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ON THE DICHOTOMY OF A LOCALLY COMPACT SEMITOPOLOGICAL BICYCLIC MONOID WITH ADJOINED ZERO

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We prove that a Hausdorff locally compact semitopological bicyclic semigroup with adjoined zero $\mathscr{C}^0$ is either compact or discrete. Also we show that the similar statement holds for a locally compact semitopological bicyclic semigroup with an adjoined compact ideal and construct an example which witnesses that a counterpart of the statements does not hold when $\mathscr{C}^0$ is a Čech-complete metrizable topological inverse semigroup.

Key words: semigroup, semitopological semigroup, topological semigroup, bicyclic monoid, locally compact space, Čech-complete space, metrizable space, zero, compact ideal.

1. Introduction and preliminaries

Further we shall follow the terminology of [7, 8, 10, 24]. Given a semigroup S, we shall denote the set of idempotents of S by $E(S)$. A semigroup S with the adjoined zero will be denoted by S^0 (cf. [8]).

A semigroup S is called *inverse* if for every $x \in S$ there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$. Later such an element y will be denoted by x^{-1} and will be called the *inverse of* x. A map $\mathsf{inv}\colon S \to S$ which assigns to every $s \in S$ its inverse is called *inversion*.

In this paper all topological spaces are Hausdorff. If Y is a subspace of a topological space X and $A \subseteq Y$, then by $\operatorname{cl}_Y(A)$ we denote the topological closure of A in Y.

A *semitopological* (*topological*) *semigroup* is a topological space with separately continuous (jointly continuous) semigroup operations. An inverse topological semigroup with continuous inversion is called a *topological inverse semigroup*.

We recall that a topological space X is:

- *locally compact* if every point x of X has an open neighbourhood $U(x)$ with the compact closure $\operatorname{cl}_X(U(x))$;

- *Čech-complete* if X is Tychonoff and there exists a compactification cX of X such that the remainder $cX \setminus c(X)$ is an F_σ-set in cX.

The *bicyclic semigroup* (or the *bicyclic monoid*) $\mathscr{C}(p,q)$ is a semigroup with the identity 1 generated by two elements p and q with only one condition $pq = 1$. The distinct elements of the bicyclic monoid are exhibited in the following array:

$$\begin{array}{ccccc}
1 & p & p^2 & p^3 & \cdots \\
q & qp & qp^2 & qp^3 & \cdots \\
q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots \\
q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The bicyclic monoid is a combinatorial bisimple F-inverse semigroup and it plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known Andersen's result [1] states that a (0–)simple semigroup with an idempotent is completely (0–)simple if and only if it does not contain an isomorphic copy of the bicyclic semigroup. The bicyclic semigroup admits only the discrete semigroup topology and if a topological semigroup S contains it as a dense subsemigroup then $\mathscr{C}(p,q)$ is an open subset of S [11]. Bertman and West in [6] extended this result for the case of semitopological semigroups. Stable and Γ-compact topological semigroups do not contain the bicyclic semigroup [2, 15]. The problem of an embedding of the bicyclic monoid into compact-like topological semigroups is discussed in [4, 5, 13].

In [11] Eberhart and Selden proved that if the bicyclic monoid $\mathscr{C}(p,q)$ is a dense subsemigroup of a topological monoid S and $I = S \setminus \mathscr{C}(p,q) \neq \varnothing$ then I is a two-sided ideal of the semigroup S. Also, there they described the closure of the bicyclic monoid $\mathscr{C}(p,q)$ in a locally compact topological inverse semigroup. The closure of the bicyclic monoid in a countably compact (pseudocompact) topological semigroups was studied in [5].

The well known A. Weil Theorem states that *every locally compact monothetic topological group G* (i.e., G contains a cyclic dense subgroup) *is either compact or discrete* (see [26]). Locally compact and compact monothetic topological semigroups was studied by Hewitt [14], Hofmann [16], Koch [18], Numakura [23] and others (see more information on this topics in the books [7] and [17]). Koch in [19] posed the following problem: "*If S is a locally compact monothetic semigroup and S has an identity, must S be compact?*" (see [7, Vol. 2, p. 144]). From the other side, Zelenyuk in [27] constructed a countable locally compact topological semigroup without unit which is neither compact nor discrete.

In this paper we prove that a Hausdorff locally compact semitopological bicyclic semigroup with adjoined zero $\mathscr{C}^0$ is either compact or discrete. Also we show that the similar statement holds for a locally compact semitopological bicyclic semigroup with an adjoined compact ideal and construct an example which witnesses that a counterpart of the statements does not hold when $\mathscr{C}^0$ is a Čech-complete metrizable topological inverse semigroup.

2. On a locally compact semitopological bicyclic semigroup with adjoined zero

The following proposition generalizes Theorem I.3 from [11].

Proposition 1. *If the bicyclic monoid $\mathscr{C}(p,q)$ is a dense subsemigroup of a semitopological monoid S and $I = S \setminus \mathscr{C}(p,q) \neq \varnothing$ then I is a two-sided ideal of the semigroup S.*

Proof. Fix an arbitrary element $y \in I$. If $xy = z \notin I$ for some $x \in \mathscr{C}(p,q)$ then there exists an open neighbourhood $U(y)$ of the point y in the space S such that $\{x\} \cdot U(y) = \{z\} \subset \mathscr{C}(p,q)$. The neighbourhood $U(y)$ contains infinitely many elements of the semigroup $\mathscr{C}(p,q)$. This contradicts Lemma I.1 [11], which states that for each $v, w \in \mathscr{C}(p,q)$ both sets $\{u \in \mathscr{C}(p,q) \colon vu = w\}$ and $\{u \in \mathscr{C}(p,q) \colon uv = w\}$ are finite. The obtained contradiction implies that $xy \in I$ for all $x \in \mathscr{C}(p,q)$ and $y \in I$. The proof of the statement that $yx \in I$ for all $x \in \mathscr{C}(p,q)$ and $y \in I$ is similar.

Suppose to the contrary that $xy = w \notin I$ for some $x, y \in I$. Then $w \in \mathscr{C}(p,q)$ and the separate continuity of the semigroup operation in S implies that there exist open neighbourhoods $U(x)$ and $U(y)$ of the points x and y in S, respectively, such that $\{x\}\cdot U(y) = \{w\}$ and $U(x)\cdot\{y\} = \{w\}$. Since both neighbourhoods $U(x)$ and $U(y)$ contain infinitely many elements of the semigroup $\mathscr{C}(p,q)$, both equalities $\{x\} \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$ contradict mentioned above Lemma I.1 from [11]. The obtained contradiction implies that $xy \in I$.

For every non-negative integer n we put

$$\mathscr{C}[q^n] = \left\{q^n p^i \in \mathscr{C}(p,q) \colon i = 0,1,2,\ldots\right\} \quad \text{and} \quad \mathscr{C}[p^n] = \left\{q^i p^n \in \mathscr{C}(p,q) \colon i = 0,1,2,\ldots\right\}.$$

Lemma 1. *Let $(\mathscr{C}^0, \tau)$ be a locally compact semitopological semigroup. Then the following assertions hold:*

1. *for every open neighbourhood $U(0)$ of zero in $(\mathscr{C}^0, \tau)$ there exists an open compact neighbourhood $V(0)$ of zero in $(\mathscr{C}^0, \tau)$ such that $V(0) \subseteq U(0)$;*
2. *for every open compact neighbourhood $U(0)$ of zero in $(\mathscr{C}^0, \tau)$ and every open neighbourhood $V(0)$ of zero in $(\mathscr{C}^0, \tau)$ the set $U(0) \cap V(0)$ is compact and open, and the set $U(0) \setminus V(0)$ is finite.*

Proof. The statements of the lemma are trivial in the case when τ is the discrete topology on $\mathscr{C}^0$, and hence later we shall assume that the topology τ is non-discrete.

(1) Let $U(0)$ be an arbitrary open neighbourhood of zero in $(\mathscr{C}^0, \tau)$. By Theorem 3.3.1 from [10] the space $(\mathscr{C}^0, \tau)$ is regular. Since it is locally compact, there exists an open neighbourhood $V(0) \subseteq U(0)$ of zero in $(\mathscr{C}^0, \tau)$ such that $\operatorname{cl}_{\mathscr{C}^0}(V(0)) \subseteq U(0)$. Since all non-zero elements of the semigroup $\mathscr{C}^0$ are isolated points in $(\mathscr{C}^0, \tau)$, $\operatorname{cl}_{\mathscr{C}^0}(V(0)) = V(0)$, and hence our assertion holds.

(2) Let $U(0)$ be an arbitrary compact open neighbourhood of zero in $(\mathscr{C}^0, \tau)$. Then for an arbitrary open neighbourhood $V(0)$ of zero in $(\mathscr{C}^0, \tau)$ the family

$$\mathscr{U} = \{V(0), \{\{x\} \colon x \in U(0) \setminus V(0)\}\}$$

is an open cover of $U(0)$. Since the family $\mathscr{U}$ is disjoint, it is finite. So the set $U(0) \setminus V(0)$ is finite and the set $U(0) \cap V(0)$ is compact.

Lemma 2. *If $(\mathscr{C}^0,\tau)$ is a locally compact non-discrete semitopological semigroup, then for each open neighbourhood $U(0)$ of zero in $(\mathscr{C}^0,\tau)$ there exist non-negative integers i and j such that both sets $\mathscr{C}[q^i]\cap U(0)$ and $\mathscr{C}[p^j]\cap U(0)$ are infinite.*

Proof. By Lemma 1(1), without loss of generality we may assume that $U(0)$ is a compact open neighbourhood of zero 0 in $(\mathscr{C}^0,\tau)$. Put

$$V_q(0)=\{x\in U(0)\colon x\cdot q\in U(0)\} \qquad \text{and} \qquad V_p(0)=\{x\in U(0)\colon p\cdot x\in U(0)\}.$$

If the set $\mathscr{C}[q^i]\cap U(0)$ is finite for any non-negative integer i, then the formula

$$q^ip^l\cdot q=\begin{cases} q^{i+1}, & \text{if } l=0;\\ q^ip^{l-1}, & \text{if } l \text{ is a positive integer,}\end{cases} \tag{1}$$

implies that the right translation $\rho_q\colon \mathscr{C}^0\to\mathscr{C}^0\colon x\mapsto x\cdot q$ shifts all non-zero elements of the neighbourhood $V_q(0)$. Then $U(0)\setminus V_q(0)$ is an infinite subset of $\mathscr{C}(p,q)$, which contradicts Lemma 1(2). Similarly, if the set $\mathscr{C}[p^j]\cap U(0)$ is finite for any non-negative integer j, then the formula

$$p\cdot q^jp^l=\begin{cases} p^{l+1}, & \text{if } j=0;\\ q^{j-1}p^l, & \text{if } j \text{ is a positive integer,}\end{cases} \tag{2}$$

implies that the left translation $\lambda_p\colon \mathscr{C}^0\to\mathscr{C}^0\colon x\mapsto p\cdot x$ shifts all non-zero elements of the neighbourhood $V_p(0)$. This implies that $U(0)\setminus V_p(0)$ is an infinite subset of $\mathscr{C}(p,q)$, which contradicts Lemma 1(2).

Lemma 3. *Let $(\mathscr{C}^0,\tau)$ be a locally compact non-discrete semitopological semigroup. Then there exist non-negative integers i and j such that $\mathscr{C}[q^i]\setminus U(0)$ and $\mathscr{C}[p^j]\setminus U(0)$ are finite for every open neighbourhood $U(0)$ of zero 0 in $(\mathscr{C}^0,\tau)$.*

Proof. Fix an arbitrary open compact neighbourhood $U_0(0)$ of zero in $(\mathscr{C}^0,\tau)$. Then Lemma 2 implies that there exist non-negative integers i and j such that both sets $\mathscr{C}[q^i]\cap U_0(0)$ and $\mathscr{C}[p^j]\cap U_0(0)$ are infinite. Let $U(0)$ be an arbitrary open neighbourhood of zero in $(\mathscr{C}^0,\tau)$. By Lemma 1(2), the set $U_0(0)\setminus U(0)$ is finite. By Lemma 1(1), there exists an open compact neighbourhood $U'(0)\subseteq U(0)$ of zero in $(\mathscr{C}^0,\tau)$.

Now, Lemma 1(1) and the separate continuity of the semigroup operation in $(\mathscr{C}^0,\tau)$ imply that there exists an open compact neighbourhood $V(0)$ of zero 0 in $(\mathscr{C}^0,\tau)$ such that

$$V(0)\subseteq U'(0), \qquad V(0)\cdot q\subseteq U'(0) \qquad \text{and} \qquad p\cdot V(0)\subseteq U'(0).$$

If the set $\mathscr{C}[q^i]\setminus U(0)$ is infinite, then formula (1) implies that the right translation $\rho_q\colon \mathscr{C}^0\to\mathscr{C}^0\colon x\mapsto x\cdot q$ shifts all non-zero elements of the neighbourhood $V(0)$ and hence the inclusion $V(0)\cdot q\subseteq U'(0)$ implies that $U'(0)\setminus V(0)$ is an infinite set, which contradicts Lemma 1(2). Hence the set $\mathscr{C}[q^i]\setminus U(0)$ is finite. Similarly, if the set $\mathscr{C}[p^j]\setminus U(0)$ is infinite, then by formula (2) we have that the left translation $\lambda_p\colon \mathscr{C}^0\to\mathscr{C}^0\colon x\mapsto p\cdot x$ shifts all non-zero elements of the neighbourhood $V(0)$ and hence the by inclusion $p\cdot V(0)\subseteq U'(0)$ we obtain that $U'(0)\setminus V(0)$ is an infinite set, which contradicts Lemma 1(2). Therefore, the set $\mathscr{C}[p^j]\setminus U(0)$ is finite as well.

Lemma 4. *Let $(\mathscr{C}^0,\tau)$ be a locally compact non-discrete semitopological semigroup. Then for every open neighbourhood $U(0)$ of zero 0 in $(\mathscr{C}^0,\tau)$ and any non-negative integer i both sets $\mathscr{C}[q^i]\setminus U(0)$ and $\mathscr{C}[p^i]\setminus U(0)$ are finite.*

Proof. By Lemma 1(1), without loss of generality we may assume that the open neighbourhood $U(0)$ is compact. By Lemma 3 there exists a non-negative integer i_0 such that $\mathscr{C}[q^{i_0}]\setminus U'(0)$ is finite for any open compact neighbourhood $U'(0)$ of zero 0 in $(\mathscr{C}^0,\tau)$.

Fix an arbitrary non-negative integer $i\neq i_0$. If $i<i_0$, then the separate continuity of the semigroup operation in $(\mathscr{C}^0,\tau)$ implies that there exists an open compact neighbourhood $V(0)\subseteq U(0)$ of zero 0 in $(\mathscr{C}^0,\tau)$ such that $p^{i_0-i}\cdot V(0)\subseteq U(0)$. Then

$$p^{i_0-i}\cdot q^{i_0}p^l=q^ip^l, \qquad \text{for any non-negative integer } l. \tag{3}$$

The set $\mathscr{C}[q^{i_0}]\setminus V(0)$ is finite, and hence by (3) the set $\mathscr{C}[q^i]\setminus U(0)\subseteq\mathscr{C}[q^i]\setminus\left(p^{i_0-i}\cdot V(0)\right)$ is finite as well.

If $i>i_0$, then the separate continuity of the semigroup operation in $(\mathscr{C}^0,\tau)$ implies that there exists an open compact neighbourhood $W(0)\subseteq U(0)$ of zero 0 in $(\mathscr{C}^0,\tau)$ such that $q^{i-i_0}\cdot W(0)\subseteq U(0)$. Then

$$q^{i-i_0}\cdot q^{i_0}p^l=q^ip^l, \qquad \text{for any non-negative integer } l, \tag{4}$$

The set $\mathscr{C}[q^{i_0}]\setminus W(0)$ is finite, and hence (4) implies that the set $\mathscr{C}[q^i]\setminus U(0)\subseteq\mathscr{C}[q^i]\setminus\left(q^{i-i_0}\cdot W(0)\right)$ is finite as well.

The proof of finiteness of the set $\mathscr{C}[p^i]\setminus U(0)$ is similar.

Lemma 5. *Let $(\mathscr{C}^0,\tau)$ be a non-discrete locally compact semitopological semigroup. Then for every open neighbourhood $U(0)$ of zero 0 in $(\mathscr{C}^0,\tau)$ the set $\mathscr{C}^0\setminus U(0)$ is finite.*

Proof. Suppose to the contrary that there exists an open neighbourhood $U(0)$ of zero 0 in $(\mathscr{C}^0,\tau)$ such that $\mathscr{C}^0\setminus U(0)$ is infinite. Lemma 1(1) implies that without loss of generality we may assume that the neighbourhood $U(0)$ is compact.

Now, the separate continuity of the semigroup operation in $(\mathscr{C}^0,\tau)$ implies that there exists an open neighbourhood $V(0)\subseteq U(0)$ of zero 0 in $(\mathscr{C}^0,\tau)$ such that $p\cdot V(0)\subseteq U(0)$. By Lemma 4 for every non-negative integer n both sets $\mathscr{C}[q^n]\setminus U(0)$ and $\mathscr{C}[p^n]\setminus U(0)$ are finite. Thus, the following conditions hold:

- (i) $U(0)\cup\bigcup_{n=0}^{m}\left(\mathscr{C}[q^n]\cup\mathscr{C}[p^n]\right)\neq\mathscr{C}^0$ for every positive integer m;
- (ii) for every positive integer k there exists a non-negative integer $k_{\max}$ such that $\left\{q^kp^j : j\geqslant k_{\max}\right\}\subset U(0)$.

We have $p\cdot q^kp^l=q^{k-1}p^k$ for any integers $k\geqslant 1$ and l. This and conditions (i) and (ii) imply that the set $U(0)\setminus V(0)$ is infinite, which contradicts Lemma 1(2). The obtained contradiction implies the statement of the lemma.

The following simple example shows that on the semigroup $\mathscr{C}^0$ there exists a topology τ_{Ac} such that $(\mathscr{C}^0,\tau_{\mathsf{Ac}})$ is a compact semitopological semigroup.

Example 1. On the semigroup $\mathscr{C}^0$ we define a topology τ_{Ac} in the following way:

- (i) every element of the bicyclic monoid $\mathscr{C}(p,q)$ is an isolated point in the space $(\mathscr{C}^0,\tau_{\mathsf{Ac}})$;
- (ii) the family $\mathscr{B}(0)=\left\{U\subseteq\mathscr{C}^0 : U\ni 0 \text{ and } \mathscr{C}(p,q)\setminus U \text{ is finite}\right\}$ determines a base of the topology τ_{Ac} at zero $0\in\mathscr{C}^0$,

i.e., τ_{Ac} is the topology of the Alexandroff one-point compactification of the discrete space $\mathscr{C}(p,q)$ with the remainder $\{0\}$. The semigroup operation in $(\mathscr{C}^0,\tau_{\mathsf{Ac}})$ is separately continuous, because all elements of the bicyclic semigroup $\mathscr{C}(p,q)$ are isolated points in the space $(\mathscr{C}^0,\tau_{\mathsf{Ac}})$.

Remark 1. In [6] Bertman and West showed that the discrete topology τ_{d} is a unique topology on the bicyclic monoid $\mathscr{C}(p,q)$ such that $\mathscr{C}(p,q)$ is a semitopological semigroup. So τ_{Ac} is the unique compact topology on $\mathscr{C}^0$ such that $(\mathscr{C}^0,\tau_{\mathsf{Ac}})$ is a compact semitopological semigroup.

Lemma 5 and Remark 1 imply the following dichotomy for a locally compact semitopological semigroup $\mathscr{C}^0$.

Theorem 1. *If $\mathscr{C}^0$ is a Hausdorff locally compact semitopological semigroup, then either $\mathscr{C}^0$ is discrete or $\mathscr{C}^0$ is topologically isomorphic to $(\mathscr{C}^0,\tau_{\mathsf{Ac}})$.*

Since the bicyclic monoid $\mathscr{C}(p,q)$ does not embeds into any Hausdorff compact topological semigroup [2], Theorem 1 implies the following corollary.

Corollary 1. *If $\mathscr{C}^0$ is a Hausdorff locally compact semitopological semigroup, then $\mathscr{C}^0$ is discrete.*

The following example shows that a counterpart of the statement of Corollary 1 does not hold when $\mathscr{C}^0$ is a Čech-complete metrizable topological inverse semigroup.

Example 2. On the semigroup $\mathscr{C}^0$ we define a topology τ_1 in the following way:

(i) every element of the bicyclic monoid $\mathscr{C}(p,q)$ is an isolated point in the space $(\mathscr{C}^0,\tau_1)$;

(ii) the family $\mathscr{B}(0)=\{U_n\colon n=0,1,2,3,\ldots\}$, where
$$U_n=\{0\}\cup\left\{q^ip^j\in\mathscr{C}(p,q)\colon i,j>n\right\},$$
determines a base of the topology τ_1 at zero $0\in\mathscr{C}^0$.

It is obvious that $(\mathscr{C}^0,\tau_1)$ is first countable space and the arguments presented in [12, p. 68] show that $(\mathscr{C}^0,\tau_1)$ is a Hausdorff topological inverse semigroup.

First we observe that each element of the family $\mathscr{B}(0)$ is an open closed subset of $(\mathscr{C}^0,\tau_1)$, and hence the space $(\mathscr{C}^0,\tau_1)$ is regular. Since the set $\mathscr{C}^0$ is countable, the definition of the topology τ_1 implies that $(\mathscr{C}^0,\tau_1)$ is second countable, and hence by Theorem 4.2.9 from [10] the space $(\mathscr{C}^0,\tau_1)$ is metrizable. Also, it is obvious that the space $(\mathscr{C}^0,\tau_1)$ is Čech-complete, as a union two Čech-complete spaces: that are the discrete space $\mathscr{C}(p,q)$ and the singleton space $\{0\}$.

3. On a locally compact semitopological bicyclic semigroup with an adjoined compact ideal

Later we need the following notions. A continuous map $f\colon X\to Y$ from a topological space X into a topological space Y is called:

- *quotient* if the set $f^{-1}(U)$ is open in X if and only if U is open in Y (see [22] and [10, Section 2.4]);
- *hereditarily quotient* or *pseudoopen* if for every $B\subset Y$ the restriction $f|_B\colon f^{-1}(B)\to B$ of f is a quotient map (see [20, 21, 3] and [10, Section 2.4]);
- *closed* if $f(F)$ is closed in Y for every closed subset F in X;

- *perfect* if X is Hausdorff, f is a closed map and all fibers $f^{-1}(y)$ are compact subsets of X [25].

Every closed map and every hereditarily quotient map are quotient [10]. Moreover, a continuous map $f\colon X\to Y$ from a topological space X onto a topological space Y is hereditarily quotient if and only if for every $y\in Y$ and every open subset U in X which contains $f^{-1}(y)$ we have that $y\in \operatorname{int}_Y(f(U))$ (see [10, 2.4.F]).

Later we need the following trivial lemma, which follows from separate continuity of the semigroup operation in semitopological semigroups.

Lemma 6. *Let S be a Hausdorff semitopological semigroup and I be a compact ideal in S. Then the Rees-quotient semigroup S/I with the quotient topology is a Hausdorff semitopological semigroup.*

Theorem 2. *Let $(\mathscr{C}_I,\tau)$ be a Hausdorff locally compact semitopological semigroup, $\mathscr{C}_I=\mathscr{C}(p,q)\sqcup I$ and I is a compact ideal of $\mathscr{C}_I$. Then either $(\mathscr{C}_I,\tau)$ is a compact semitopological semigroup or the ideal I open.*

Proof. Suppose that I is not open. By Lemma 6 the Rees-quotient semigroup $\mathscr{C}_I/I$ with the quotient topology τ_{q} is a semitopological semigroup. Let $\pi\colon \mathscr{C}_I\to\mathscr{C}_I/I$ be the natural homomorphism which is a quotient map. It is obvious that the Rees-quotient semigroup $\mathscr{C}_I/I$ is isomorphic to the semigroup $\mathscr{C}^0$ and the image $\pi(I)$ is zero of $\mathscr{C}^0$. Now we shall show that the natural homomorphism $\pi\colon \mathscr{C}_I\to\mathscr{C}_I/I$ is a hereditarily quotient map. Since $\pi(\mathscr{C}(p,q))$ is a discrete subspace of $(\mathscr{C}_I/I,\tau_{\mathsf{q}})$, it is sufficient to show that for every open neighbourhood $U(I)$ of the ideal I in the space $(\mathscr{C}_I,\tau)$ we have that the image $\pi(U(I))$ is an open neighbourhood of the zero 0 in the space $(\mathscr{C}_I/I,\tau_{\mathsf{q}})$. Indeed, $\mathscr{C}_I\setminus U(I)$ is a closed-and-open subset of $(\mathscr{C}_I,\tau)$, because the elements of the bicyclic monoid $\mathscr{C}(p,q)$ are isolated point of $(\mathscr{C}_I,\tau)$. Also, since the restriction $\pi|_{\mathscr{C}(p,q)}\colon \mathscr{C}(p,q)\to\pi(\mathscr{C}(p,q))$ of the natural homomorphism $\pi\colon \mathscr{C}_I\to\mathscr{C}_I/I$ is one-to-one, $\pi(\mathscr{C}_I\setminus U(I))$ is a closed-and-open subset of $(\mathscr{C}_I/I,\tau_{\mathsf{q}})$. So $\pi(U(I))$ is an open neighbourhood of the zero 0 of the semigroup $(\mathscr{C}_I/I,\tau_{\mathsf{q}})$, and hence the natural homomorphism $\pi\colon \mathscr{C}_I\to\mathscr{C}_I/I$ is a hereditarily quotient map. Since I is a compact ideal of the semitopological semigroup $(\mathscr{C}_I,\tau)$, $\pi^{-1}(y)$ is a compact subset of $(\mathscr{C}_I,\tau)$ for every $y\in\mathscr{C}_I/I$. By Din' N'e T'ong's Theorem (see [9] or [10, 3.7.E]), $(\mathscr{C}_I/I,\tau_{\mathsf{q}})$ is a Hausdorff locally compact space. If I is not open then by Theorem 1 the semitopological semigroup $(\mathscr{C}_I/I,\tau_{\mathsf{q}})$ is topologically isomorphic to $(\mathscr{C}^0,\tau_{\mathsf{Ac}})$ and hence it is compact. Next we shall prove that the space $(\mathscr{C}_I,\tau)$ is compact. Let $\mathscr{U}=\{U_\alpha\colon \alpha\in\mathscr{I}\}$ be an arbitrary open cover of $(\mathscr{C}_I,\tau)$. Since I is compact, there exist $U_{\alpha_1},\ldots,U_{\alpha_n}\in\mathscr{U}$ such that $I\subseteq U_{\alpha_1}\cup\cdots\cup U_{\alpha_n}$. Put $U=U_{\alpha_1}\cup\cdots\cup U_{\alpha_n}$. Then $\mathscr{C}_I\setminus U$ is a closed-and-open subset of $(\mathscr{C}_I,\tau)$. Also, since the restriction $\pi|_{\mathscr{C}(p,q)}\colon \mathscr{C}(p,q)\to\pi(\mathscr{C}(p,q))$ of the natural homomorphism $\pi\colon \mathscr{C}_I\to\mathscr{C}_I/I$ is one-to-one, $\pi(\mathscr{C}_I\setminus U(I))$ is a closed-and-open subset of $(\mathscr{C}_I/I,\tau_{\mathsf{q}})$, and hence the image $\pi(\mathscr{C}_I\setminus U(I))$ is finite, because the semigroup $(\mathscr{C}_I/I,\tau_{\mathsf{q}})$ is compact. Thus, the set $\mathscr{C}_I\setminus U$ is finite and hence the space $(\mathscr{C}_I,\tau)$ is compact as well.

Corollary 2. *If $(\mathscr{C}_I,\tau)$ is a locally compact topology topological semigroup, $\mathscr{C}_I=\mathscr{C}(p,q)\sqcup I$ and I is a compact ideal of $\mathscr{C}_I$, then the ideal I is open.*

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ПРО ДИХОТОМІЮ ЛОКАЛЬНО КОМПАКТНОГО НАПІВТОПОЛОГІЧНОГО БІЦИКЛІЧНОГО МОНОЇДА З ПРИЄДНАНИМ НУЛЕМ

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Доведено таке: що гаусдорфова локально компактна напівтопологічна біциклічна напівгрупа з приєднаним нулем $\mathscr{C}^0$ є або компактною, або дискретною. Також доведено, що аналогічне твердження виконується для локально компактного напівтопологічного біциклічного моноїда з приєднаним компактним ідеалом, і побудовано приклад, який доводить, що аналог цих тверджень не виконується, коли $\mathscr{C}^0$ — повна за Чехом метризовна топологічна інверсна напівгрупа.

Ключові слова: напівгрупа, напівтопологічна напівгрупа, топологічна напівгрупа, біциклічний моноїд, локально компактний простір, повний за Чехом простір, метризовний простір, нуль, компактний ідеал.