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ON PROPERTIES OF RADICALS AND SPECTRUM OF FINITE HOMOMORPHIC IMAGES OF A COMMUTATIVE BEZOUT DOMAIN

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The properties of finite homomorphic images of commutative Bezout domains under certain conditions on the Jacobson radical and maximal ideals are investigated. We describe the structure of maximal ideals in the case of semisimple rings. We also describe the rings R and R/aR provided that R/aR is a Kasch ring.

Key words: Bezout ring, adequate ring, Kasch ring, pure ideal.

1. Introduction

Finite homomorphic images of a commutative Bezout domain are a source of examples and counterexamples in the current research in the ring theory. In the article [3] it is proved that any finite homomorphic image of a commutative Bezout domain is an IF-ring, i.e. a ring over which any injective module is flat, and in the article [9] it is proved that it is a P-injective ring. Furthermore, it is known that a commutative domain is hereditary if and only if every homomorphic image of the ring is self-injective Artinian ring [7]. In addition, a commutative domain is a Dedekind ring if every homomorphic image of this domain is a quasi-Frobenius ring [7]. In [14], there are investigated commutative Bezout domains by calculating their homological dimensions. In [13] it is proved that the finite homomorphic images of a commutative Bezout ring R are semiregular if and only if R is an adequate domain. In [2] it is proved that a Noetherian ring whose Jacobson radical is projective has an Artinian quotient ring. In [5] there are investigated the rings whose Jacobson radicals are coherent (flat, injective). Note that a finite homomorphic image of the ring will be understood as factor ring on principal ideal.

Throughout this paper R is assumed to be a commutative ring with $1 \neq 0$. By a Bezout ring we mean a ring in which all finitely generated ideals are principal. By $J(R)$ and $\mathrm{rad}(R)$ we denote Jacobson radical and nilradical of the ring R.

Definition 1. *A nonzero element a of a ring R is called adequate if for every element $b \in R$, there exist $r, s \in R$ such that*
1. $a = r \cdot s$,
2. $(r, b) = 1$,
3. for every nonunit divisor s' of s, we have $(s', b) \neq 1$.

A commutative Bezout ring with identity is said to be adequate if every nonzero element is adequate [15].

First note the following result.

Proposition 1. *Let R be a commutative Bezout domain and $a \in R\backslash(0)$. If the element a is not adequate then* $\mathrm{rad}(R/aR) \neq (0)$.

Proof. Note that if $a = b \cdot c$, where $(b, c) = 1$ for all such representations, then a is an adequate element. Hence, $a = b \cdot c$, and $bR + cR = dR, d \notin U(R)$, otherwise the element a will be adequate. Then we have $c = c_0 d, b = b_0 d$ for some elements $c_0, b_0 \in R$. Denote $\alpha = c_0 b_0 d$. It is evident that $aR \subset c_0 b_0 dR = \alpha R$ and $\alpha^2 = a c_0 b_0$. That is $aR \subsetneq \alpha R$ and $\alpha^2 R \subset aR$. Hence $\bar{\alpha} \in \mathrm{rad}(R/aR)$ and $\bar{\alpha} \neq \bar{0}$, and we obtain $\mathrm{rad}(R/aR) \neq (0)$.

Note that the condition $\mathrm{rad}(R/aR) \neq (0)$ or, more generally, $J(R/aR) \neq (0)$ does not imply that the element a is not an adequate element. For example, $J(\mathbb{Z}/12\mathbb{Z}) \neq \bar{0}$, but 12 is an adequate element in the ring $\mathbb{Z}$.

The answer to the question for an adequate element a is given by Theorem.

Theorem 1. *Let R be a commutative Bezout domain and $a \in R\backslash 0$. Then the element a is adequate if and only if R/aR is a semiregular ring. [13]*

2. Main Results

For finite homomorphic images of a commutative Bezout domain we describe the ring in which the Jacobson radical is a principal ideal.

Theorem 2. *Let R be a commutative Bezout domain and for $a \in R\backslash(0)$: $J(R/aR)$ is a principal ideal. Then the element a is contained only in a finite number of maximal ideals which are principal ideals.*

Proof. Denote $\bar{R} = R/aR$ and $\bar{b} = b + aR$. According to [12] the annihilator $Ann(\bar{b})$ of any element $\bar{b}$ is a principal ideal. By virtue of [9] the ring $\bar{R}$ satisfies the condition that $Ann(Ann(\bar{b})) = \bar{b}\bar{R}$. If $J(\bar{R}) = \bar{b}\bar{R}$, then $J(\bar{R}) = Ann(\bar{c})$ for some element $\bar{c} \in \bar{R}$. According to [1], $J(\bar{R})$ is a principal ideal if and only if the element a is contained in a finite set of maximal ideals which are principal. The proof is complete.

Definition 2. *An ideal I of a ring is called pure if for every $a \in I$ there exists an element $b \in I$ such that $ab = a$ [4].*

Note that if I is a pure ideal and $I \subseteq J(R)$, then $I = (0)$.

Let us investigate whether the Jacobson radical is a pure ideal for any finite homomorphic image of a commutative Bezout domain.

Definition 3. *An ideal I is called dense if $Ann(I) = (0)$ [6].*

Remark that the following results are obvious.

Proposition 2. *Let R be a commutative ring and $J(R) \neq (0)$ but $Ann(J(R)) = (0)$. Then every maximal ideal of the ring R is dense.*

Proof. Let M be a maximal ideal of the ring R. Then $J(R) \subset M$, and hence $Ann(M) \subset Ann(J(R)) = (0)$. That is $Ann(M) = (0)$, which was to be shown.

Proposition 3. *Let R be a commutative ring and $J(R) \neq (0)$ and there exists a maximal ideal M such that $Ann(M) \neq (0)$. Then $Ann(J(R)) \neq (0)$.*

Proof. Since $J(R) \subset M$, then $(0) \neq Ann(M) \subset Ann(J(R))$.

Proposition 4. *Let R be a commutative semiprime (i.e. reduced) ring. Then in the ring R does not exist an ideal M such that $(0) \neq Ann(M) \subset M$.*

Proof. Let M be an ideal in R such that $(0) \neq Ann(M) \subset M$. Then $(Ann(M))^2 = (0)$. Since R is semiprime ring, we obtain that $Ann(M) = (0)$. Contradiction.

The proposition is proved.

By a semisimple ring we mean a ring R with $J(R) = (0)$.

As a consequence we obtain the following result.

Theorem 3. *Let R be a commutative semisimple ring. Then for any maximal ideal M of the ring R we have $Ann(M) = (0)$ or $M = mR, (m^2 = m)$.*

Proof. Let M be a maximal ideal of the ring R. If $Ann(M) \neq (0)$ then by Proposition 2.4, $Ann(M) \not\subseteq M$ and we obtain $M + Ann(M) = R$. Whence $m + n = 1$, where $m \in M, n \in Ann(M)$. Then $m^2 + mn = m$ and $m^2 = m$, hence $mR \subset M$. Then for any $s \in M$ we have $sm + sn = s$ and $s = sm$ that is $M \subset mR$. Hence $M = mR$ and $m^2 = m$. The proof is complete.

Thus, we have proved that any commutative ring with pure Jacobson radical is a ring with projective socle, i.e., so-called PS-ring [7].

Theorem 4. *Let R be a commutative semisimple ring. Then R is a PS-ring.*

Definition 4. *A commutative ring in which any arbitrary maximal ideal is dense is called a Kasch ring [6].*

By an atom we mean a non-unit which is not a product of non-units.

Theorem 5. *Let R be commutative Bezout domain and for $a \in R\backslash(0)$ $\bar{R} = R/aR$ is a Kasch ring. Then an arbitrary maximal ideal $\bar{M}$ of the ring $\bar{R}$ has the form $\bar{M} = \bar{e}\bar{R}$, where $\bar{e}^2 = \bar{e}$ or $\bar{M} = \bar{p}\bar{R}$, where $\bar{p}$ is an atom.*

Proof. If $Ann(\bar{M}) \not\subseteq \bar{M}$ then, as we have shown, $\bar{M} = \bar{e}\bar{R}$. If $Ann(\bar{M}) \subset \bar{M}$ then since $\bar{R}$ is a Kasch ring we have $\bar{M} = \bar{p}\bar{R}$. We prove that the element $\bar{p}$ is an atom. Let $\bar{p} = \bar{b}\bar{c}$. If there is $\bar{b}\bar{R} = \bar{R}$, it is proved. Let $\bar{b}\bar{R} \neq \bar{R}$. The inclusion $\bar{p}\bar{R} \subset \bar{b}\bar{R}$ and that $\bar{p}\bar{R}$ is a maximal ideal of the ring $\bar{R}$ imply $\bar{p}\bar{R} = \bar{b}\bar{R}$, and then $\bar{b} = \bar{p}\bar{t}$. Hence $\bar{p} = \bar{p}\bar{b}\bar{t}$ and $\bar{p}(\bar{1} - \bar{b}\bar{t}) = \bar{0}$. We get that $\bar{1} - \bar{b}\bar{t} \in Ann(\bar{M})$. Because of what is proven above, for any $\bar{x} \in ann\bar{M}$ it follows that $\bar{x}^2 = \bar{0}$ and then $\bar{b}\bar{t} = \bar{1} + \bar{r}$, where $\bar{r} \in \mathrm{rad}(\bar{R})$. We obtain that the element $\bar{b}\bar{t}$ is a unit and that $\bar{b}\bar{R} = \bar{R}$, which proves the theorem.

Corollary 1. *Let R be a commutative Bezout domain and, for $a \in R\backslash(0)$; $\bar{R} = R/aR$ is a Kasch ring and let $\bar{M}$ be a maximal ideal of the ring $\bar{R}$ such that $\bar{M} = \bar{p}\bar{R}$, where $\bar{p}^2 \neq \bar{p}$. Then the ideal $\bar{M}$ is not pure.*

Proof. Suppose that the ideal $\bar{M}$ is pure. According to the definition of a pure ideal, for any element $\bar{p} \in \bar{M}$ there exists an element $\bar{q} \in \bar{M}$ such that $\bar{p}\bar{q} = \bar{p}$. Since p is an atom, this is possible only when $\bar{b} \in U(\bar{R})$, which contradicts to the fact that $\bar{q} \in \bar{M}$. The corollary is proved.

Corollary 2. *Let R be a commutative Bezout domain and, for $a \in R\backslash(0)$; $\bar{R} = R/aR$ is a Kasch ring. The ring $\bar{R}$ is a finite direct sum of fields if and only if all maximal ideals of the ring $\bar{R}$ are pure.*

Proof. According to Theorem 2.4 and Corollary 2.1, any maximal ideal of the ring $\bar{R}$ has the form $\bar{M} = \bar{e}\bar{R}$, where $\bar{e}^2 = \bar{e}$. Based on [10], we obtain the proof of our Corollary.

Theorem 6. *Let R be a commutative Bezout domain and, for $a \in R\backslash(0)$; $\bar{R} = R/aR$ is a Kasch ring and let a maximal ideal $\bar{M}$ of the ring $\bar{R}$ be flat. Then there exists an idempotent $\bar{e}$ of the ring $\bar{R}$ such that $\bar{M} = \bar{e}\bar{R}$.*

Proof. Since $\bar{R}$ is a Kasch ring, then $\bar{M} = \bar{p}\bar{R}$. And since $Ann(\bar{M}) \subset \mathrm{rad}(\bar{R})$, then according to [11] $\bar{M}$ is a projective module that is generated by an idempotent. The theorem is proved.

Similarly to the Corollary 2.2, by virtue of Theorem 2.5, we obtain the following result.

Corollary 3. *Let R be a commutative Bezout domain and, for $a \in R\backslash(0)$, $\bar{R} = R/aR$ is a Kasch ring. The ring $\bar{R}$ is a finite direct sum of fields if and only if all maximal ideals of the ring $\bar{R}$ are flat.*

We formulate a similar question (about the purity and flatness) for the case of annihilator of arbitrary maximal ideal of the ring $\bar{R}$.

According to [4], the pure ideal contained in the Jacobson radical is zero and therefore preliminary results we obtain the following result.

Corollary 4. *Let R be a commutative Bezout domain and, for $a \in R\backslash(0)$, $\bar{R} = R/aR$ is a Kasch ring. The ring $\bar{R}$ is a finite direct sum of fields if and only if an annihilator of every maximal ideal of the ring $\bar{R}$ is pure.*

As a consequence of previos results we obtain the following theorem.

Theorem 7. *Let R be a commutative Bezout domain in which an arbitrary maximal ideal is principal and let an element $a \in R\backslash(0)$ be not adequate. Then any maximal ideal $\bar{M} = M/aR$ of the ring $\bar{R} = R/aR$ (here M is a maximal ideal containing the element a) cannot be neither flat, pure, nor injective.*

Proof. Since an arbitrary maximal ideal of the ring R is a principal ideal, then by [12] the annihilator of an arbitrary element $\bar{b} \in \bar{R}$ is a principal ideal. By virtue of the article [7] in the ring $\bar{R}$ the following holds: $Ann(Ann(\bar{b})) = \bar{b}\bar{R}$. Then from [8] we obtain that $\bar{R}$ is a Kasch ring. According to Proposition 1.1 and Corollaries 2.1, 2.4 we obtain that

$\bar{M}$ is neither pure nor flat $\bar{R}$-module. Since the ring $\bar{R}$ is an IF-ring, $\bar{M}$ cannot be an injective $\bar{R}$-module. The theorem is proved.

References

1. V. Camillo, W.K. Nicholson, Z. Wang, Left quasi-morphic rings, *J. Algebra Appl.*, **7** (2008), pp. 725–733.
2. A. W. Chatters and C. R. Hajarnavis, Noetherian rings with projective Jacobson radical, *Comm. Algebra*, **3(6)** (1985), pp. 1359–1366.
3. Colby R., Rings which have flat injective modules, *J. Algebra*, **35** (1975), pp. 239–252.
4. De Marco G., Projectivity of pure ideals, *Rendiconti del Seminario Matematico della Universita di Padova*, **69** (1983), pp. 289–304.
5. N. Ding, Y. Li, L. Mao, J-coherent rings, *J. Algebra Appl.*, **8** (2009), pp. 139–155.
6. Faith C., Annihilator ideals associated primes and Kasch-McCoy commutative rings, *Comm. Algebra*, **19(7)** (1991), pp. 1867–1892.
7. G. A. Garkusha, FP-injective and weakly Quasi-Frobenius rings, *Zap. Nauchn. Sem. S.-Peterburg Otdel. Mat. Inst. Steklov. (POMI)*, **265** (1999), pp. 110-–129 (in Russian); translation in *J. Math. Sci. (New York)* **112** (2002), no. 3, pp. 4303–4312.
8. Nicholsow W.K.,Watter J.F., Rings with projective socle, *Proc. Amer. Math. Soc.* **102 (3)** (1988), pp. 443–450.
9. Nicholson W.K., Yousif M.F., On Principally Injective Rings, *J. Algebra*, **174** (1995), pp. 77–93.
10. M. Satyanarayana, Rings with primary ideals as maximal ideals, *Math.Scand.*, **20** (1967) pp. 52–54.
11. W. W. Smith, Projective ideals of finite type, *Canad. J. Math.*, **21** (1969), pp. 1057–1061.
12. Zabavsky B.V., Fractionally regular Bezout rings, *Math. Stud.*, **32** (2009), pp. 76–80.
13. B. V. Zabavsky, S. I. Bilavska Decomposition of finitely generated projective modules over Bezout ring, *Math. Stud.*, **40(1)** (2013), pp. 104–107.
14. B. V. Zabavsky, S. I. Bilavska Weakly global dimension of finite homomorfic images of commutative Bezout domain, *Applied prob. of mech. and math.*, **10** (2013), pp. 71–73. (in Ukrainian)
15. B. V. Zabavsky, M. Ya. Komarnitskiy, Remarks on adequate rings, *Visnyk Lviv. univ., ser. phys.-math.*, **27** (1988), pp. 39–43.(in Ukrainian)

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ПРО ВЛАСТИВОСТІ РАДИКАЛІВ І СПЕКТРА СКІНЧЕННИХ ГОМОМОРФНИХ ОБРАЗІВ КОМУТАТИВНОЇ ОБЛАСТІ БЕЗУ

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Досліджено властивості скінченних гомоморфних образів комутативних областей Безу за деяких умов на радикал Джекобсона і на максимальні ідеали. У випадку напівпростого кільця описано структуру максимальних ідеалів. Описано кільця R і R/aR за умови, що R/aR є кільцем Каша.

Ключові слова: кільце Безу, адекватне кільце, кільце Каша, чистий ідеал.