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RELATIVE GROWTH OF HADAMARD COMPOSITIONS OF ENTIRE DIRICHLET SERIES

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Let $F(s)=\sum\limits_{n=1}^{\infty}a_n\exp\{s\lambda_n\}$ and $F_j(s)=\sum\limits_{n=1}^{\infty}a_{n,j}\exp\{s\lambda_n\}$ be entire Dirichlet series with exponents $0\le\lambda_n\uparrow+\infty$. The function F is called Hadamard composition of the genus $m\ge1$ of the functions F_j if $a_n=P(a_{n,1},\dots,a_{n,p})$, where $P(x_1,\dots,x_p)=\sum\limits_{k_1+\dots+k_p=m}c_{k_1\dots k_p}x_1^{k_1}\cdot\ldots\cdot x_p^{k_p}$ is a homogeneous polynomial of degree $m\ge1$. The growth of the function F with respect to the entire Dirichlet series $G(s)=\sum\limits_{n=1}^{\infty}g_n\exp\{s\lambda_n\}$ is identified with the growth of the function $M_G^{-1}(M_F(\sigma))$, where $M_F(\sigma)=\sup\{|F(\sigma+it)|:\ t\in\mathbb{R}\}$. The dependence of the growth of a function $M_G^{-1}(M_F(\sigma))$ on the growth of functions $M_G^{-1}(M_{F_j}(\sigma))$ is studied in terms of generalized orders and a generalized convergence class.

Key words: Dirichlet series, Hadamard composition, generalized order, generalized convergence class.

1. Introduction

Let f and g be entire transcendental functions and

$$M_f(r)=\max\{|f(z)|:\ |z|=r\}.$$

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For the study of relative growth of the functions f and g Ch. Roy [1] used the order

$$\varrho_g[f] = \varlimsup_{r\to+\infty} \ln M_g^{-1}(M_f(r))/\ln r$$

and the lower order

$$\lambda_g[f] = \varliminf_{r\to+\infty} \ln M_g^{-1}(M_f(r))/\ln r$$

of the function f with respect to the function g. Research of relative growth of entire functions was continued by S. K. Data, T. Biswas and other mathematicians (see, for example, [2], [3], [4], [5]) in terms of maximal terms, Nevanlinna characteristic function and k-logarithmic orders. In [6] it is considered a relative growth of entire functions of two complex variables and in [7] the relative growth of entire Dirichlet series is studied in the terms of R-orders.

Suppose that $\Lambda = (\lambda_n)$ is an increasing to $+\infty$ sequence of non-negative numbers and by $S(\Lambda)$ we denote a class of entire Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f_n \exp\{s\lambda_n\}, \quad s = \sigma + it. \tag{1}$$

For $\sigma < +\infty$ we put $M_F(\sigma) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ and remark that the function $M_F(\sigma)$ is continuous and increasing to $+\infty$ on $(-\infty, +\infty)$ and, therefore, there exists the function $M_F^{-1}(x)$ inverse to $M_F(\sigma)$, which increase to $+\infty$ on $(x_0, +\infty)$. Thus, the growth of the Dirichlet series (1) with respect to the entire Dirichlet series

$$G(s) = \sum_{n=1}^{\infty} g_n \exp\{s\lambda_n\}$$

can be studied in terms of the growth of the function $M_G^{-1}(M_F(\sigma))$. Articles [8], [9], [10] are devoted to this problem.

Let $f_j(z) = \sum\limits_{n=0}^{\infty} a_{n,j} z^n$ $(j = 1, 2)$ be entire transcendental functions. The function $(f_1 * f_2)(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is called [11] Hadamard composition (product) of the functions f_j if $a_n = a_{n,1}a_{n,2}$ for all n. Obtained by J. Hadamard properties of this composition find the applications [12], [13] in the theory of the analytic continuation of the functions represented by power series. For a Dirichlet series, the usual Hadamard composition is defined in a similar way.

Now suppose that $F_j \in S(\Lambda)$,

$$F_j(s) = \sum_{n=1}^{\infty} f_{n,j} \exp\{s\lambda_n\}, \quad j = 1, 2, \ldots, p, \tag{2}$$

and recall that a polynomial is named homogeneous if all monomials with nonzero coefficients have the identical degree. A polynomial $P(x_1, \ldots, x_p)$ is homogeneous to the degree m if and only if $P(tx_1, \ldots, tx_p) = t^m P(x_1, \ldots, x_p)$ for all t from the field above that a polynomial is defined. Dirichlet series (1) is called [14], [15] a Hadamard composition of genus m of Dirichlet series (2) if $a_n = P(a_{n,1}, \dots, a_{n,p})$, where

$$P(x_1, \dots, x_p) = \sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p} x_1^{k_1} \cdot \dots \cdot x_p^{k_p}$$

is a homogeneous polynomial of degree $m \geq 1$. Therefore, if the function $F = (F_1 * \dots * F_p)$ is Hadamard composition of genus $m \geq 1$ of the functions F_j then for all n

$$f_n = \sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p} f_{n,1}^{k_1} \cdot \dots \cdot f_{n,p}^{k_p}. \tag{3}$$

Using this equality we will study the asymptotic behavior of Hadamard compositions of entire Dirichlet series.

2. Growth estimates from above

To characterize the growth of entire Dirichlet series we will use generalized orders. For this purpose, we denote by L a class of positive continuous functions α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$ and $0 < \alpha(x) \uparrow +\infty$ as $x_0 \leq x \uparrow +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i. e. α is slowly increasing function. Clearly, $L_{si} \subset L^0$.

Lemma 1 ([16]). *If $\beta \in L$ and $B(\delta) = \varlimsup\limits_{x\to+\infty} \dfrac{\beta((1+\delta)x)}{\beta(x)}$, $\delta > 0$, then in order that $\beta \in L^0$, it is necessary and sufficient that $B(\delta) \to 1$ as $\delta \to +0$. In addition, if $\beta \in L^0$ then β is RO-varying function [17], that is for every $\lambda \geq 1$ there exists $K = K(\lambda) > 0$ such that $\beta(\lambda x) \leq K\beta(x)$ for all $x \geq x_0$.*

If $\alpha \in L$, $\beta \in L$ and $F \in S(\Lambda)$ then the quantities

$$\varrho_{\alpha,\beta}[F] = \varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)} \qquad \text{and} \qquad \lambda_{\alpha,\beta}[F] = \varliminf_{\sigma\to+\infty} \frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)}$$

are called [18], [19] the *generalized (α, β)-order* and the *generalized lower (α, β)-order* of F accordingly. Similarly, the generalized (α, β)-order $\varrho_{\alpha,\beta}[F]_G$ and the generalized lower (α, β)-order $\lambda_{\alpha,\beta}[F]_G$ of the function $F \in S(\Lambda)$ with respect to a function $G \in S(\Lambda)$, given by Dirichlet series $G(s) = \sum\limits_{n=1}^{\infty} g_n \exp\{s\lambda_n\}$ we define as follows

$$\varrho_{\alpha,\beta}[F]_G = \varlimsup_{\sigma\to+\infty} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)},$$

$$\lambda_{\alpha,\beta}[F]_G = \varliminf_{\sigma\to+\infty} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}.$$

We need also the following statement [20], [21, p. 184].

Lemma 2. *Let $\mu_F(\sigma) = \max\{|f_n| \exp\{\sigma\lambda_n\} : n \geq 0\}$ be the maximal term of series (1) and $\tau = \varlimsup\limits_{n\to\infty} \dfrac{\ln n}{\lambda_n}$. Then $\mu_F(\sigma) \leq M_F(\sigma) \leq \mu_F(\sigma + \tau + \varepsilon)$ for every $\varepsilon > 0$ and all $\sigma \geq \sigma_0(\varepsilon)$.*

The following theorem is true.

Theorem 1. *Let F is Hadamard composition of genus $m \geq 1$ of the functions F_j. If either $m = 1$, $\tau = 0$, $\alpha \in L^0$ and $\beta(\ln x) \in L^0$ or $m \geq 2$, $\tau < +\infty$, $\beta \in L$ and $\alpha(M_G^{-1}(e^x)) \in L_{si}$ then*

$$\varrho_{\alpha,\beta}[F]_G \leq \varrho = \max\{\varrho_{\alpha,\beta}[F_j]_G : 1 \leq j \leq p\}.$$

Proof. From (3) we obtain

$$|f_n| \leq \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}||f_{n,1}|^{k_1} \cdot \ldots \cdot |f_{n,p}|^{k_p},$$

whence

$$\mu_F(m\sigma) \leq \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|\mu_{F_1}(\sigma)^{k_1} \cdot \ldots \cdot \mu_{F_p}(\sigma)^{k_p}.$$

By Lemma 2 $M_F(\sigma - \tau - \varepsilon) \leq \mu_F(\sigma) \leq M_F(\sigma)$ and, therefore, we get

$$M_F(m\sigma - \tau - \varepsilon) \leq \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|M_{F_1}(\sigma)^{k_1} \cdot \ldots \cdot M_{F_p}(\sigma)^{k_p}. \tag{4}$$

Let

$$\varrho = \max\{\varrho_{\alpha,\beta}[F_j]_G : 1 \leq j \leq p\} < +\infty.$$

Then

$$M_{F_j}(\sigma) \leq M_G(\alpha^{-1}((\varrho + \varepsilon)\beta(\sigma)))$$

for every $\varepsilon > 0$ all $\sigma \geq \sigma_0(\varepsilon)$ and all j. Therefore, (4) implies

$$M_F(m\sigma - \tau - \varepsilon) \leq CM_G^m(\alpha^{-1}((\varrho + \varepsilon)\beta(\sigma))), \quad C = \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|. \tag{5}$$

If $m = 1$ and $\tau = 0$ then from (5) we have

$$M_F(\sigma - \varepsilon) \leq CM_G(\alpha^{-1}((\varrho + \varepsilon)\beta(\sigma))),$$

i.e.,

$$M_F(\sigma) \leq CM_G(\alpha^{-1}((\varrho + \varepsilon)\beta(\sigma + \varepsilon)))$$

for every $\varepsilon > 0$ and all σ large enough. Since the function $\ln M_G(\sigma)$ is convex on $(-\infty, +\infty)$, the function $(\ln M_G(\sigma))'$ is increasing (in points where the derivative does not exist $(\ln M_G(\sigma))'$ mean right-hand derivative), i.e., $\dfrac{d\ln M_G(\sigma)}{d\ln \sigma} \uparrow +\infty$ as $\sigma \to +\infty$.

Therefore, $\dfrac{d\ln M_G^{-1}(x)}{d\ln x}\to 0$ as $x\to+\infty$ and, thus, $M_G^{-1}\in L_{si}$. Since $\alpha\in L^0$, we get

$$\begin{aligned}\varrho_{\alpha,\beta}[F]_G &= \varlimsup_{\sigma\to+\infty}\frac{\alpha(M_G^{-1}(CM_F(\sigma)/C))}{\beta(\sigma)}\le\\ &\le\varlimsup_{\sigma\to+\infty}\frac{\alpha((1+o(1))M_G^{-1}(M_G(\alpha^{-1}((\varrho+\varepsilon)\beta(\sigma+\varepsilon)))))}{\beta(\sigma)}=\\ &=(\varrho+\varepsilon)\varlimsup_{\sigma\to+\infty}\frac{\beta(\sigma+\varepsilon)}{\beta(\sigma)}=\\ &=(\varrho+\varepsilon)\varlimsup_{x\to+\infty}\frac{\beta(\ln(xe^{\varepsilon}))}{\beta(\ln x)}=\\ &=(\varrho+\varepsilon)\varlimsup_{x\to+\infty}\frac{\beta(\ln(x(1+\delta)))}{\beta(\ln x)},\end{aligned}$$

where $\delta=e^{\varepsilon}-1\to 0$ as $\varepsilon\to 0$. Therefore, $\varrho_{\alpha,\beta}[F]_G\le(\varrho+\varepsilon)B(\delta)$ and in view of the arbitrariness of ε and of Lemma 1 $\varrho_{\alpha,\beta}[F]_G\le\varrho$, Q. E. D.

Now let $m\ge 2$. Then from the convexity of $\ln M_F(\sigma)$ it follows that

$$M_F(m\sigma-\tau-\varepsilon)\ge CM_F(\sigma)$$

for all σ enough large and, thus, in view of (5)

$$M_F(\sigma)\le M_G^m(\alpha^{-1}((\varrho+\varepsilon)\beta(\sigma))),$$

i.e.,

$$M_G^{-1}(M_F(\sigma)^{1/m})\le\alpha^{-1}((\varrho+\varepsilon)\beta(\sigma)).$$

Since $\alpha(M_G^{-1}(e^x))\in L_{si}$, for every $q>0$ we have

$$\begin{aligned}\alpha(M_G^{-1}(x^q))&=\alpha(M_G^{-1}(e^{q\ln x}))=(1+o(1))\alpha(M_G^{-1}(e^{\ln x}))=\\ &=(1+o(1))\alpha(M_G^{-1}(x))\end{aligned}$$

as $x\to+\infty$. Thus,

$$\begin{aligned}\alpha(M_G^{-1}(M_F(\sigma))&=(1+o(1))\alpha(M_G^{-1}(M_F(\sigma)^{1/m}))\le\\ &\le(1+o(1))(\varrho+\varepsilon)\beta(\sigma)),\quad\sigma\to+\infty,\end{aligned}$$

whence it follows that $\varrho_{\alpha,\beta}[F]_G\le\varrho$. Theorem 1 is proved. □

Remark 1. The condition $\alpha(M_G^{-1}(e^x))\in L_{si}$ holds if $\alpha\in L_{si}$ and for every $c\ge 1$ there exists $K(c)>1$ such that $M_G^{-1}(e^{cx})\le K(c)M_G^{-1}(e^x)$ for all $x\ge x_0$, i.e., $c\ln M_G(\sigma)\le$ $\le\ln M_G(K(c)\sigma)$ for all $\sigma\ge\sigma_0$. Suppose that $\dfrac{d\ln\ln M_G(\sigma)}{d\ln\sigma}\ge h>0$ for all σ enough large. Then

$$\ln\ln M_G(K(c)\sigma)-\ln\ln M_G(\sigma)=\int\limits_{\sigma}^{K(c)\sigma}\frac{d\ln\ln M_G(t)}{d\ln t}d\ln t\ge h\ln K(c),$$

whence it follows that if $K(c)=c^{1/h}$ then $c\ln M_G(\sigma)\le\ln M_G(K(c)\sigma)$. Therefore, $\alpha(M_G^{-1}(e^x))\in L_{si}$ if $\alpha\in L_{si}$ and $\dfrac{d\ln\ln M_G(\sigma)}{d\ln\sigma}\ge h>0$ for all σ enough large. Note that for every entire Dirichlet series $\ln\ln M_G(\sigma) \geq \ln\sigma$, and from the last condition it follows that $\ln\ln M_G(\sigma) \geq h(1+o(1))\ln\sigma$ as $\sigma \to +\infty$.

By choosing $\alpha(x) = \ln x$ and $\beta(x) = x$ for $x \geq x_0$ from the definition of $\varrho_{\alpha,\beta}[F]_G$ we obtain the definition of the relative R-order $\varrho_R[F]_G$. Theorem 1 implies the following statement.

Corollary 1. *Let F is Hadamard composition of genus $m \geq 1$ of the functions F_j. If either $m = 1$ and $\tau = 0$ or $m \geq 2$, $\tau < +\infty$ and $\dfrac{d\ln\ln M_G(\sigma)}{d\ln\sigma} \geq h > 0$ for $\sigma \geq \sigma_0$ then*

$$\varrho_R[F]_G \leq \max\{\varrho_R[F_j]_G : 1 \leq j \leq p\}.$$

Suppose that $\varrho_R[F]_G = \varrho_R[F_j]_G = \varrho \in (0,+\infty)$ for all $1 \leq j \leq p$ and choose $\alpha(x) = x$ and $\beta(x) = e^{\varrho x}$. Then we obtain the definition of the relative R-type $T_R[F]_G$, and from Theorem 1 we get the following corollary.

Corollary 2. *Let F is Hadamard composition of genus $m \geq 1$ of the functions F_j. If either $m = 1$ and $\tau = 0$ or $m \geq 2$, $\tau < +\infty$ and $\dfrac{d\ln\ln M_G(\sigma)}{d\ln\sigma} \to +\infty$ as $\sigma \to +\infty$ then*

$$T_R[F]_G \leq \max\{T_R[F_j]_G : 1 \leq j \leq p\}.$$

3. Asymptotic behavior of a certain class of Hadamard compositions

It is clear that in the general case it is impossible to estimate of $\varrho_{\alpha,\beta}[F_1 * \ldots * F_p]_G$ from below. Therefore, additional conditions are needed for the coefficients $f_{n,j}$. For example, suppose that $|c_{m0\ldots0}||f_{n,1}|^m > 0$ for all n and $|f_{n,j}| = o(|f_{n,1}|)$ as $n \to \infty$ for all $2 \leq j \leq p$. We put

$$\Sigma'_n = \sum_{k_1+\cdots+k_p=m,\, k_1\neq m} c_{k_1\ldots k_p}(f_{n,1})^{k_1}\cdot\ldots\cdot(f_{n,p})^{k_p} =$$
$$= \sum_{k_1+\ldots+k_p=m} c_{k_1\ldots k_p}(f_{n,1})^{k_1}\cdot\ldots\cdot(f_{n,p})^{k_p} - c_{m0\ldots0}(f_{n,1})^m.$$

Since for each monomial of the polynomial Σ'_n the sum of the exponents is equal to m, we have

$$\frac{|f_{n,1}|^{k_1}\cdot\ldots\cdot|f_{n,p}|^{k_p}}{|f_{n,1}|^m} = \frac{|f_{n,2}|^{k_2}\cdot\ldots\cdot|f_{n,p}|^{k_p}}{|f_{n,1}|^{m-k_1}} \to 0, \quad n \to \infty$$

and, thus $\Sigma'_n = o(|f_{n,1}|^m)$ as $n \to \infty$. Since

$$|c_{m0\ldots0}||f_{n,1}|^m - |\Sigma'_n| \leq |f_n| \leq |c_{m0\ldots0}||f_{n,1}|^m + |\Sigma'_n|,$$

we have

$$|f_n| = (1+o(1))|c_{m0\ldots0}||f_{n,1}|^m, \quad n \to \infty. \tag{6}$$

Since $\mu_F(\sigma) \to +\infty$ as $\sigma \to +\infty$, from (6) it follows that

$$\mu_F(m\sigma) = (1+o(1))|c_{m0\ldots0}|\mu_{F_1}(\sigma)^m$$

as $\sigma \to +\infty$. Therefore, as above

$$\begin{aligned} M_F(m\sigma - \tau - \varepsilon) \le \mu_F(m\sigma) = \\ = (1+o(1))|c_{m0\ldots 0}|\mu_{F_1}(\sigma)^m \le \\ \le (1+o(1))|c_{m0\ldots 0}|M_{F_1}(\sigma)^m \end{aligned}$$

and

$$\begin{aligned} M_{F_1}(\sigma - \tau - \varepsilon)^m \le \mu_{F_1}(\sigma)^m = \\ = \frac{1+o(1)}{|c_{m0\ldots 0}|}\mu_F(m\sigma) \le \\ \le \frac{1+o(1)}{|c_{m0\ldots 0}|}M_F(m\sigma) \end{aligned}$$

as $\sigma \to +\infty$, i.e.,

$$\begin{aligned} M_F(m\sigma - \tau - \varepsilon) \le (1+o(1))|c_{m0\ldots 0}|M_{F_1}(\sigma)^m \le \\ \le M_F(m\sigma + m\tau + m\varepsilon), \quad \sigma \to +\infty. \end{aligned} \tag{7}$$

Using (7) we prove the following theorem.

Theorem 2. *Let F be Hadamard composition of genus $m \ge 1$ of the functions F_j, $|c_{m0\ldots 0}||f_{n,1}|^m > 0$ for all n and $|f_{n,j}| = o(|f_{n,1}|)$ as $n \to \infty$ for all $2 \le j \le p$. If either $m = 1$, $\tau = 0$, $\alpha \in L^0$ and $\beta(\ln x) \in L^0$ or $m \ge 2$, $\tau < +\infty$, $\beta \in L_{si}$ and $\alpha(M_G^{-1}(e^x)) \in L_{si}$ then $\varrho_{\alpha,\beta}[F]_G = \varrho_{\alpha,\beta}[F_1]_G$.*

Proof. If $m = 1$ and $\tau = 0$ then (7) implies

$$M_F(\sigma - \varepsilon) \le (1+o(1))|c_{m0\ldots 0}|M_{F_1}(\sigma) \le M_F(\sigma + \varepsilon)$$

and, since $M_G^{-1} \in L_{si}$, we have

$$M_G^{-1}(M_F(\sigma - \varepsilon)) \le (1+o(1))M_G^{-1}(M_{F_1}(\sigma)) \le M_G^{-1}(M_F(\sigma + \varepsilon)),$$

whence in view of conditions $\alpha \in L^0$ and $\beta(\ln x) \in L^0$ as above we get

$$\varrho_{\alpha,\beta}[F]_G = \varrho_{\alpha,\beta}[F_1]_G.$$

If $m \ge 2$ and $\tau < +\infty$ then (7) implies

$$\begin{aligned} \frac{\alpha(M_G^{-1}(M_F(m\sigma - \tau - \varepsilon)))}{\beta(\sigma)} \le (1+o(1))\frac{\alpha(M_G^{-1}(M_{F_1}(\sigma)^m))}{\beta(\sigma)} \le \\ \le \frac{\alpha(M_G^{-1}(M_F(m\sigma + m\tau + m\varepsilon)))}{\beta(\sigma)}, \end{aligned}$$

whence in view of conditions $\beta \in L_{si}$ and $\alpha(M_G^{-1}(e^x)) \in L_{si}$ as above we get

$$\varrho_{\alpha,\beta}[F]_G = \varrho_{\alpha,\beta}[F_1]_G,$$

Q. E. D. □

Remark 2. If the conditions of Theorem 2 hold then

$$\lambda_{\alpha,\beta}[F]_G = \lambda_{\alpha,\beta}[F_1]_G.$$

4. Relative convergence class

For an entire functions f of the order $\varrho \in (0, +\infty)$ G. Valiron [22, p. 18] defined the convergence class by condition

$$\int_{r_0}^{\infty} \frac{\ln M_f(r)}{r^{\varrho+1}} dr < +\infty.$$

According to P. Kampthan [23], an entire Dirichlet series (1) of the R-order $\varrho_R \in (0, +\infty)$ belongs, by definition, to the convergence class if

$$\int_{\sigma_0}^{\infty} e^{-\varrho_R \sigma} \ln M_F(\sigma) d\sigma < +\infty.$$

For entire Dirichlet series and functions $\alpha \in L$, $\beta \in L$ the generalized convergence $\alpha\beta$-class is defined [24], [25] by the condition

$$\int_{\sigma_0}^{\infty} \frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)} d\sigma < +\infty.$$

By analogy, we define a relative generalized convergence $\alpha\beta$-class by the condition

$$\int_{\sigma_0}^{\infty} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)} d\sigma < +\infty.$$

Theorem 3. *Let F is Hadamard composition of genus $m \geq 1$ of the functions F_j, $\tau < +\infty$, $\beta \in L^0$ and $\alpha(M_G^{-1}(e^x)) \in L^0$. If F_j belongs to relative generalized convergence $\alpha\beta$-class for all j then F belongs to the same class.*

Proof. For σ enough large from (4) we get

$$\alpha(M_G^{-1}(M_F(m\sigma - \tau - \varepsilon))) \leq$$

$$\leq \alpha\left(M_G^{-1}\left(\sum_{k_1+\ldots+k_p=m} |c_{k_1\ldots k_p}| M_{F_1}(\sigma)^{k_1} \cdot \ldots \cdot M_{F_p}(\sigma)^{k_p}\right)\right) =$$

$$= \alpha\left(M_G^{-1}\left(\exp\left\{\ln \sum_{k_1+\ldots+k_p=m} |c_{k_1\ldots k_p}| M_{F_1}(\sigma)^{k_1} \cdot \ldots \cdot M_{F_p}(\sigma)^{k_p}\right\}\right)\right) \leq$$

$$= \alpha\left(M_G^{-1}\left(\exp\left\{\sum_{k_1+\ldots+k_p=m} (k_1 \ln M_{F_1}(\sigma) + \ldots + k_p \ln M_{F_p}(\sigma)) + K_1\right\}\right)\right),$$

where

$$K_1 = \ln p + \sum_{k_1+\ldots+k_p=m} \ln |c_{k_1\ldots k_p}|.$$

Since $\alpha(M_G^{-1}(e^x)) \in L^0$, by Lemma 1 we have

$$\alpha(M_G^{-1}(M_F(m\sigma-\tau-\varepsilon)))\le$$
$$\le K_2\alpha\left(M_G^{-1}\left(\exp\left\{m\sum_{k_1+\cdots+k_p=m}(\ln M_{F_1}(\sigma)+\cdots+\ln M_{F_p}(\sigma))\right\}\right)\right)\le$$
$$\le K_2\alpha\left(M_G^{-1}\left(\exp\left\{m\sum_{k_1+\cdots+k_p=m}p\cdot\max\{\ln M_{F_j}(\sigma):\,1\le j\le p\}\right\}\right)\right)=$$
$$=K_2\alpha\left(M_G^{-1}\left(\exp\left\{mpK_3\max\{\ln M_{F_j}(\sigma):\,1\le j\le p\}\right\}\right)\right)\le$$
$$\le K_4\alpha\left(M_G^{-1}\left(\exp\left\{\max\{\ln M_{F_j}(\sigma):\,1\le j\le p\}\right\}\right)\right)=$$
$$=K_4\max\{\alpha\left(M_G^{-1}\left(M_{F_j}(\sigma)\right)\right):\,1\le j\le p\}=K_4\sum_{j=1}^{p}\alpha\left(M_G^{-1}\left(M_{F_j}(\sigma)\right)\right),$$

where K_j are some positive constants. Thus, since $\beta\in L^0$, we obtain

$$\int\limits_{\sigma_0}^{\infty}\frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}d\sigma\le K_5\int\limits_{\sigma_0}^{\infty}\frac{\alpha(M_G^{-1}(M_F(m\sigma-\tau-\varepsilon)))}{\beta(\sigma)}d\sigma\le$$
$$\le K_6\sum_{j=1}^{p}\int\limits_{\sigma_0}^{\infty}\frac{\alpha(M_G^{-1}(M_{F_j}(\sigma)))}{\beta(\sigma)}d\sigma,$$

Q. E. D. □

References

1. Ch. Roy, *On the relative order and lower order of an entire function*, Bull. Calcutta Math. Soc. **102** (2010), no.1, 17–26.
2. S. K. Data and A. R. Maji, *Relative order of entire functions in terms of their maximum terms*, Int. J. Math. Anal., Ruse **5** (2011), no. 41-44, 2119–2126.
3. S. K. Data, T. Biswas, and Ch. Ghosh, *Growth analysis of entire functions concerning generalized relative type and generalized relative weak type*, Facta Univ., Ser. Math. Inf. **30** (2015), no. 3, 295–324.
4. S. K. Data, T. Biswas, and A. Hoque, *Some results on the growth analysis of entire function using their maximum terms and relative L^*-order*, J. Math. Ext. **10** (2016), no. 2, 59–73.
5. S. K. Data, T. Biswas, and P. Das, *Some results on generalized relative order of meromorohic functions*, Ufa Math. J. **8** (2016), no. 2, 95–103.
6. S. K. Data and T. Biswas, *Growth analysis of entire functions of two complex variables*, Sahand Commun. Math. Anal. **3** (2016), no. 2, 13–22. DOI: 20.1001.1.23225807.2016.03.2.2.7
7. S. K. Data and T. Biswas, *Some growth analysis of entire functions in the form of vector valued Dirichlet series on the basis on their relative Ritt L^*-order and relative Ritt L^*-lower order*, New Trends in Msth. Sci. **5** (2017), no. 2, 97–103. DOI: 10.20852/ntmsci.2017.159
8. O. M. Mulyava and M. M. Sheremeta, *Relative growth of Dirichlet series*, Mat. Stud. **49** (2018), no. 2, 158–164. DOI: 10.15330/ms.49.2.158-164

9. O. M. Mulyava and M. M. Sheremeta, *Remarks to relative growth of entire Dirichlet series*, Visnyk of Lviv Univ. Ser. Mech.-Math. **87** (2019), 73–81. DOI: 10.30970/vmm.2019.87.073-081
10. O. M. Mulyava and M. M. Sheremeta, *Relative growth of entire Dirichlet series with different generalized orders*, Bukovyn. Mat. Zh. **9** (2021), no. 2, 22–34. DOI: 10.31861/bmj2021.02.02 5
11. J. Hadamard, *Théorème sur le séries entières*, Acta Math. **22** (1899), 55–63. DOI: 10.1007/BF02417870 4
12. J. Hadamard, *La série de Taylor et son prolongement analitique.* Paris, C. Naud (Ed.). Scientia, Phys.-Math. **12**, 1901. 102pp.
13. L. Bieberbach, *Analytische Fortsetzung*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 3, Springer-Verlag, Berlin, 1955. DOI: 10.1007/978-3-662-01270-3
14. O. M. Mulyava and M. M. Sheremeta, *Compositions of Dirichlet series similar to the Hadamard compositions, and convergence classes.* Mat. Stud. **51** (2019), no. 1, 25–34. DOI: 10.15330/ms.51.1.25-34.
15. A. I. Bandura, O. M. Mulyava, and M. M. Sheremeta, *On Dirichlet series similar to Hadamard compositions in half-plane*, Carpatian Math. Publ. **15** (2023), no. 1, 180–195. DOI: 10.15330/cmp.15.1.180-195.
16. M. M. Sheremeta, *On two classes of positive functions and the belonging to them of main characteristics of entire functions*, Mat. Stud. **19** (2003), no. 1, 75–82.
17. E. Seneta, *Regularly varying functions*, Lect, Notes Math., **508**, Springer-Verlag, Berlin, 1976. DOI: 10.1007/BFb0079658
18. Ya. D. Pyanylo and M. N. Sheremeta, *On the growth of entire fuctions given by Dirichlet series*, Izv. Vyssh. Uchebn. Zaved., Mat. (1975), no. 10, 91–93 (in Russian).
19. M. M. Sheremeta, *Entire Dirichlet series*, ISDO, Kyiv, 1993 (in Ukrainian).
20. K. Sugimura, *Übertragung einiger Sätze aus der Theorie der ganzen Funktionen auf Dirichletschen Reihen*, Math. Z. **29** (1929), 264–277. DOI: 10.1007/BF01180529
21. A. F. Leont'ev, *Series of exponents*, Nauka, Moscow, 1976 (in Russian).
22. G. Valiron, *General theory of integral functions.* Toulouse, 1923.
23. P. K. Kampthan, *A theorem of step functions, II*, Instanbul univ. fen. fac.mecm. A **26** (1963), 65–69.
24. O. M. Mulyava, *Convergence classes in theory of Dirichlet series*, Dopov. Nats. Akad. Nauk Ukr., Mat. Pryr. Tekh. Nauky (1999), no. 3, 35–39. (in Ukrainian)
25. O. M. Mulyava and M. M. Sheremeta, *Convergence classes of analytic functions*, Publishing Lira-K, Kyiv, 2020.

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ВІДНОСНЕ ЗРОСТАННЯ АДАМАРОВИХ КОМПОЗИЦІЙ ЦІЛИХ РЯДІВ ДІРІХЛЕ

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Нехай $F(s) = \sum\limits_{n=1}^{\infty} a_n \exp\{s\lambda_n\}$ і $F_j(s) = \sum\limits_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}$ — цілі ряди Діріхле з показниками $0 \le \lambda_n \uparrow +\infty$. Функція F називається адамаровою композицією роду $m \ge 1$ функцій F_j, якщо $a_n = P(a_{n,1}, \dots, a_{n,p})$, де $P(x_1, \dots, x_p) = \sum\limits_{k_1+\dots+k_p=m} c_{k_1\dots k_p} x_1^{k_1} \cdot \ldots \cdot x_p^{k_p}$ — однорідний поліном степеня $m \ge 1$. Зростання функції F відносно ряду Діріхле $G(s) = \sum\limits_{n=1}^{\infty} g_n \exp\{s\lambda_n\}$ ототожнено зі зростанням функції $M_G^{-1}(M_F(\sigma))$, де $M_F(\sigma) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$. У термінах узагальнених порядків і узагальненого класу збіжності вивчено залежність зростання функції $M_G^{-1}(M_F(\sigma))$ від зростання функцій $M_G^{-1}(M_{F_j}(\sigma))$.

Ключові слова: ряд Діріхле, Адамарова композиція, узагальнений порядок, узагальнений клас збіжності.