$\underset{\gamma\leq\alpha<\xi}{\oplus} M_\alpha = g(\underset{\gamma\leq\alpha<\xi}{\oplus} M_\alpha) \subseteq L.$$

Therefore

$$L \in F_\sigma.$$

Let $f_\theta(\theta < \xi)$ be an element of $End(M)$ such that $f_\theta(m) = m$ for every $m \in M_\theta$ and $f_\theta(m) = 0$ for every $m \in M_\alpha$, where $\alpha < \xi$ and $\alpha \neq \theta$.

Put

$$F_{\sigma,\theta} = \{f_\theta(L)\,|L \in F_\sigma\}.$$

Let $\theta < \sigma$ and $S \leq M_\theta$. Then $\underset{\theta+1\leq\alpha<\xi}{\oplus} M_\alpha \subseteq \underset{\theta+1\leq\alpha<\xi}{\oplus} M_\alpha + S$, because $\theta + 1 < \sigma$. Hence $\underset{\theta+1\leq\alpha<\xi}{\oplus} M_\alpha + S \in F_\sigma$. Thus $S = f_\theta(\underset{\theta+1\leq\alpha<\xi}{\oplus} M_\alpha + S) \in F_{\sigma,\theta}$. We obtain $F_{\sigma,\theta} = \{S\,|S \leq M_\theta\}$ for any $\theta < \sigma$.

Let $\sigma \leq \theta < \xi$ and let H be an arbitrary element of $F_{\sigma,\theta}$. Then there exists $K \in F_\sigma$ such that $f_\theta(K) = H$. Hence $\underset{\chi\leq\alpha<\xi}{\oplus} M_\alpha \subseteq K$ for some $\chi < \sigma$. Since $\sigma \leq \theta < \xi$ and $\chi < \sigma$, $\chi < \theta < \xi$. Therefore $M_\theta \subseteq \underset{\chi\leq\alpha<\xi}{\oplus} M_\alpha \subseteq K$. Hence, $M_\theta = f_\theta(M_\theta) \subseteq f_\theta(K) \subseteq M_\theta$. We obtain $H = M_\theta$.

Therefore $\{\sum_{\alpha<\xi} H_\alpha\,|H_\alpha \in F_{\sigma,\alpha}\} = \{T \leq M\,|\bigcap F_\sigma \subseteq T\} \neq F_\sigma$ (see Proposition 6 (i)).

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РАДИКАЛЬНІ ФІЛЬТРИ НАПІВПРОСТИХ МОДУЛІВ ЗІ СКІНЧЕННОЮ КІЛЬКІСТЮ ОДНОРІДНИХ КОМПОНЕНТ

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Описано радикальні фільтри напівпростих модулів зі скінченною кількістю однорідних компонент.

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