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ON THE RELATIVE GROWTH OF ENTIRE DIRICHLET SERIES

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Let $\Lambda = (\lambda_n)$ be an increasing to $+\infty$ sequence of non-negative numbers, $S(\Lambda)$ be a class of entire Dirichlet series $F(s) = \sum\limits_{n=1}^{\infty} f_n \exp\{s\lambda_n\}$ and $M_F(\sigma) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$. By L we denote a class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. For $\alpha \in L$, $\beta \in L$ and $F \in S(\Lambda)$ the quantities $\varrho_{\alpha,\beta}[F] = \varlimsup\limits_{\sigma\to+\infty} \alpha(\ln M_F(\sigma))/\beta(\sigma)$ and $\lambda_{\alpha,\beta}[F] = \varliminf\limits_{\sigma\to+\infty} \alpha(\ln M_F(\sigma))/\beta(\sigma)$ are called the generalized (α, β)-order and the generalized lower (α, β)-order of F. Define the generalized (α, β)-order and the generalized lower (α, β)-order of the function $F \in S(\Lambda)$ with respect to a function $G(s) = \sum\limits_{n=1}^{\infty} g_n \exp\{s\lambda_n\} \in S(\Lambda)$ as follows $\varrho_{\alpha,\beta}[F]_G = \varlimsup\limits_{\sigma\to+\infty} \alpha(M_G^{-1}(M_F(\sigma)))/\beta(\sigma)$ and $\lambda_{\alpha,\beta}[F]_G = \varliminf\limits_{\sigma\to+\infty} \alpha(M_G^{-1}(M_F(\sigma)))/\beta(\sigma)$.
Under the condition $0 < \lambda_{\alpha,\alpha}[G] = \varrho_{\alpha,\alpha}[G] < +\infty$, formulas have been found for calculating $\varrho_{\alpha,\beta}[F]_G$ and $\lambda_{\alpha,\beta}[F]_G$ through the coefficients f_n and g_n.

Key words: Dirichlet series, relative growth, generalized order.

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1. Introduction

Let f and g be entire transcendental functions and $M_f(r) = \max\{|f(z)| : |z| = r\}$. For the study of relative growth of the functions f and g Ch. Roy [1] used the order

$$\varrho_g[f] = \varlimsup_{r\to+\infty} \frac{\ln M_g^{-1}(M_f(r))}{\ln r}$$

and the lower order

$$\lambda_g[f] = \varliminf_{r\to+\infty} \frac{\ln M_g^{-1}(M_f(r))}{\ln r}$$

of the function f with respect to the function g. Research of relative growth of entire functions was continued by S.K. Data, T. Biswas and other mathematicians (see, for example, [2 - 5]) in terms of maximal terms, Nevanlinna characteristic function and k-logarithmic orders. In [6] it is considered a relative growth of entire functions of two complex variables and in [7] the relative growth of entire Dirichlet series is studied in the terms of R-orders.

Suppose that $\Lambda = (\lambda_n)$ is an increasing to $+\infty$ sequence of non-negative numbers and by $S(\Lambda)$ we denote a class of entire Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f_n \exp\{s\lambda_n\}, \quad s = \sigma + it. \tag{1}$$

For $\sigma < +\infty$ we put

$$M_F(\sigma) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}.$$

We remark that the function $M_F(\sigma)$ is continuous and increasing to $+\infty$ on $(-\infty, +\infty)$ and, therefore, there exists the function $M_F^{-1}(x)$ inverse to $M_F(\sigma)$, which increase to $+\infty$ on $(x_0, +\infty)$.

By L we denote a class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i. e. α is a slowly increasing function. Clearly, $L_{si} \subset L^0$.

For $\alpha \in L$, $\beta \in L$ and $F \in S(\Lambda)$ the quantities

$$\varrho_{\alpha,\beta}[F] := \varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)},$$
$$\lambda_{\alpha,\beta}[F] := \varliminf_{\sigma\to+\infty} \frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)}$$

are called [8, 9] the *generalized* (α, β)*-order* and the *generalized lower* (α, β)*-order* of F accordingly. We say that F has the generalized regular (α, β)-growth, if

$$0 < \lambda_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[F] < +\infty.$$

The generalized (α, β)-order $\varrho_{\alpha,\beta}[F]_G$ and the generalized lower (α, β)-order $\lambda_{\alpha,\beta}[F]_G$ of the function $F \in S(\Lambda)$ with respect to a function $G \in S(\Lambda)$ given by Dirichlet series

$G(s)=\sum\limits_{n=1}^{\infty} g_n\exp\{s\lambda_n\}$ we define as follows

$$\varrho_{\alpha,\beta}[F]_G:=\varlimsup_{\sigma\to+\infty}\frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)},$$
$$\lambda_{\alpha,\beta}[F]_G:=\varliminf_{\sigma\to+\infty}\frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}.$$

Connections between $\varrho_{\alpha,\beta}[F]_G$ and $\lambda_{\alpha,\beta}[F]_G$ from one side and $\varrho_{\alpha,\beta}[F]$, $\lambda_{\alpha,\beta}[F]$, $\varrho_{\alpha,\beta}[G]$ and $\lambda_{\alpha,\beta}[F]$ on the other hand investigated in the articles [10, 11]. In particular in [11] the following theorems are proved.

Theorem A. *If* $\alpha\in L$ *and* $\beta\in L$ *then for each function* $\gamma\in L$ *such that*

$$0<\lambda_{\gamma,\alpha}[G]\le\varrho_{\gamma,\alpha}[G]<+\infty$$

the estimates

$$\varrho_{\gamma,\beta}[F]/\varrho_{\gamma,\alpha}[G]\le\varrho_{\alpha,\beta}[F]_G\le\varrho_{\gamma,\beta}[F]/\lambda_{\gamma,\alpha}[G]$$

and

$$\lambda_{\gamma,\beta}[F]/\varrho_{\gamma,\alpha}[G]\le\lambda_{\alpha,\beta}[F]_G\le\lambda_{\gamma,\beta}[F]/\lambda_{\gamma,\alpha}[G]$$

hold.

Theorem B. *Let* $\alpha\in L^0$, $\beta\in L^0$, $\gamma\in L_{si}$, $\dfrac{d\alpha^{-1}(c\gamma(x))}{d\ln x}=O(1)$ *and* $\dfrac{d\beta^{-1}(c\gamma(x))}{d\ln x}=O(1)$ *as* $x\to+\infty$ *for each* $c\in(0,+\infty)$. *Suppose that* $\ln n=o(\lambda_n\alpha^{-1}(c\gamma(\lambda_n)))$ *and* $\ln n=o(\lambda_n\beta^{-1}(c\gamma(\lambda_n)))$ *as* $n\to\infty$ *for each* $c\in(0,+\infty)$. *If the function* G *has generalized regular* (γ,α)*-growth,* $\gamma(\lambda_{n+1})\sim\gamma(\lambda_n)$ *and* $\kappa_n[G]:=\dfrac{\ln|g_n|-\ln|g_{n+1}|}{\lambda_{n+1}-\lambda_n}\nearrow+\infty$ *as* $n_0\le n\to\infty$ *then*

$$\varrho_{\alpha,\beta}[F]_G=P_{\alpha,\beta}:=\varlimsup_{n\to\infty}\frac{\alpha\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|g_n|}\right)}{\beta\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|f_n|}\right)}.\tag{2}$$

If, moreover, $\kappa_n[F]\nearrow+\infty$ *as* $n_0\le n\to\infty$ *then*

$$\lambda_{\alpha,\beta}[F]_G=p_{\alpha,\beta}:=\varliminf_{n\to\infty}\frac{\alpha\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|g_n|}\right)}{\beta\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|f_n|}\right)}.\tag{3}$$

Remark that $\sigma=o(\ln M_G(\sigma))$ as $\sigma\to+\infty$ for each function $G\in S(\Lambda)$, whence it follows that $\lambda_{\alpha,\alpha}[G]\ge1$. Therefore, the function G has the generalized regular (α,α)-growth if

$$1\le\lambda_{\alpha,\alpha}[G]=\varrho_{\alpha,\alpha}[G]<+\infty.$$

Let us choose $\gamma(x)=\alpha(x)$ and assume that the function G has the generalized regular (α,α)-growth. Then by Theorem A we have $\varrho_{\alpha,\beta}[F]_G=\dfrac{\varrho_{\alpha,\beta}[F]}{\varrho_{\alpha,\alpha}[G]}$ and $\lambda_{\alpha,\beta}[F]_G=\dfrac{\lambda_{\alpha,\beta}[F]}{\varrho_{\alpha,\alpha}[G]}$.

It is clear that the function $\alpha\in L$ does not satisfy the condition $\dfrac{d\alpha^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ for each $c\in(0,+\infty)$. Therefore, we cannot use Theorem B in the case when $\gamma(x)=\alpha(x)$, and this note is devoted to the study of this case.

2. Auxiliary lemmas

To prove the analogue of Theorem B, we need several lemmas.

Lemma 1 ([9, 13]). *Let $\alpha\in L_{si}$, $\beta\in L^0$ and $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ for each $c\in(0,+\infty)$. If $\ln n=o(\lambda_n\beta^{-1}(c\alpha(\lambda_n)))$ as $n\to\infty$ for each $c\in(0,+\infty)$ and $G\in S(\Lambda)$ then*

$$\varrho_{\alpha,\beta}[G]=\varlimsup_{n\to\infty}\frac{\alpha(\lambda_n)}{\beta\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|g_n|}\right)}.$$

If, moreover, $\varrho_{\alpha,\beta}[G]<+\infty$, $\alpha(\lambda_{n+1})\sim\alpha(\lambda_n)$ and $\kappa_n[G]\nearrow+\infty$ as $n_0\le n\to\infty$ then

$$\lambda_{\alpha,\beta}[G]=\varliminf_{n\to\infty}\frac{\alpha(\lambda_n)}{\beta\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|g_n|}\right)}.$$

Lemma 2 ([9, 12]). *Let $\mu_F(\sigma)=\max\{|f_n|\exp\{\sigma\lambda_n\}:\ n\ge0\}$ be the maximal term of Dirichlet series (1) and $h_0=\varlimsup\limits_{n\to\infty}\dfrac{\ln n}{-\ln|f_n|}<1$. Then for every $\varepsilon\in(0,h_0)$ there exists $A_0(\varepsilon)>0$ such that for all $\sigma\ge0$ the inequality*

$$M_F(\sigma)\le A_0(\varepsilon)\mu_F\left(\frac{\sigma}{1-h_0-\varepsilon}\right)$$

holds.

Let Ω be a class of positive unbounded on $(-\infty,+\infty)$ functions Φ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(-\infty,+\infty)$. From now on, we denote by φ the inverse function to Φ', and let $\Psi(x)=x-\dfrac{\Phi(x)}{\Phi'(x)}$ be the function associated with Φ in the sense of Newton.

Lemma 3 ([9, 12, 13]). *Let $\Phi\in\Omega$. In order that $\ln\mu_F(\sigma)\le\Phi(\sigma)$ for all $\sigma\ge\sigma_0$, it is necessary and sufficient that $\ln|f_n|\le-\lambda_n\Psi(\varphi(\lambda_n))$ for all $n\ge n_0$.*

Using Lemmas 2 and 3, we prove the following lemma.

Lemma 4. *Let $\alpha(e^x)\in L_{si}$, $\ln n=o(\lambda_n\alpha^{-1}(c\alpha(\lambda_n)))$ as $n\to\infty$ for each $c\in(0,+\infty)$ and $G\in S(\Lambda)$. Then*

$$\varrho_{\alpha,\alpha}[G]=K_{\alpha,\alpha}[G]:=\varlimsup_{n\to\infty}\frac{\alpha(\lambda_n)}{\alpha\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|g_n|}\right)}. \tag{4}$$

If, moreover, $\varrho_{\alpha,\alpha}[G] < +\infty$, $\alpha(\lambda_{n+1}) \sim \alpha(\lambda_n)$ *and* $\kappa_n[G] \nearrow +\infty$ *as* $n_0 \le n \to \infty$ *then*

$$\lambda_{\alpha,\alpha}[G] = k_{\alpha,\alpha}[G] := \varliminf_{n\to\infty} \frac{\alpha(\lambda_n)}{\alpha\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|g_n|}\right)}. \tag{5}$$

Proof. Let $\varrho_{\alpha,\alpha}[G] < +\infty$. Then for every $\varrho > \varrho_{\alpha,\alpha}[G]$ and all $\sigma \ge \sigma_0(\varrho)$ we have $\ln M_G(\sigma) \le \alpha^{-1}(\varrho\alpha(\sigma))$ and in view of Cauchy inequality $\ln |g_n| \le \alpha^{-1}(\varrho\alpha(\sigma)) - \sigma\lambda_n$ for all $\sigma \ge \sigma_0(\varrho)$ and $n \ge 1$. For $n \ge 1$ we choose $\sigma_n = \alpha^{-1}(\alpha(\lambda_n)/\varrho)$. Then $\sigma_n \ge \sigma_0$ for $n \ge n_0$ and, thus,

$$\ln |g_n| \le \lambda_n - \sigma_n\lambda_n = -\lambda_n(\alpha^{-1}(\alpha(\lambda_n)/\varrho) - 1), \quad n \ge n_0. \tag{6}$$

Hence it follows that

$$h_0 = \varlimsup_{n\to\infty} \frac{\ln n}{-\ln |g_n|} \le \varlimsup_{n\to\infty} \frac{\ln n}{\lambda_n\alpha^{-1}(\alpha(\lambda_n)/\varrho)} = 0$$

and by Lemma 2

$$\ln M_F(\sigma) \le \ln A_0(\varepsilon) + \ln \mu_F\left(\frac{\sigma}{1-\varepsilon}\right),$$

whence in view of Cauchy inequality $\mu_G(\sigma) \le M_G(\sigma)$ and of the condition $\alpha \in L_{si}$ we get

$$\varrho_{\alpha,\alpha}[G] = \varrho_{\alpha,\alpha}[\mu_G] := \varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln \mu_G(\sigma))}{\alpha(\sigma)},$$

and

$$\lambda_{\alpha,\alpha}[G] = \lambda_{\alpha,\alpha}[\mu_G] := \varliminf_{\sigma\to+\infty} \frac{\alpha(\ln \mu_G(\sigma))}{\alpha(\sigma)}.$$

From (6) also it follows that $K_{\alpha,\alpha}[G] \le \varrho$, and in view of the arbitrariness of ϱ we obtain $K_{\alpha,\alpha}[G] \le \varrho_{\alpha,\alpha}[G]$. If $\varrho_{\alpha,\alpha}[G] = +\infty$ then the last inequality is trivial.

On the other hand, we suppose on the contrary that $K_{\alpha,\alpha}[G] < \varrho_{\alpha,\alpha}[G]$. Then for every $k \in (K_{\alpha,\alpha}[G], \varrho_{\alpha,\alpha}[G])$ and all $n \ge n_0(k)$ we have $\ln |g_n| \le -\lambda_n\alpha^{-1}(\alpha(\lambda_n)/k)$. Now we put

$$\Phi(\sigma) = \int_{\sigma_0}^{\sigma} \alpha^{-1}(k\alpha(x))dx + \text{const}, \quad \sigma_0 > 0. \tag{7}$$

Then $\varphi(x) = \alpha^{-1}(\alpha(x)/k)$ and since

$$(x\Psi(\varphi(x)))' = (x\varphi(x) - \Phi(\varphi(x)))' = \varphi(x),$$

we obtain

$$x\Psi(\varphi(x)) = \int_{x_0}^{x} \alpha^{-1}(\alpha(x)/k)dx + \text{const}. \tag{8}$$

Therefore, by Lemma 3

$$\ln \mu_G(\sigma) \le \int_{\sigma_0}^{\sigma} \alpha^{-1}(k\alpha(x))dx + \text{const}, \quad \sigma \ge \sigma_0(\varrho),$$

if and only if

$$\ln |g_n| \le -\int\limits_{x_0}^{\lambda_n} \alpha^{-1}(\alpha(x)/k)dx + \text{const}.$$

Since

$$\int\limits_{x_0}^{\lambda_n} \alpha^{-1}(\alpha(x)/k)dx \le \lambda_n(\alpha^{-1}(\alpha(\lambda_n)/k)),$$

from hence it follows that

$$\ln \mu_G(\sigma) \le \int\limits_{\sigma_0}^{\sigma} \alpha^{-1}(k\alpha(x))dx + \text{const} \le (1+o(1))\sigma\alpha^{-1}(k\alpha(\sigma)), \quad \sigma \to +\infty,$$

whence in view of condition $\alpha(e^x) \in L_{si}$ we obtain

$$\begin{aligned} \varrho_{\alpha,\alpha}[G] = \varrho_{\alpha,\alpha}[\mu_G] \le \\ &\le \varlimsup_{\sigma\to+\infty} \frac{\alpha(\sigma\alpha^{-1}(k\alpha(\sigma)))}{\alpha(\sigma)} = \\ &= \varlimsup_{\sigma\to+\infty} \frac{\alpha(\exp\{\ln \sigma + \ln \alpha^{-1}(k\alpha(\sigma))\})}{\alpha(\sigma)} \le \\ &\le \varlimsup_{\sigma\to+\infty} \frac{\alpha(\exp\{2\max\{\ln \sigma, \ln \alpha^{-1}(k\alpha(\sigma))\}\})}{\alpha(\sigma)} = \\ &= \varlimsup_{\sigma\to+\infty} \frac{\alpha(\exp\{\max\{\ln \sigma, \ln \alpha^{-1}(k\alpha(\sigma))\}\})}{\alpha(\sigma)} = \\ &= \varlimsup_{\sigma\to+\infty} \frac{\max\{\alpha(\sigma), k\alpha(\sigma)\})}{\alpha(\sigma)} = \max\{1,k\}, \end{aligned}$$

which is impossible. Equality (4) is proved.

Now we prove equality (5). Suppose that $k_{\alpha,\alpha}[G] > 0$. Then for every $k \in (0, k_{\alpha,\alpha}[G])$ and all $n \ge n_0$ we have $\ln |g_n| \ge -\lambda_n\alpha^{-1}(\alpha(\lambda_n)/k))$ and, therefore,

$$\ln \mu_G(\sigma) \ge -\lambda_n\alpha^{-1}(\alpha(\lambda_n)/k)) + \sigma\lambda_n$$

for all σ and $n \ge n_0$. We choose $\sigma_n = \alpha^{-1}(\alpha(\lambda_n)/k) + 1$. Then for $n \ge n_0$

$$\ln \mu_G(\sigma_n) \ge -\lambda_n\alpha^{-1}(\alpha(\lambda_n)/k)) + \alpha^{-1}(\alpha(\lambda_n)/k)\lambda_n + \lambda_n = \lambda_n.$$

If $\sigma_n \le \sigma \le \sigma_{n+1}$ then

$$\begin{aligned} \frac{\alpha(\ln \mu_G(\sigma)}{\alpha(\sigma)} \ge \frac{\alpha(\ln \mu_G(\sigma_n)}{\alpha(\sigma_{n+1})} &= \\ &= \frac{\alpha(\lambda_n)}{\alpha(\alpha^{-1}(\alpha(\lambda_{n+1})/k) + 1)} = \\ &= (1+o(1))k\frac{\alpha(\lambda_n)}{\alpha(\lambda_{n+1})} = \\ &= (1+o(1))k, \quad \sigma \to +\infty. \end{aligned}$$

In view of the arbitrariness of k we obtain $k_{\alpha,\alpha}[G] \le \lambda_{\alpha,\alpha}[G]$. If $k_{\alpha,\alpha}[G] = 0$ then the last inequality is trivial.

On the other hand, we suppose on the contrary that $k_{\alpha,\alpha}[G] < \lambda_{\alpha,\alpha}[G]$. Then for every $k \in (k_{\alpha,\alpha}[G], \lambda_{\alpha,\alpha}[G])$ there exists an increasing sequence (n_j) such that

$$\ln |g_{n_j}| \le -\lambda_{n_j}\alpha^{-1}(\alpha(\lambda_{n_j})/k).$$

Since $\kappa_n[G] \nearrow +\infty$ as $n_0 \le n \to \infty$, we have

$$\ln \mu_G(\kappa_n[G]) = \ln |g_n| + \lambda_n\kappa_n[G]$$

and, thus,

$$\ln \mu_G(\kappa_{n_j}[G]) \le -\lambda_{n_j}\alpha^{-1}(\alpha(\lambda_{n_j})/k) + \lambda_{n_j}\kappa_{n_j}[G] \le \\ \le \ln \mu(\kappa_{n_j}[G]),$$

where

$$\ln \mu(\sigma) = \max\{-\lambda_n\alpha^{-1}(\alpha(\lambda_n)/k) + \lambda_n\sigma : n \ge n_0\}.$$

Using Lemma 3 as above we get the inequality $\ln \mu(\sigma) \le (1 + o(1))\sigma\alpha^{-1}(k\alpha(\sigma))$ as $\sigma \to +\infty$, i.e.,

$$\ln \mu_G(\kappa_{n_j}[G]) \le (1 + o(1))\kappa_{n_j}[G]\alpha^{-1}(k\alpha(\kappa_{n_j}[G])), \quad j \to \infty.$$

Hence in view of condition $\alpha(e^x) \in L_{si}$ as above we obtain

$$\lambda_{\alpha,\alpha}[G] = \lambda_{\alpha,\alpha}[\mu_G] \le \varliminf_{j\to\infty} \frac{\alpha(\sigma\alpha^{-1}(k\alpha(\kappa_{n_j}[G]))))}{\alpha(\kappa_{n_j}[G]))} \le \max\{1, k\},$$

which is impossible. Equality (5) is proved. □

3. Main result

The following theorem is an analogue of Theorem B.

Theorem 1. *Let $\alpha(e^x) \in L_{si}$, $\beta \in L^0$, $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ as $x \to +\infty$ and $\ln n = o(\lambda_n\beta^{-1}(c\alpha(\lambda_n)))$ as $n \to \infty$ for each $c \in (0, +\infty)$.*

If the function G has generalized regular (α, α)-growth, $\alpha(\lambda_{n+1}) \sim \alpha(\lambda_n)$ and $\kappa_n[G] \nearrow +\infty$ as $n_0 \le n \to \infty$ then (2) holds.

If, moreover, $\kappa_n[F] \nearrow +\infty$ as $n_0 \le n \to \infty$ then (3) holds.

Proof. At first we remark that from the condition $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ as $x \to +\infty$ for each $c \in (0, +\infty)$ we obtain $\beta^{-1}(\alpha(x)) \le c_0 \ln x$, i.e.,

$$\beta^{-1}(x) \le c_0 \ln \alpha^{-1}(x) \le c_0\alpha^{-1}(x)$$

for some $c_0 > 0$ and all $x \ge x_0$. Therefore, the condition $\ln n = o(\lambda_n\beta^{-1}(c\alpha(\lambda_n)))$ as $n \to \infty$ for each $c \in (0, +\infty)$ implies the condition $\ln n = o(\lambda_n\alpha^{-1}(c\alpha(\lambda_n)))$ as $n \to \infty$ for each $c \in (0, +\infty)$. Thus, all conditions of Lemmas 1 and 4 hold.

Since G has generalized regular (α, α)-growth, we have

$$0 < \varrho_{\alpha,\alpha}[G] = \varrho_{\alpha,\alpha}[G] < +\infty,$$

$\varrho_{\alpha,\beta}[F]_G = \dfrac{\varrho_{\alpha,\beta}[F]}{\varrho_{\alpha,\alpha}[G]}$ and $\lambda_{\alpha,\beta}[F]_G = \dfrac{\lambda_{\alpha,\beta}[F]}{\varrho_{\alpha,\alpha}[G]}$. Therefore, by Lemmas 1 and 4

$$\varrho_{\alpha,\beta}[F] = \varlimsup_{n\to\infty} \alpha(\lambda_n)/\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right),$$
$$\lambda_{\alpha,\alpha}[G] = \varrho_{\alpha,\alpha}[G] = \lim_{n\to\infty} \alpha(\lambda_n)/\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right).$$

and

$$\begin{aligned}\varrho_{\alpha,\beta}[F]_G &= \varlimsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)} \lim_{n\to\infty}\frac{\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\alpha(\lambda_n)} = \\ &= \varlimsup_{n\to\infty}\left(\frac{\alpha(\lambda_n)}{\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)}\frac{\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\alpha(\lambda_n)}\right) = \\ &= P_{\alpha,\beta}.\end{aligned}$$

Similarly,

$$\begin{aligned}\lambda_{\alpha,\beta}[F]_G &= \varliminf_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)} \lim_{n\to\infty}\frac{\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\alpha(\lambda_n)} = \\ &= \varliminf_{n\to\infty}\left(\frac{\alpha(\lambda_n)}{\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)}\frac{\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\alpha(\lambda_n)}/\right) = \\ &= p_{\alpha,\beta}.\end{aligned}$$

Theorem 1 is proved. □

Remark. Let $\beta(x) = x$ and $\alpha(x) = \ln_k x$ for $x \geq x_0$, where $\ln_k x$ is the kth iteration of the logarithm, i.e., $\ln_1 x = \ln x$ and $\ln_k x = \ln\ln_{k-1} x$ for $k \geq 2$. If $k \geq 2$ then these functions satisfy the conditions of Theorem 1. The functions $\beta(x) = x$ and $\alpha(x) = \ln x$ for $x \geq e$ does not satisfy these conditions. In this case $\varrho_R[F] := \varlimsup\limits_{\sigma\to+\infty}\dfrac{\ln\ln M_F(\sigma)}{\sigma}$ and $\lambda_R[F] := \varliminf\limits_{\sigma\to+\infty}\dfrac{\ln\ln M_F(\sigma)}{\sigma}$ are the R-order and the lower R-order F respectively [14]. Moreover, let $\varrho_l[F] := \varlimsup\limits_{\sigma\to+\infty}\dfrac{\ln\ln M_F(\sigma)}{\ln\sigma}$ and $\lambda_l[F] := \varliminf\limits_{\sigma\to+\infty}\dfrac{\ln\ln M_F(\sigma)}{\ln\sigma}$ be the logarithmic order and the lower logarithmic order respectively. We put

$$\varrho_{R,l}[F]_G := \varlimsup_{\sigma\to+\infty} \frac{\ln M_G^{-1}(M_F(\sigma))}{\sigma},$$

$$\lambda_{R,l}[F]_G := \varliminf_{\sigma\to+\infty} \frac{\ln M_G^{-1}(M_F(\sigma))}{\sigma}.$$

In [11] the following result is obtained.

Proposition. *If the function G has regular logarithmic growth (i.e., $0 < \lambda_l[F] = \varrho_l[F] < +\infty$), $\ln n = o(\lambda_n \ln \lambda_n)$, $\ln \lambda_{n+1} \sim \ln \lambda_n$ and $\varkappa_n[G] \nearrow +\infty$ as $n_0 \le n \to \infty$ then*

$$\varrho_{R,l}[F]_G = \varlimsup_{n\to\infty} \frac{\lambda_n \ln \lambda_n \ln (1/|g_n|}{\ln (1/|f_n|) \ln \ln (1/|g_n|)}.$$

If, moreover, $\varkappa_n[F] \nearrow +\infty$ as $n_0 \le n \to \infty$ then

$$\lambda_{R,l}[F]_G = \varliminf_{n\to\infty} \frac{\lambda_n \ln \lambda_n \ln (1/|g_n|)}{\ln (1/|f_n|) \ln \ln (1/|g_n|)}.$$

This result does not follow from Theorem 1.

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ПРО ВІДНОСНЕ ЗРОСТАННЯ ЦІЛИХ РЯДІВ ДІРІХЛЕ

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Нехай $\Lambda = (\lambda_n)$ — зростаюча до $+\infty$ послідовність невід'ємних чисел, $S(\Lambda)$ — клас цілих рядів Діріхле $F(s) = \sum\limits_{n=1}^{\infty} f_n \exp\{s\lambda_n\}$ і $M_F(\sigma) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$. Через L позначимо клас таких неперервних невід'ємних на $(-\infty, +\infty)$ функцій α, що $\alpha(x) = \alpha(x_0) \geq 0$ для $x \leq x_0$ і $\alpha(x) \uparrow +\infty$ при $x_0 \leq x \to +\infty$. Для $\alpha \in L$, $\beta \in L$ і $F \in S(\Lambda)$ величини $\varrho_{\alpha,\beta}[F] = \varlimsup\limits_{\sigma\to+\infty} \alpha(\ln M_F(\sigma))/\beta(\sigma)$ і $\lambda_{\alpha,\beta}[F] = \varliminf\limits_{\sigma\to+\infty} \alpha(\ln M_F(\sigma))/\beta(\sigma)$ називаються узагальненим (α, β)-порядком і узагальненим нижнім (α, β)-порядком функції F. Означимо узагальнений (α, β)-порядок і узагальнений нижній (α, β)-порядок функції $F \in S(\Lambda)$ відносно функції $G(s) = \sum\limits_{n=1}^{\infty} g_n \exp\{s\lambda_n\} \in S(\Lambda)$ рівностями $\varrho_{\alpha,\beta}[F]_G = \varlimsup\limits_{\sigma\to+\infty} \alpha(M_G^{-1}(M_F(\sigma)))/\beta(\sigma)$ і $\lambda_{\alpha,\beta}[F]_G = \varliminf\limits_{\sigma\to+\infty} \alpha(M_G^{-1}(M_F(\sigma)))/\beta(\sigma)$.
За умови $0 < \lambda_{\alpha,\alpha}[G] = \varrho_{\alpha,\alpha}[G] < +\infty$ знайдено формули для обчислення $\varrho_{\alpha,\beta}[F]_G$ і $\lambda_{\alpha,\beta}[F]_G$ через коефіцієнти f_n і g_n.

Ключові слова: ряд Діріхле, відносне зростання, узагальнений порядок.