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ON THE SEMIGROUP OF NON-INJECTIVE MONOID ENDOMORPHISMS OF THE SEMIGROUP $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ WITH A TWO-ELEMENT FAMILY $\mathscr{F}$ OF INDUCTIVE NONEMPTY SUBSETS OF ω

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We study the semigroup of non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive nonempty subsets of ω. We describe the structure of elements of the semigroup $\boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. In particular we show that its subsemigroup $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of non-injective non-annihilating monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the direct product of the two-element left-zero semigroup and the multiplicative semigroup of positive integers and describe Green's relations on $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Key words: bicyclic monoid, inverse semigroup, bicyclic extension, monoid endomorphism, non-injective, Green's relations, left-zero semigroup, direct product.

We shall follow the terminology of [1, 2, 9]. By ω we denote the set of all non-negative integers, by $\mathbb{N}$ the set of all positive integers, and by $\mathbb{Z}$ the set of all integers.

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F \in \mathscr{P}(\omega)$ and $n \in \mathbb{Z}$ we put $n + F = \{n + k : k \in F\}$ if $F \neq \varnothing$ and $n + \varnothing = \varnothing$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called *ω-closed* if $F_1 \cap (-n + F_2) \in \mathscr{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathscr{F}$. For any $a \in \omega$ we denote $[a) = \{x \in \omega : x \geqslant a\}$.

A subset A of ω is said to be *inductive*, if $i \in A$ implies $i + 1 \in A$. Obviously, $\varnothing$ is an inductive subset of ω.

Remark 1 ([5]). (1) By Lemma 6 from [4] a nonempty subset $F \subseteq \omega$ is inductive in ω if and only $(-1 + F) \cap F = F$.

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(2) Since the set ω with the usual order is well-ordered, for any nonempty inductive subset F in ω there exists a nonnegative integer $n_F \in \omega$ such that $[n_F) = F$.
(3) Statement (2) implies that the intersection of an arbitrary finite family of nonempty inductive subsets in ω is a nonempty inductive subset of ω.

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents.

If S is an inverse semigroup then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the *natural partial order* on S [12].

If S is a semigroup, then we shall denote the Green relations on S by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ (see [1, Section 2.1]):

$$
\begin{aligned}
&a\mathscr{R}b \text{ if and only if } aS^1 = bS^1;\\
&a\mathscr{L}b \text{ if and only if } S^1a = S^1b;\\
&a\mathscr{J}b \text{ if and only if } S^1aS^1 = S^1bS^1;\\
&\mathscr{D} = \mathscr{L}\circ\mathscr{R} = \mathscr{R}\circ\mathscr{L};\\
&\mathscr{H} = \mathscr{L}\cap\mathscr{R}.
\end{aligned}
$$

The $\mathscr{L}$-class [$\mathscr{R}$-class, $\mathscr{H}$-class, $\mathscr{D}$-class, $\mathscr{J}$-class] of the semigroup S containing the element $a \in S$ will be denoted by $\boldsymbol{L}_a$ [$\boldsymbol{R}_a$, $\boldsymbol{H}_a$, $\boldsymbol{D}_a$, $\boldsymbol{J}_a$].

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$
q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.
$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [1].

On the set $\boldsymbol{B}_{\omega} = \omega\times\omega$ we define the semigroup operation "$\cdot$" in the following way

$$
(i_1, j_1)\cdot(i_2, j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2), & \text{if } j_1 \leqslant i_2;\\ (i_1, j_1 - i_2 + j_2), & \text{if } j_1 \geqslant i_2. \end{cases} \tag{1}
$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is isomorphic to the semigroup $\boldsymbol{B}_{\omega}$ by the mapping $\mathfrak{h}\colon \mathscr{C}(p,q) \to \boldsymbol{B}_{\omega}$, $q^k p^l \mapsto (k,l)$, $k,l \in \omega$ (see: [1, Section 1.12] or [11, Exercise IV.1.11(ii)]). Later we identify the bicyclic monoid $\mathscr{C}(p,q)$ with the semigroup $\boldsymbol{B}_{\omega}$ by the mapping $\mathfrak{h}$.

Next we shall describe the construction which is introduced in [4].

Let $\boldsymbol{B}_\omega$ be the bicyclic monoid and $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_\omega \times \mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(2)\qquad (i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 \geqslant i_2. \end{cases}$$

In [4] is proved that if the family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is ω-closed then $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I} = \{(i, j, \varnothing) \colon i, j \in \omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$. For any ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_\omega^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)/\boldsymbol{I}, & \text{if } \varnothing \in \mathscr{F}; \\ (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot), & \text{if } \varnothing \notin \mathscr{F} \end{cases}$$

is defined in [4]. The semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in [4] that $\boldsymbol{B}_\omega^{\mathscr{F}}$ is a combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_\omega^{\mathscr{F}}$ and its set of idempotents are described. Here, the criteria when the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is simple, 0-simple, bisimple, 0-bisimple, or it has the identity, are given. In particularly in [4] it is proved that the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega{\times}\omega$-matrix units if and only if $\mathscr{F}$ consists of a singleton set and the empty set, and $\boldsymbol{B}_\omega^{\mathscr{F}}$ is isomorphic to the bicyclic monoid if and only if $\mathscr{F}$ consists of a non-empty inductive subset of ω.

Group congruences on the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ and its homomorphic retracts in the case when an ω-closed family $\mathscr{F}$ consists of inductive non-empty subsets of ω are studied in [5]. It is proven that a congruence $\mathfrak{C}$ on $\boldsymbol{B}_\omega^{\mathscr{F}}$ is a group congruence if and only if its restriction on a subsemigroup of $\boldsymbol{B}_\omega^{\mathscr{F}}$, which is isomorphic to the bicyclic semigroup, is not the identity relation. Also in [5], all non-trivial homomorphic retracts and isomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ are described. In [6] it is proved that an injective endomorphism ε of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is the indentity transformation if and only if ε has three distinct fixed points, which is equivalent to existence non-idempotent element $(i, j, [p)) \in \boldsymbol{B}_\omega^{\mathscr{F}}$ such that $(i, j, [p))\varepsilon = (i, j, [p))$.

In [3, 10] the algebraic structure of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is established in the case when ω-closed family $\mathscr{F}$ consists of atomic subsets of ω.

It is well-known that every automorphism of the bicyclic monoid $\boldsymbol{B}_\omega$ is the identity self-map of $\boldsymbol{B}_\omega$ [1], and hence the group $\mathbf{Aut}(\boldsymbol{B}_\omega)$ of automorphisms of $\boldsymbol{B}_\omega$ is trivial. In [8] it is proved that the semigroup $\mathbf{End}(\boldsymbol{B}_\omega)$ of all endomorphisms of the bicyclic semigroup $\boldsymbol{B}_\omega$ is isomorphic to the semidirect products $(\omega, +) \rtimes_\varphi (\omega, *)$, where $+$ and $*$ are the usual addition and the usual multiplication on ω.

In the paper [7] we study injective endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive nonempty subsets of ω. We describe the elements of the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ of all injective monoid endomorphisms of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$. In particular we show that every element of the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ has a form either

$\alpha_{k,p}$ or $\beta_{k,p}$, where the endomorphism $\alpha_{k,p}$ is defined by the formulae

$$(i,j,[0])\alpha_{k,p} = (ki,kj,[0]),$$
$$(i,j,[1])\alpha_{k,p} = (p+ki,p+kj,[1]),$$

for an arbitrary positive integer k and any $p \in \{0,\dots,k-1\}$, and the endomorphism $\beta_{k,p}$ is defined by the formulae

$$(i,j,[0])\beta_{k,p} = (ki,kj,[0]),$$
$$(i,j,[1])\beta_{k,p} = (p+ki,p+kj,[0]),$$

an arbitrary positive integer $k \geqslant 2$ and any $p \in \{1,\dots,k-1\}$. In [7] we describe the product of elements of the semigroup $\boldsymbol{End}_{*}^{1}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$:

$$\alpha_{k_1,p_1}\alpha_{k_2,p_2} = \alpha_{k_1k_2,p_2+k_2p_1};$$
$$\alpha_{k_1,p_1}\beta_{k_2,p_2} = \beta_{k_1k_2,p_2+k_2p_1};$$
$$\beta_{k_1,p_1}\beta_{k_2,p_2} = \beta_{k_1k_2,k_2p_1};$$
$$\beta_{k_1,p_1}\alpha_{k_2,p_2} = \beta_{k_1k_2,k_2p_1}.$$

Also, here we prove that Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$ on $\boldsymbol{End}_{*}^{1}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ coincide with the equality relation.

Later we assume that an ω-closed family $\mathscr{F}$ consists of two nonempty inductive nonempty subsets of ω.

This paper is a continuation of [7]. We study non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. We describe the structure of elements of the semigroup $\boldsymbol{End}_{0}^{*}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all non-injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. In particular we show that its subsemigroup $\boldsymbol{End}^{*}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all non-injective non-annihilating monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the direct product the two-element left-zero semigroup and the multiplicative semigroup of positive integers and describe Green's relations on $\boldsymbol{End}^{*}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Remark 2. By Proposition 1 of [5] for any ω-closed family $\mathscr{F}$ of inductive subsets in $\mathscr{P}(\omega)$ there exists an ω-closed family $\mathscr{F}^*$ of inductive subsets in $\mathscr{P}(\omega)$ such that $[0] \in \mathscr{F}^*$ and the semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^*}$ are isomorphic. Hence without loss of generality we may assume that the family $\mathscr{F}$ contains the set $[0)$.

If $\mathscr{F}$ is an arbitrary ω-closed family $\mathscr{F}$ of inductive subsets in $\mathscr{P}(\omega)$ and $[s) \in \mathscr{F}$ for some $s \in \omega$ then

$$\boldsymbol{B}_{\omega}^{\{[s)\}} = \{(i,j,[s)) \colon i,j \in \omega\}$$

is a subsemigroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ [5] and by Proposition 3 of [4] the semigroup $\boldsymbol{B}_{\omega}^{\{[s)\}}$ is isomorphic to the bicyclic semigroup.

Lemma 1. *Let $\mathscr{F} = \{[0),[1)\}$ and let $\mathfrak{e}$ be a monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. If $(i_1,j_1,F)\mathfrak{e} = (i_2,j_2,F)\mathfrak{e}$ for distinct two elements $(i_1,j_1,F),(i_2,j_2,F)$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for some $F \in \mathscr{F}$ then $\mathfrak{e}$ is the annihilating endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.*

Proof. By Theorem 1 of [5] the image $(\boldsymbol{B}_{\omega}^{\mathscr{F}})\mathfrak{e}$ is a subgroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. By Theorem 4(*iii*) of [4] every $\mathscr{H}$-class in $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a singleton, and hence $\mathfrak{e}$ is the annihilating monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. □

Lemma 2. *Let $\mathscr{F}=\{[0),[1)\}$. Then $(\boldsymbol{B}_{\omega}^{\mathscr{F}})\mathfrak{e}\subseteq \boldsymbol{B}_{\omega}^{\{[0)\}}$ for any non-injective monoid endomorphism $\mathfrak{e}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.*

Proof. By Proposition 3 of [4] the subsemigroup $\boldsymbol{B}_{\omega}^{\{[0)\}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the bicyclic semigroup and hence by Corollary 1.32 of [1] the image $(\boldsymbol{B}_{\omega}^{\{[0)\}})\mathfrak{e}$ either is isomorphic to the bicyclic semigroup or is a cyclic subgroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. Since $(0,0,[0))\mathfrak{e}=(0,0,[0))$, Proposition 4 from [5] implies that $(\boldsymbol{B}_{\omega}^{\{[0)\}})\mathfrak{e}\subseteq \boldsymbol{B}_{\omega}^{\{[0)\}}$ in the case when the image $(\boldsymbol{B}_{\omega}^{\{[0)\}})\mathfrak{e}$ is isomorphic to the bicyclic semigroup. In the other case we have that the equality $(0,0,[0))\mathfrak{e}=(0,0,[0))$ implies that

$$(\boldsymbol{B}_{\omega}^{\{[0)\}})\mathfrak{e}\subseteq \{(0,0,[0))\}\subseteq \boldsymbol{B}_{\omega}^{\{[0)\}},$$

because by Theorem 4(*iii*) of [4] every $\mathscr{H}$-class in $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a singleton.

Next, by Proposition 3 of [4] the subsemigroup $\boldsymbol{B}_{\omega}^{\{[1)\}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the bicyclic semigroup and hence by Corollary 1.32 of [1] the image $(\boldsymbol{B}_{\omega}^{\{[1)\}})\mathfrak{e}$ either is isomorphic to the bicyclic semigroup or is a cyclic subgroup of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. Suppose that the image $(\boldsymbol{B}_{\omega}^{\{[1)\}})\mathfrak{e}$ is isomorphic to the bicyclic semigroup and $(\boldsymbol{B}_{\omega}^{\{[1)\}})\mathfrak{e}\subseteq \boldsymbol{B}_{\omega}^{\{[1)\}}$. Then monoid endomorphism $\mathfrak{e}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is injective. Indeed, injectivity of the restriction $\mathfrak{e}\restriction_{\boldsymbol{B}_{\omega}^{\{[1)\}}}\boldsymbol{B}_{\omega}^{\{[1)\}}\to \boldsymbol{B}_{\omega}^{\{[1)\}}$, Proposition 4 of [5], Corollary 1.32 of [1], Theorem 4(*iii*) of [4], and the equality $(0,0,[0))\mathfrak{e}=(0,0,[0))$ imply that either the restriction $\mathfrak{e}\restriction_{\boldsymbol{B}_{\omega}^{\{[0)\}}}\boldsymbol{B}_{\omega}^{\{[0)\}}\to \boldsymbol{B}_{\omega}^{\{[0)\}}$ is an injective mapping or is an annihilating endomorphism. In the case when the restriction $\mathfrak{e}\restriction_{\boldsymbol{B}_{\omega}^{\{[0)\}}}\boldsymbol{B}_{\omega}^{\{[0)\}}\to \boldsymbol{B}_{\omega}^{\{[0)\}}$ is an injective mapping we get that the endomorphism $\mathfrak{e}$ is injective. If the image $(\boldsymbol{B}_{\omega}^{\{[0)\}})\mathfrak{e}$ is a singleton then by Lemma 1 we have that $\mathfrak{e}$ is the annihilating monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. In the both cases we obtain that $(\boldsymbol{B}_{\omega}^{\mathscr{F}})\mathfrak{e}\subseteq \boldsymbol{B}_{\omega}^{\{[0)\}}$. □

Example 1. Let $\mathscr{F}=\{[0),[1)\}$ and k be an arbitrary non-negative integer. We define a map $\gamma_k\colon \boldsymbol{B}_{\omega}^{\mathscr{F}}\to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ by the formulae

$$(i,j,[0))\gamma_k=(i,j,[1))\gamma_k=(ki,kj,[0))$$

for all $i,j\in\omega$.

We claim that $\gamma_k\colon \boldsymbol{B}_{\omega}^{\mathscr{F}}\to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ is an endomorphism. Example 2 and Proposition 5 from [5] imply that the map $\gamma_1\colon \boldsymbol{B}_{\omega}^{\mathscr{F}}\to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a homomorphic retraction of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, and hence it is a monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. By Lemma 2 of [8] every monoid endomorphism $\mathfrak{h}$ of the semigroup $\boldsymbol{B}_{\omega}$ has the following form

$$(i,j)\mathfrak{h}=(ki,kj),\qquad \text{for some}\quad k\in\omega.$$

This implies that the map γ_k is a monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.

Example 2. Let $\mathscr{F}=\{[0),[1)\}$ and k be an arbitrary non-negative integer. We define a map $\delta_k\colon \boldsymbol{B}_{\omega}^{\mathscr{F}}\to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ by the formulae

$$(i,j,[0))\delta_k=(ki,kj,[0))\qquad \text{and}\qquad (i,j,[1))\delta_k=(k(i+1),k(j+1),[0))$$

for all $i, j \in \omega$.

Proposition 1. *Let $\mathscr{F} = \{[0), [1)\}$. Then for any $k \in \omega$ the map δ_k is an endomorphism of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.*

Proof. Since by Proposition 3 of [4] the subsemigroups $\boldsymbol{B}_{\omega}^{\{[0)\}}$ and $\boldsymbol{B}_{\omega}^{\{[1)\}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ are isomorphic to the bicyclic semigroup, by Lemma 2 of [8] the restrictions $\delta_k\restriction_{\boldsymbol{B}_{\omega}^{\{[0)\}}} \colon \boldsymbol{B}_{\omega}^{\{[0)\}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ and $\delta_k\restriction_{\boldsymbol{B}_{\omega}^{[1)}} \colon \boldsymbol{B}_{\omega}^{\{[1)\}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ of δ_k are homomorphisms. Hence it sufficient to show that the following equalities

$$(i_1, j_1, [0))\delta_k \cdot (i_2, j_2, [1))\delta_k = ((i_1, j_1, [0)) \cdot (i_2, j_2, [1)))\delta_k;$$
$$(i_1, j_1, [1))\delta_k \cdot (i_2, j_2, [0))\delta_k = ((i_1, j_1, [1)) \cdot (i_2, j_2, [0)))\delta_k,$$

hold for any $i_1, j_1, i_2, j_2 \in \omega$.

We observe that the above equalities are trivial in the case when $k = 0$. Hence later we assume that k is a positive integer.

Then we have that

$$\begin{aligned}
(i_1,j_1,[0))\delta_k \cdot (i_2,j_2,[1))\delta_k &= (ki_1, kj_1, [0)) \cdot (k(i_2+1), k(j_2+1), [0)) = \\
&= \begin{cases} (ki_1+k(i_2+1)-kj_1, k(j_2+1), (kj_1-k(i_2+1)+[0)) \cap [0)), & \text{if } kj_1 < k(i_2+1); \\ (ki_1, k(j_2+1), [0) \cap [0)), & \text{if } kj_1 = k(i_2+1); \\ (ki_1, kj_1+k(j_2+1)-k(i_2+1), [0) \cap (k(i_2+1)-kj_1+[0))), & \text{if } kj_1 > k(i_2+1) \end{cases} \\
&= \begin{cases} (k(i_1+i_2+1-j_1), k(j_2+1), [0)), & \text{if } j_1 < i_2+1; \\ (ki_1, k(j_2+1), [0)), & \text{if } j_1 = i_2+1; \\ (ki_1, k(j_1+j_2-i_2), [0)), & \text{if } j_1 > i_2+1 \end{cases} \\
&= \begin{cases} (k(i_1+i_2+1-j_1), k(j_2+1), [0)), & \text{if } j_1 < i_2; \\ (k(i_1+i_2+1-j_1), k(j_2+1), [0)), & \text{if } j_1 = i_2; \\ (ki_1, k(j_2+1), [0)), & \text{if } j_1 = i_2+1; \\ (ki_1, k(j_1+j_2-i_2), [0)), & \text{if } j_1 > i_2+1 \end{cases} \\
&= \begin{cases} (k(i_1+i_2+1-j_1), k(j_2+1), [0)), & \text{if } j_1 < i_2; \\ (k(i_1+1), k(j_2+1), [0)), & \text{if } j_1 = i_2; \\ (ki_1, k(j_2+1), [0)), & \text{if } j_1 = i_2+1; \\ (ki_1, k(j_1+j_2-i_2), [0)), & \text{if } j_1 > i_2+1, \end{cases}
\end{aligned}$$

$$\begin{aligned}
((i_1, j_1, [0)) \cdot (i_2, j_2, [1)))\delta_k &= \begin{cases} (i_1+i_2-j_1, j_2, (j_1-i_2+[0)) \cap [1))\delta_k, & \text{if } j_1 < i_2; \\ (i_1, j_2, [0) \cap [1))\delta_k, & \text{if } j_1 = i_2; \\ (i_1, j_1+j_2-i_2, [0) \cap (i_2-j_1+[1)))\delta_k, & \text{if } j_1 > i_2 \end{cases} \\
&= \begin{cases} (i_1+i_2-j_1, j_2, [1))\delta_k, & \text{if } j_1 < i_2; \\ (i_1, j_2, [1))\delta_k, & \text{if } j_1 = i_2; \\ (i_1, j_1+j_2-i_2, [0))\delta_k, & \text{if } j_1 > i_2 \end{cases} \\
&= \begin{cases} (k(i_1+i_2-j_1+1), k(j_2+1), [0)), & \text{if } j_1 < i_2; \\ (k(i_1+1), k(j_2+1), [0)), & \text{if } j_1 = i_2; \\ (ki_1, k(j_1+j_2-i_2), [0)), & \text{if } j_1 > i_2 \end{cases}
\end{aligned}$$

$$
= \begin{cases} (k(i_1+i_2-j_1+1), k(j_2+1), [0)), & \text{if } j_1 < i_2; \\ (k(i_1+1), k(j_2+1), [0)), & \text{if } j_1 = i_2; \\ (ki_1, k(j_1+j_2-i_2), [0)), & \text{if } j_1 = i_2+1; \\ (ki_1, k(j_1+j_2-i_2), [0)), & \text{if } j_1 > i_2+1 \end{cases}
$$

$$
= \begin{cases} (k(i_1+i_2+1-j_1), k(j_2+1), [0)), & \text{if } j_1 < i_2; \\ (k(i_1+1), k(j_2+1), [0)), & \text{if } j_1 = i_2; \\ (ki_1, k(j_2+1), [0)), & \text{if } j_1 = i_2+1; \\ (ki_1, k(j_1+j_2-i_2), [0)), & \text{if } j_1 > i_2+1, \end{cases}
$$

and

$$
(i_1, j_1, [1))\delta_k \cdot (i_2, j_2, [0))\delta_k = (k(i_1+1), k(j_1+1), [0)) \cdot (ki_2, kj_2, [0)) =
$$

$$
= \begin{cases} (k(i_1+1)+ki_2-k(j_1+1), kj_2, (k(j_1+1)-ki_2+[0)) \cap [0)), & \text{if } k(j_1+1) < ki_2; \\ (k(i_1+1), kj_2, [0) \cap [0)), & \text{if } k(j_1+1) = ki_2; \\ (k(i_1+1), k(j_1+1)+kj_2-ki_2, [0) \cap (ki_2-k(j_1+1)+[0))), & \text{if } k(j_1+1) > ki_2 \end{cases}
$$

$$
= \begin{cases} (k(i_1+i_2-j_1), kj_2, [0)), & \text{if } j_1+1 < i_2; \\ (k(i_1+1), kj_2, [0)), & \text{if } j_1+1 = i_2; \\ (k(i_1+1), k(j_1+1+j_2-i_2), [0)), & \text{if } j_1+1 > i_2 \end{cases}
$$

$$
= \begin{cases} (k(i_1+i_2-j_1), kj_2, [0)), & \text{if } j_1+1 < i_2; \\ (k(i_1+1), kj_2, [0)), & \text{if } j_1+1 = i_2; \\ (k(i_1+1), k(j_1+1+j_2-i_2), [0)), & \text{if } j_1 = i_2; \\ (k(i_1+1), k(j_1+1+j_2-i_2), [0)), & \text{if } j_1+1 > i_2 \end{cases}
$$

$$
= \begin{cases} (k(i_1+i_2-j_1), kj_2, [0)), & \text{if } j_1+1 < i_2; \\ (k(i_1+1), kj_2, [0)), & \text{if } j_1+1 = i_2; \\ (k(i_1+1), k(j_2+1), [0)), & \text{if } j_1 = i_2; \\ (k(i_1+1), k(j_1+j_2-i_2+1), [0)), & \text{if } j_1 > i_2, \end{cases}
$$

$$
((i_1, j_1, [1)) \cdot (i_2, j_2, [0)))\delta_k = \begin{cases} (i_1+i_2-j_1, j_2, (j_1-i_2+[1)) \cap [0))\delta_k, & \text{if } j_1 < i_2; \\ (i_1, j_2, [1) \cap [0))\delta_k, & \text{if } j_1 = i_2; \\ (i_1, j_1+j_2-i_2, [1) \cap (i_2-j_1+[0)))\delta_k, & \text{if } j_1 > i_2 \end{cases}
$$

$$
= \begin{cases} (i_1+i_2-j_1, j_2, [0))\delta_k, & \text{if } j_1 < i_2; \\ (i_1, j_2, [1))\delta_k, & \text{if } j_1 = i_2; \\ (i_1, j_1+j_2-i_2, [1))\delta_k, & \text{if } j_1 > i_2 \end{cases}
$$

$$
= \begin{cases} (k(i_1+i_2-j_1), kj_2, [0)), & \text{if } j_1+1 < i_2; \\ (k(i_1+i_2-j_1), kj_2, [0)), & \text{if } j_1+1 = i_2; \\ (k(i_1+1), k(j_2+1), [0)), & \text{if } j_1 = i_2; \\ (k(i_1+1), k(j_1+j_2-i_2+1), [0)), & \text{if } j_1 > i_2 \end{cases}
$$

$$
= \begin{cases} (k(i_1+i_2-j_1), kj_2, [0)), & \text{if } j_1+1 < i_2; \\ (k(i_1+1), kj_2, [0)), & \text{if } j_1+1 = i_2; \\ (k(i_1+1), k(j_2+1), [0)), & \text{if } j_1 = i_2; \\ (k(i_1+1), k(j_1+j_2-i_2+1), [0)), & \text{if } j_1 > i_2. \end{cases}
$$

This completes the proof of the statement of the proposition. □

Remark 3. It obvious that if $\mathfrak{e}$ is the annihilating endomorphism of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ then $\mathfrak{e} = \gamma_0 = \delta_0$.

By $\boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we denote the semigroup of all non-injective monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for the family $\mathscr{F} = \{[0), [1)\}$.

Theorems 1 and 2 describe the algebraic structure of the semigroup $\boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Theorem 1. *If $\mathscr{F} = \{[0), [1)\}$, then for any non-injective monoid endomorphism $\mathfrak{e}$ of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ only one of the following conditions holds:*

(1) *$\mathfrak{e}$ is the annihilating endomorphism, i.e., $\mathfrak{e} = \gamma_0 = \delta_0$;*
(2) *$\mathfrak{e} = \gamma_k$ for some positive integer k;*
(3) *$\mathfrak{e} = \delta_k$ for some positive integer k.*

Proof. Fix an arbitrary non-injective monoid endomorphism $\mathfrak{e}$ of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. If $\mathfrak{e}$ is the annihilating endomorphism then statement (1) holds. Hence, later we assume that the endomorphism $\mathfrak{e}$ is not annihilating.

By Lemma 1 the restriction $\mathfrak{e}\restriction_{\boldsymbol{B}_{\omega}^{\{[0)\}}}\boldsymbol{B}_{\omega}^{\{[0)\}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ of the endomorphism $\mathfrak{e}$ is an injective mapping. Since by Proposition 3 of [4] the subsemigroup $\boldsymbol{B}_{\omega}^{\{[0)\}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ are isomorphic to the bicyclic semigroup, the injectivity of the restriction $\mathfrak{e}\restriction_{\boldsymbol{B}_{\omega}^{\{[0)\}}}$ of the endomorphism $\mathfrak{e}$, Proposition 4 of [5], and Lemma 2 of [8] imply that there exists a positive integer k such that

$$(i, j, [0))\mathfrak{e} = (ki, kj, [0)), \tag{3}$$

for all $i, j \in \omega$.

By Lemma 1 the restriction $\mathfrak{e}\restriction_{\boldsymbol{B}_{\omega}^{\{[1)\}}}\boldsymbol{B}_{\omega}^{\{[1)\}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}}$ of the endomorphism $\mathfrak{e}$ is an injective mapping, and by Lemma 2 we have that $(\boldsymbol{B}_{\omega}^{\{[1)\}})\mathfrak{e} \subseteq \boldsymbol{B}_{\omega}^{\{[0)\}}$. By Proposition 1.4.21(6) of [9] a homomorphism of inverse semigroups preserves the natural partial order, and hence the following inequalities

$$(1, 1, [0)) \preccurlyeq (0, 0, [1)) \preccurlyeq (0, 0, [0)),$$

Lemma 2, and Propositions 2 of [5] imply that

$$\begin{aligned}(k, k, [0)) &= (1, 1, [0))\mathfrak{e} \preccurlyeq \\ &\preccurlyeq (s, s, [0)) = \\ &= (0, 0, [1))\mathfrak{e} \preccurlyeq \\ &\preccurlyeq (0, 0, [0)) = \\ &= (0, 0, [0))\mathfrak{e}\end{aligned}$$

for some $s \in \{0, 1, \ldots, k\}$. Again by Proposition 1.4.21(6) of [9] and by Lemma 2 we get that

$$(1, 1, [1))\mathfrak{e} = (s + p, s + p, [0))$$

for some non-negative integer p. If $p = 0$ then $(1, 1, [1))\mathfrak{e} = (0, 0, [1))\mathfrak{e}$. By Lemma 1 the endomorphism $\mathfrak{e}$ is annihilating. Hence we assume that p is a positive integer.

Let $(0,1,[1])\mathfrak{e} = (x,y,[0])$ for some $x,y \in \omega$. By Proposition 1.4.21(1) of [9] and Lemma 4 of [4] we have that

$$\begin{aligned}(1,0,[1])\mathfrak{e} &= ((0,1,[1])^{-1})\mathfrak{e} =\\ &= ((0,1,[1])\mathfrak{e})^{-1} =\\ &= (x,y,[0])^{-1} =\\ &= (y,x,[0]).\end{aligned}$$

Since

$$(0,1,[1])\cdot(1,0,[1]) = (0,0,[1]) \qquad \text{and} \qquad (1,0,[1])\cdot(0,1,[1]) = (1,1,[1]),$$

the equalities $(0,0,[1])\mathfrak{e} = (s,s,[0])$ and $(1,1,[1])\mathfrak{e} = (s+p,s+p,[0])$ imply that

$$\begin{aligned}(s,s,[0]) &= (0,0,[1])\mathfrak{e} =\\ &= ((0,1,[1])\cdot(1,0,[1]))\mathfrak{e} =\\ &= (0,1,[1])\mathfrak{e}\cdot(1,0,[1])\mathfrak{e} =\\ &= (x,y,[0])\cdot(y,x,[0]) =\\ &= (x,x,[0])\end{aligned}$$

and

$$\begin{aligned}(s+p,s+p,[0]) &= (1,1,[1])\mathfrak{e} =\\ &= ((1,0,[1])\cdot(0,1,[1]))\mathfrak{e} =\\ &= (1,0,[1])\mathfrak{e}\cdot(0,1,[1])\mathfrak{e} =\\ &= (y,x,[0])\cdot(x,y,[0]) =\\ &= (y,y,[0]).\end{aligned}$$

This and the definition of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ imply that

$$(0,1,[1])\mathfrak{e} = (s,s+p,[0]) \qquad \text{and} \qquad (1,0,[1])\mathfrak{e} = (s+p,s,[0]).$$

Then for any positive integers n_1 and n_2 by usual calculations we get that

$$\begin{aligned}(0,n_1,[1])\mathfrak{e} &= (\underbrace{(0,1,[1])\cdot\ldots\cdot(0,1,[1])}_{n_1\text{-times}})\mathfrak{e} =\\ &= \underbrace{(0,1,[1])\mathfrak{e}\cdot\ldots\cdot(0,1,[1])\mathfrak{e}}_{n_1\text{-times}} =\\ &= (s,s+p,[0])^{n_1} =\\ &= (s,s+n_1p,[0])\end{aligned}$$

and

$$\begin{aligned}(n_2,0,[1])\mathfrak{e} &= (\underbrace{(1,0,[1])\cdot\ldots\cdot(1,0,[1])}_{n_2\text{-times}})\mathfrak{e} =\\ &= \underbrace{(1,0,[1])\mathfrak{e}\cdot\ldots\cdot(1,0,[1])\mathfrak{e}}_{n_2\text{-times}} =\end{aligned}$$

$$= (s+p, s, [0])^{n_2} =$$
$$= (s+n_2p, s, [0]),$$

and hence

(4) $$(n_1, n_2, [1])\mathfrak{e} = (s+n_1p, s+n_2p, [0]).$$

The definition of the natural partial order on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ (see Proposition 4 of [5]) imply that for any positive integer m we have that

$$(m+1, m+1, [0]) \preccurlyeq (m, m, [1]) \preccurlyeq (m, m, [0]).$$

Then by equalities (3), (4), and Proposition 1.4.21(6) of [9] we obtain that

$$\begin{aligned}(k(m+1), k(m+1), [0]) &= (m+1, m+1, [0])\mathfrak{e} \preccurlyeq \\ &\preccurlyeq (s+pm, s+pm, [0]) = \\ &= (m, m, [1])\mathfrak{e} \preccurlyeq \\ &\preccurlyeq (m, m, [0])\mathfrak{e} = \\ &= (km, km, [0]).\end{aligned}$$

The above inequalities and the definition of the natural partial order on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ (see Proposition 4 of [5]) imply that $km \leqslant s+pm \leqslant k(m+1)$ for any positive integer m. This implies that

$$k \leqslant \frac{s}{m} + p \leqslant k + \frac{1}{m},$$

and since p is a positive integer we get that $p = k$. Hence by (4) we get that

(5) $$(n_1, n_2, [1])\mathfrak{e} = (s+n_1k, s+n_2k, [0]),$$

for all $n_1, n_2 \in \omega$.

It is obvious that if $s \in \{1, \ldots, k-1\}$ then $\mathfrak{e}$ is an injective monoid endomorphism of the semigroup. Hence we have that either $s = 0$ or $s = k$, Simple verifications show that

$$\mathfrak{e} = \begin{cases} \gamma_k, & \text{if } s = 0; \\ \delta_k, & \text{if } s = k. \end{cases}$$

This completes the proof of the theorem. □

Theorem 2. *Let $\mathscr{F} = \{[0], [1]\}$. Then for all positive integers k_1 and k_2 the following conditions hold:*

(1) $\gamma_{k_1}\gamma_{k_2} = \gamma_{k_1k_2}$;
(2) $\gamma_{k_1}\delta_{k_2} = \gamma_{k_1k_2}$;
(3) $\delta_{k_1}\gamma_{k_2} = \delta_{k_1k_2}$;
(4) $\delta_{k_1}\delta_{k_2} = \delta_{k_1k_2}$.

Proof. (1) For any $i, j \in \omega$ we have that

$$\begin{aligned}(i, j, [0])\gamma_{k_1}\gamma_{k_2} &= (k_1i, k_1j, [0])\gamma_{k_2} = \\ &= (k_1k_2i, k_1k_2j, [0]),\end{aligned}$$

and $(i, j, [1])\gamma_{k_1} = (i, j, [0])\gamma_{k_1}$. This implies that $\gamma_{k_1}\gamma_{k_2} = \gamma_{k_1k_2}$.

(2) Since

$$\begin{aligned}(i,j,[0))\gamma_{k_1}\delta_{k_2} &= (k_1 i, k_1 j, [0))\delta_{k_2} = \\ &= (k_1k_2 i, k_1k_2 j, [0)),\end{aligned}$$

and $(i,j,[1))\gamma_{k_1} = (i,j,[0))\gamma_{k_1}$ for all $i,j\in\omega$, we get that $\gamma_{k_1}\delta_{k_2} = \gamma_{k_1k_2}$.

(3) For any $i,j\in\omega$ we have that

$$\begin{aligned}(i,j,[0))\delta_{k_1}\gamma_{k_2} &= (k_1 i, k_1 j, [0))\gamma_{k_2} = \\ &= (k_1k_2 i, k_1k_2 j, [0)),\end{aligned}$$

and

$$\begin{aligned}(i,j,[1))\delta_{k_1}\gamma_{k_2} &= (k_1(i+1), k_1(j+1), [0))\gamma_{k_2} = \\ &= (k_1k_2(i+1), k_1k_2(j+1), [0)),\end{aligned}$$

and hence $\delta_{k_1}\gamma_{k_2} = \delta_{k_1k_2}$.

(4) For any $i,j\in\omega$ we have that

$$\begin{aligned}(i,j,[0))\delta_{k_1}\delta_{k_2} &= (k_1 i, k_1 j, [0))\delta_{k_2} = \\ &= (k_1k_2 i, k_1k_2 j, [0)),\end{aligned}$$

and

$$\begin{aligned}(i,j,[1))\delta_{k_1}\delta_{k_2} &= (k_1(i+1), k_1(j+1), [0))\delta_{k_2} = \\ &= (k_1k_2(i+1), k_1k_2(j+1), [0)),\end{aligned}$$

and hence $\delta_{k_1}\delta_{k_2} = \delta_{k_1k_2}$. ☐

By $\mathfrak{e}_0$ we denote the annihilating monoid endomorphism of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for the family $\mathscr{F}=\{[0),[1)\}$, i.e., $(i,j,[p))\mathfrak{e}_0=(0,0,[0))$ for all $i,j\in\omega$ and $p=0,1$. We put $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}}) = \boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})\setminus\{\mathfrak{e}_0\}$. Theorem 2 implies that $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ is a subsemigroup of $\boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Theorem 2 implies the following corollary.

Corollary 1. *If $\mathscr{F}=\{[0),[1)\}$, then the elements γ_1 and δ_1 are unique idempotents of the semigroup $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.*

Next, by $\mathfrak{LZ}_2$ we denote the left zero semigroup with two elements and by $\mathbb{N}_u$ the multiplicative semigroup of positive integers.

Proposition 2. *Let $\mathscr{F}=\{[0),[1)\}$. Then the semigroup $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ is isomorphic to the direct product $\mathfrak{LZ}_2\times\mathbb{N}_u$.*

Proof. Put $LZ_2=\{c,d\}$. We define a map $\mathfrak{I}\colon \boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})\to\mathfrak{LZ}_2\times\mathbb{N}_u$ by the formula

$$(\mathfrak{e})\mathfrak{I}=\begin{cases}(c,k), & \text{if } \mathfrak{e}=\gamma_k;\\ (d,k), & \text{if } \mathfrak{e}=\delta_k.\end{cases}$$

It is obvious that such defined map $\mathfrak{I}$ is bijective, and by Theorem 2 it is a homomorphism. ☐

Theorem 3 describes Green's relations on the semigroup $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$. Later by $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})^1$ we denote the semigroup $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ with adjoined identity element.

Theorem 3. *Let $\mathscr{F} = \{[0),[1)\}$. Then the following statements hold:*

(1) $\gamma_{k_1}\mathscr{R}\gamma_{k_2}$ *in* $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *if and only if* $k_1 = k_2$*;*
(2) $\gamma_{k_1}\mathscr{R}\delta_{k_2}$ *does not hold in* $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *for any* $\gamma_{k_1}, \delta_{k_2}$*;*
(3) $\delta_{k_1}\mathscr{R}\delta_{k_2}$ *in* $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *if and only if* $k_1 = k_2$*;*
(4) $\gamma_{k_1}\mathscr{L}\gamma_{k_2}$ *in* $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *if and only if* $k_1 = k_2$*;*
(5) $\gamma_{k_1}\mathscr{L}\delta_{k_2}$ *in* $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *if and only if* $k_1 = k_2$*;*
(6) $\delta_{k_1}\mathscr{L}\delta_{k_2}$ *in* $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *if and only if* $k_1 = k_2$*;*
(7) $\mathscr{H}$ *is the identity relation on* $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$*;*
(8) $\mathfrak{e}_1\mathscr{D}\mathfrak{e}_2$ *in* $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ *if and only if* $\mathfrak{e}_1 = \mathfrak{e}_2$ *or there exists a positive integer* k *such that* $\mathfrak{e}_1, \mathfrak{e}_2 \in \{\gamma_k, \delta_k\}$*;*
(9) $\mathscr{D} = \mathscr{J}$ *in* $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Proof. (1) ($\Rightarrow$) Suppose that $\gamma_{k_1}\mathscr{R}\gamma_{k_2}$ in $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$. Then there exist $\mathfrak{e}_1, \mathfrak{e}_2 \in \boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})^1$ such that $\gamma_{k_1} = \gamma_{k_2}\mathfrak{e}_1$ and $\gamma_{k_2} = \gamma_{k_1}\mathfrak{e}_2$. The equality $\gamma_{k_1} = \gamma_{k_2}\mathfrak{e}_1$ and Theorem 2 imply that there exists a positive integer p such that either $\mathfrak{e}_1 = \gamma_p$ or $\mathfrak{e}_1 = \delta_p$. In both above cases by Theorem 2 we have that

$$\gamma_{k_1} = \gamma_{k_2}\mathfrak{e}_1 = \gamma_{k_2}\gamma_p = \gamma_{k_2}\delta_p = \gamma_{k_2 p},$$

and hence $k_2|k_1$. The proof of the statement that $\gamma_{k_2} = \gamma_{k_1}\mathfrak{e}_2$ implies that $k_1|k_2$ is similar. Therefore we get that $k_1 = k_2$.

Implication ($\Leftarrow$) is trivial.

Statement (2) follows from Theorem 2(2).

The proof of statement (3) is similar to (1).

(4) ($\Rightarrow$) Suppose that $\gamma_{k_1}\mathscr{L}\gamma_{k_2}$ in $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$. Then there exist $\mathfrak{e}_1, \mathfrak{e}_2 \in \boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})^1$ such that $\gamma_{k_1} = \mathfrak{e}_1\gamma_{k_2}$ and $\gamma_{k_2} = \mathfrak{e}_2\gamma_{k_1}$. The equality $\gamma_{k_1} = \mathfrak{e}_1\gamma_{k_2}$ and Theorem 2 imply that there exists a positive integer p such that $\mathfrak{e}_1 = \gamma_p$. Then we have that

$$\gamma_{k_1} = \mathfrak{e}_1\gamma_{k_2} = \gamma_p\gamma_{k_2} = \gamma_{pk_2},$$

and hence $k_2|k_1$. The proof of the statement that $\gamma_{k_2} = \mathfrak{e}_2\gamma_{k_1}$ implies that $k_1|k_2$ is similar. Therefore we get that $k_1 = k_2$.

Implication ($\Leftarrow$) is trivial.

(5) ($\Rightarrow$) Suppose that $\gamma_{k_1}\mathscr{L}\delta_{k_2}$ in $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$. Then there exist $\mathfrak{e}_1, \mathfrak{e}_2 \in \boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})^1$ such that $\gamma_{k_1} = \mathfrak{e}_1\delta_{k_2}$ and $\delta_{k_2} = \mathfrak{e}_2\gamma_{k_1}$. The equality $\gamma_{k_1} = \mathfrak{e}_1\delta_{k_2}$ and Theorem 2 imply that there exists a positive integer p such that $\mathfrak{e}_1 = \gamma_p$. Then we have that

$$\gamma_{k_1} = \mathfrak{e}_1\delta_{k_2} = \gamma_p\delta_{k_2} = \gamma_{pk_2},$$

and hence $k_2|k_1$. The equality $\delta_{k_2} = \mathfrak{e}_2\gamma_{k_1}$ and Theorem 2 imply that there exists a positive integer q such that $\mathfrak{e}_1 = \delta_q$. Then we have that

$$\delta_{k_2} = \mathfrak{e}_2\gamma_{k_1} = \delta_q\gamma_{k_1} = \gamma_{qk_1},$$

and hence $k_1|k_2$. Thus we get that $k_1 = k_2$.

Implication ($\Leftarrow$) is trivial.

(6) ($\Rightarrow$) Suppose that $\delta_{k_1}\mathscr{L}\delta_{k_2}$ in $\boldsymbol{End}^*(\boldsymbol{B}_\omega^{\mathscr{F}})$. Then there exist $\mathfrak{e}_1,\mathfrak{e}_2\in\boldsymbol{End}^*(\boldsymbol{B}_\omega^{\mathscr{F}})^1$ such that $\delta_{k_1}=\mathfrak{e}_1\delta_{k_2}$ and $\delta_{k_2}=\mathfrak{e}_2\delta_{k_1}$. The equality $\delta_{k_1}=\mathfrak{e}_1\delta_{k_2}$ and Theorem 2 imply that there exists a positive integer p such that $\mathfrak{e}_1=\delta_p$. Then we have that

$$\delta_{k_1}=\mathfrak{e}_1\delta_{k_2}=\delta_p\delta_{k_2}=\delta_{pk_2},$$

and hence $k_2|k_1$. The proof of the statement that $\delta_{k_2}=\mathfrak{e}_2\delta_{k_1}$ implies that $k_1|k_2$ is similar. Hence we get that $k_1=k_2$.

Implication ($\Leftarrow$) is trivial.

(7) By statements (1), (2), and (3), $\mathscr{R}$ is the identity relation on the semigroup $\boldsymbol{End}^*(\boldsymbol{B}_\omega^{\mathscr{F}})$. Then so is $\mathscr{H}$, because $\mathscr{H}\subseteq\mathscr{R}$.

Statement (8) follows from statements (1)–(6).

(9) Suppose to the contrary that $\mathscr{D}\neq\mathscr{J}$ in $\boldsymbol{End}^*(\boldsymbol{B}_\omega^{\mathscr{F}})$. Since $\mathscr{D}\subseteq\mathscr{J}$, statement (8) implies that there exist $\mathfrak{e}_1,\mathfrak{e}_2\in\boldsymbol{End}^*(\boldsymbol{B}_\omega^{\mathscr{F}})^1$ such that $\mathfrak{e}_1\mathscr{J}\mathfrak{e}_2$ and $\mathfrak{e}_1,\mathfrak{e}_2\notin\{\gamma_k,\delta_k\}$ for any positive integer k. Then there exist distinct positive integers k_1 and k_2 such that $\mathfrak{e}_1\in\{\gamma_{k_1},\delta_{k_1}\}$ and $\mathfrak{e}_2\in\{\gamma_{k_2},\delta_{k_2}\}$. Without loss of generality we may assume that $k_1<k_2$. Since $\mathfrak{e}_1\mathscr{J}\mathfrak{e}_2$ there exist $\mathfrak{e}_1',\mathfrak{e}_2',\mathfrak{e}_1'',\mathfrak{e}_2''\in\boldsymbol{End}^*(\boldsymbol{B}_\omega^{\mathscr{F}})^1$ such that $\mathfrak{e}_1=\mathfrak{e}_1'\mathfrak{e}_2\mathfrak{e}_1''$ and $\mathfrak{e}_2=\mathfrak{e}_2'\mathfrak{e}_2\mathfrak{e}_2''$. Since $\mathfrak{e}_1\in\{\gamma_{k_1},\delta_{k_1}\}$ and $\mathfrak{e}_2\in\{\gamma_{k_2},\delta_{k_2}\}$, the equality $\mathfrak{e}_1=\mathfrak{e}_1'\mathfrak{e}_2\mathfrak{e}_1''$, Theorems 1 and 2 imply that $k_2|k_1$. This contradicts the inequality $k_1<k_2$. The obtained contradiction implies the requested statement □

Remark 4. Since $\mathfrak{e}_\mathbf{0}$ is zero of the semigroup $\boldsymbol{End}_0^*(\boldsymbol{B}_\omega^{\mathscr{F}})$, the classes of equivalence of Green's relations of non-zero elements of $\boldsymbol{End}_0^*(\boldsymbol{B}_\omega^{\mathscr{F}})$ in the semigroup $\boldsymbol{End}_0^*(\boldsymbol{B}_\omega^{\mathscr{F}})$ coincide with their corresponding classes of equivalence in $\boldsymbol{End}^*(\boldsymbol{B}_\omega^{\mathscr{F}})$, and moreover we have that

$$\boldsymbol{L}_{\mathfrak{e}_\mathbf{0}}=\boldsymbol{R}_{\mathfrak{e}_\mathbf{0}}=\boldsymbol{H}_{\mathfrak{e}_\mathbf{0}}=\boldsymbol{D}_{\mathfrak{e}_\mathbf{0}}=\boldsymbol{J}_{\mathfrak{e}_\mathbf{0}}=\{\mathfrak{e}_\mathbf{0}\}$$

in the semigroup $\boldsymbol{End}_0^*(\boldsymbol{B}_\omega^{\mathscr{F}})$.

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ПРО НАПІВГРУПУ НЕІН'ЄКТИВНИХ МОНОЇДАЛЬНИХ ЕНДОМОРФІЗМІВ НАПІВГРУПИ $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ З ДВОЕЛЕМЕНТНОЮ СІМ'ЄЮ $\mathscr{F}$ ІНДУКТИВНИХ НЕПОРОЖНІХ ПІДМНОЖИН У ω

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Вивчено напівгрупу неін'єктивних моноїдальних ендоморфізмів напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ з двоелементною сім'єю $\mathscr{F}$ індуктивних непорожніх підмножин у ω. Описано елементи напівгрупи $\boldsymbol{End}_0^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ усіх неін'єктивних моноїдальних енодоморфізмів напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. Зокрема, доведено, що її піднапівгрупа $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ усіх неін'єктивних неанулюючих моноїдальних ендоморфізмів напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ ізоморфна прямому добутку двоелементної напівгрупи з лівим нульовим множенням і мультиплікативної напівгрупи натуральних чисел. Також описанмо відношення Ґріна на напівгрупі $\boldsymbol{End}^*(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Ключові слова: біциклічнй моноїд, інверсна напівгрупа, біциклічне розширення, моноїдальний ендоморфізм, неін'ктивний, відношення Ґріна, напівгрупа лівих нулів, прямий добуток.