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# FINITE SEMIGROUPS OF ORDERED-DECREASING TRANSFORMATIONS

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We study semigroups of partially defined order-decreasing transformations of partially ordered sets and semigroups of partially defined order-decreasing transformations of the lexicographic and direct product of partially ordered sets.

*Key words:* semigroup, partially defined transformation, order-decreasing transformation, lexicographic product, direct product.

## 1. Introduction

Let $(M, \leqslant)$ be a partially ordered set. A transformation $\varphi : M \to M$ (generally speaking, partial) is called *order-decreasing* if for any $a$ from the domain of the transformation $\varphi$ the inequality $\varphi(a) \leqslant a$ holds. The set of all partially defined order-decreasing transformations of the set $(M, \leqslant)$ is denoted by $\mathrm{PDecr}(M, \leqslant)$, and all everywhere defined order-decreasing transformations are denoted by $\mathrm{Decr}(M, \leqslant)$. If it is clear which partial order is being referred to, we will simply write $\mathrm{PDecr}(M)$ and $\mathrm{Decr}(M)$.

Each of these sets forms a semigroup with respect to the composition of transformations.

Sometimes (for example, when studying nilpotent semigroups, see [4], [3], [5]) it is convenient to consider semigroups of *strictly order-decreasing* transformations $\varphi$, which for any $a$ from the domain satisfy the inequality $\varphi(a) < a$. For a finite set $(M, \leqslant)$, such a transformation will always be only partially defined. The corresponding subsemigroup from $\mathrm{PDecr}(M)$ will be denoted $\mathrm{PSDecr}(M)$.

Instead of order-decreasing transformations, one can also study the dual concept of *order-increasing* transformations $\varphi : M \to M$ such that for all $a \in M$ the inequality

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$\varphi(a) \geqslant a$ holds. They form the semigroups $\mathrm{Incr}(M)$ and $\mathrm{PIncr}(M)$ with respect to the composition of transformations. If the partial order $\leqslant$ is self-dual (as in many important cases), then these semigroups will be isomorphic to the corresponding semigroups of order-decreasing transformations.

The semigroup $\mathrm{Decr}(L_n)$ of all order-decreasing transformations of an $n$-element linearly ordered set $L_n$ first appears, perhaps, in [13] in connection with the study of formal languages. In 1992, Howie [8] drew attention to the importance of studying semigroups of order-decreasing transformations. Deeper study of the semigroups $\mathrm{PDecr}(L_n)$ and $\mathrm{Decr}(L_n)$ began in the 1990s in the works of Umar [15, 16, 17]. Later, the combinatorial properties of some other semigroups of order-decreasing transformations were studied by A. Laradji and A. Umar [10].

Currently, there are several dozen works in which the semigroups of order-decreasing transformations of the set $L_n$ and some of their special subsemigroups are studied (see, for example, [18], [19], [10], [11], [9] and the bibliography in [4]). However, semigroups of order-decreasing transformations of other partially ordered sets have been little studied so far ([6], [14], [12]).

The symbol $\mathtt{N}$ denotes the set $1, 2, \ldots, n$, and $\mathfrak{B}_n$ denotes the set of all subsets of the set $\mathtt{N}$, ordered by the inclusion relation.

## 2. Semigroups of order-decreasing transformations of some partially ordered sets

The lower cone of an element $a$ in a partially ordered set $(M, \leqslant)$ is defined as the set $a_\triangle = \{x \in M \mid x \leqslant a\}$.

The proposition directly follows from the definitions.

**Proposition 1.** *For a finite partially ordered set* $(M, \leqslant)$,

$$|\mathrm{Decr}(M)| = |\mathrm{PSDecr}(M)| = \prod_{a \in M} |a_\triangle|,$$

$$|\mathrm{PDecr}(M)| = \prod_{a \in M} (|a_\triangle| + 1).$$

**Proposition 2** ([15])**.** *For an $n$-element linearly ordered set $L_n$,*

$$|\mathrm{Decr}\, L_n| = n!, \quad |\mathrm{PDecr}\, L_n| = (n+1)!\,.$$

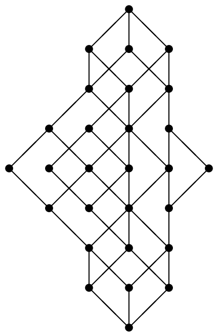
**Proposition 3.** *a) For a 3-generated free modular lattice $F_M(3)$,*

$$|\mathrm{Decr}\, F_M(3)| = 28 \cdot 22^3 \cdot 18^3 \cdot 15 \cdot 12^3 \cdot 10^3 \cdot 8 \cdot 6^3 \cdot 5^3 \cdot 4^3 \cdot 2^3 =$$
$$= 11^3 \cdot 7 \cdot 5^7 \cdot 3^{13} \cdot 2^{32} = 4984278472584069120000000.$$

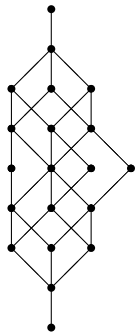
*b) For a 3-generated free distributive lattice $F_D(3)$,*

$$|\mathrm{Decr}\, F_D(3)| = 20 \cdot 19 \cdot 14^3 \cdot 11^3 \cdot 9 \cdot 6^3 \cdot 5^3 \cdot 3^3 \cdot 2 =$$
$$= 19 \cdot 11^3 \cdot 7^3 \cdot 5^4 \cdot 3^8 \cdot 2^9 = 18211503119040000.$$

*Proof.* The Hasse diagrams of the 3-generated free modular lattice $F_M(3)$ and the distributive lattice $F_D(3)$, are as follows, respectively:



and



(see [7]). The proposition follows directly from the appearance of these diagrams and Proposition 4. □

**Theorem 1.** *a)* $|\mathrm{Decr}\,\mathfrak{B}_n| = 2^{n\cdot 2^{n-1}}$;

*b)* $|\mathrm{PDecr}\,\mathfrak{B}_n| = \prod\limits_{k=0}^{n}(2^k+1)^{\binom{n}{k}}$.

*Proof.* *a)* For a $k$-element subset $A \in \mathfrak{B}_n$, the lower cone $A_\triangle$ has a size of $2^k$. Therefore, according to Proposition 4,

$$|\mathrm{Decr}\,\mathfrak{B}_n| = \prod_{k=0}^{n}(2^k)^{\binom{n}{k}} = 2^{\sum_k k\binom{n}{k}} = 2^{n\sum_k \binom{n-1}{k-1}} = 2^{n\cdot 2^{n-1}}.$$

Statement *b)* Follows from Proposition 4. □

Let $\mathcal{L}(n,q)$ denote the set of all subspaces of the $n$-dimensional vector space $\mathbb{F}_q^n$ over the $q$-element finite field $\mathbb{F}_q$, ordered by the inclusion relation. The Gaussian binomial coefficient $\binom{n}{k}_q$ (is referred to as the Gauss number) is the number of all $k$-dimensional subspaces of an $n$-dimensional vector space over the field $\mathbb{F}_q$. The number $G_n(q)$, representing all subspaces of an $n$-dimensional space over the field $\mathbb{F}_q$, is referred to as the Galois number.

**Theorem 2.** *For the set ordered by inclusion, $\mathcal{L}(n,q)$*

$$|\mathrm{Decr}\,\mathcal{L}(n,q)| = \prod_{k=0}^{n}(G_k(q))^{\binom{n}{k}_q} = \prod_{k=0}^{n}\Big(\sum_{i=0}^{k}\binom{k}{i}_q\Big)^{\binom{n}{k}_q}.$$

*Proof.* It follows from Proposition 4 since the lower cone of a $k$-dimensional subspace from $\mathcal{L}(n,q)$ contains $\binom{n}{k}_q$ elements. □

There is a natural one-to-one correspondence between the partitions of the set N and the equivalence relations on the set N. The set $\mathrm{Eq}_n$ of all equivalence relations on the set N is naturally ordered by the inclusion relation. This order induces an order relation on the set $\mathrm{Part}_n$ of partitions of the set N.

A partition $\tau$ is said to have type $\langle l_1, l_2, \ldots, l_n\rangle$ if it contains $l_1$ blocks of length 1, $l_2$ blocks of length 2, …, $l_n$ blocks of length $n$. Obviously,

$$l_1 + 2l_2 + \cdots + nl_n = n.$$

It is easy to understand that the number $P(l_1, l_2, \dots, l_n)$ of partitions of the set $\mathtt{N}$ of type $\langle l_1, l_2, \dots, l_n\rangle$ is equal to

$$(1) \qquad P(l_1, l_2, \dots, l_n) = \frac{n!}{(1!)^{l_1}(2!)^{l_2}\cdots(n!)^{l_n} l_1! l_2! \cdots l_n!}.$$

**Theorem 3.** *For the lattice of partitions* $\mathrm{Part}_n$,

$$|\mathrm{Decr}\,\mathrm{Part}_n| = \prod_{l_1+2l_2+\cdots+nl_n=n} P(l_1, l_2, \dots, l_n) B_1^{l_1} B_2^{l_2} \cdots B_n^{l_n},$$

*where $B_k$ is the $k$-th Bell number.*

*Proof.* It is evident that when two partitions have the same type, their lower cones are of equal power. Let's consider the structure of the lower cone of a partition $\rho$ of type $\langle l_1, l_2, \dots, l_n\rangle$. If a partition $\tau$ belongs to the lower cone $\rho_\triangle$, then each block of the partition $\tau$ is contained in one of the blocks of the partition $\rho$. Therefore, the partition $\tau$ induces on each block $M$ of the partition $\rho$ a certain partition $\tau_M$. The set of these induced partitions can be viewed as an element of the set

$$\mathrm{Part}_1^{l_1} \times \cdots \times \mathrm{Part}_n^{l_n}.$$

Conversely, each element from $\mathrm{Part}_1^{l_1} \times \cdots \times \mathrm{Part}_n^{l_n}$ can be considered as a set of partitions of the blocks of the partition $\rho$, that is, as a partition of the set $\mathtt{N}$ belonging to the lower cone $\rho_\triangle$. Therefore,

$$\rho_\triangle \simeq \mathrm{Part}_1^{l_1} \times \cdots \times \mathrm{Part}_n^{l_n}, \quad \text{and} \quad |\rho_\triangle| = |\mathrm{Part}_1|^{l_1} \cdots |\mathrm{Part}_n|^{l_n}.$$

The proof is completed by referring to Proposition 4 and noting that the number of partitions of a $k$-element set is the $k$-th Bell number $B_k$. □

For any group $G$, let $\mathcal{L}(G)$ denote the lattice of its subgroups ordered by inclusion. For a subgroup $H \leqslant G$, let $\mathcal{L}(H, G)$ denote the lattice $\{Q \in \mathcal{L}(G) \mid H \leqslant Q\}$ of its overgroups.

Recall that according to the Fricke-Klein Theorem (see [2]), each subgroup $H$ of the direct product of groups $P \times Q$ is uniquely determined by 5 parameters: subgroups $A_1 \lhd A \leqslant P$, $B_1 \lhd B \leqslant Q$ such that $A/A_1 \simeq B/B_1$, and an isomorphism $\Phi : A/A_1 \to B/B_1$. Here,

$$H = \{(a, b) \in A \times B \mid \Phi(\overline{a}) = \overline{b}\},$$

where $\overline{x}$ denotes the corresponding element of the quotient group. The subgroup $H$ with parameters $(A, A_1, B, B_1, \Phi)$ is denoted by

$$A/A_1 \underset{\Phi}{\times} B/B_1.$$

*Remark* 1. From the Fricke-Klein theorem, it follows that

$$|A/A_1 \underset{\Phi}{\times} B/B_1| = |A_1| \cdot |B_1| \cdot |A/A_1|.$$

**Proposition 4.** *For any natural numbers $r$, $s$, and a prime number $p$,*

$$(2) \qquad |\mathcal{L}(C_{p^r} \times C_{p^s})| = \sum_{k=0}^{\min(r,s)} (r - k + 1)(s - k + 1)\varphi(p^k),$$

*where $\varphi(p^k)$ is the Euler's function.*

*Proof.* Every subsemigroup $H$ of $C_{p^r} \times C_{p^s}$ is determined by 5 parameters: subsemigroups

$$A_1 \leqslant A \leqslant C_{p^r}, \quad B_1 \leqslant B \leqslant C_{p^s}$$

such that $|A/A_1| = |B/B_1|$, and an isomorphism $\Phi : A/A_1 \to B/B_1$. If

$$|A/A_1| = |B/B_1| = p^k,$$

then the exponent $k$ can be any integer in $[0, \min(r, s)]$, the pair $A_1 \leqslant A \leqslant C_{p^r}$ can be chosen in $r - k + 1$ ways, the pair

$$B_1 \leqslant B \leqslant C_{p^s}$$

in $s - k + 1$ ways, and the cyclic group $C_{p^k}$ has $\varphi(p^k)$ automorphisms. □

Let's denote the right-hand side of the equality (2) as $N_p(r, s)$.

**Lemma 1.** *Let a subgroup $H$ of*

$$C_{p^n} \times C_{p^m}$$

*be defined by the parameters $(A, A_1, B, B_1, \Phi)$, where*

$$A \simeq C_{p^r}, \quad A_1 \simeq C_{p^{r'}}, \quad B \simeq C_{p^t}, \quad B_1 \simeq C_{p^{t'}},$$

*and $n \geqslant r \geqslant r'$, $m \geqslant t \geqslant t'$. Then*

$$H \simeq C_{p^{\max(r,t)}} \times C_{p^{\min(r',t')}}.$$

*Proof.* As a subgroup of a 2-generated abelian group, $H$ must have a generating set with $\leqslant 2$ elements. Therefore,

$$H \simeq C_{p^k} \times C_{p^l} \quad \text{for some} \quad k \geqslant l \geqslant 0.$$

In particular, the maximum order of an element in $H$ equals $p^k$. On the other hand, as a subgroup of

$$A \times B \simeq C_{p^r} \times C_{p^t},$$

$H$ cannot have elements of an order greater than $p^{\max(r,t)}$. However, since the projection of $H$ onto each of the factors $A$ and $B$ coincides with these factors, it follows that the subgroup $H$ does have an element of order $p^{\max(r,t)}$. Therefore, $k = \max(r, t)$.

Furthermore, from Remark 1 and the equality $r - r' = t - t'$, it follows that

$$H = p^{\max(r,t)} \cdot p^{\min(r',t')}.$$

Hence, $l$ must coincide with $\min(r', t')$. □

*Remark* 2. From the equality $r - r' = t - t'$, it follows that the subgroup $H$ from Lemma 1 is uniquely determined by 4 independent parameters $r$, $t$, $v = r - r' = t - t'$, and $\Phi$.

**Theorem 4.** *For the lattice of subgroups $\mathcal{L}(C_{p^n} \times C_{p^m})$ of the group $C_{p^n} \times C_{p^m}$ ordered by inclusion*

$$|\mathrm{Decr}\, \mathcal{L}(C_{p^n} \times C_{p^m})| = \sum_{r \leqslant n} \sum_{t \leqslant m} \sum_{v \leqslant \min r,t} \varphi(p^v) N_p(\max(r,t), \min(r,t) - v).$$

*Proof.* From Lemma 1, it follows that a subgroup $H \leqslant C_{p^n} \times C_{p^m}$ with parameters $r$, $t$, $v$, $\Phi$ is isomorphic to the group

$$C_{p^{\max(r,t)}} \times C_{p^{\min(r,t)-v}}.$$

Therefore, the type of subgroup $H$ is completely determined by the first three parameters $r$, $t$, $v$, and the number of subgroups with such parameters equals $\varphi(p^v)$. According to Proposition 4, the lower cone $H_\triangle$ of the subgroup $H$ with parameters $r$, $t$, $v$ has a cardinality of

$$N_p(\max(r,t), \min(r,t) - v).$$

The statement of the theorem now follows from Proposition 4 and the fact that $r \leqslant n$, $t \leqslant m$, and $v \leqslant \min(r,t)$. □

*Remark* 3. Unlike the Klein-Fricke Theorem, a good description of subgroups for the direct product of more than two factors is still unknown. Therefore, the question about the order of the semigroup Decr, $\mathcal{L}(G)$, even for the group

$$G = C_{p^n} \times C_{p^m} \times C_{p^k},$$

remains open.

## 3. Connection with operations over partially ordered sets

**Theorem 5.** *For the lexicographic product $M_1 \circ M_2$ of partially ordered sets $M_1$ and $M_2$*

$$|\mathrm{Decr}\,(M_1 \circ M_2)| = \prod_{a\in M_1,\ b\in B_2} \big(|b_\triangle| + (|a_\triangle| - 1)|M_2|\big).$$

*Proof.* For elements $(a,b)$ and $(x,y)$ from $M_1 \circ M_2$, the inequality $(a,b) \geqslant (x,y)$ holds if and only if either $a = x$ and $b \geqslant y$ (there are $|b_\triangle|$ such elements $(x,y)$), or $a > x$ (there are $(|a_\triangle| - 1)|M_2|$ such elements $(x,y)$). Therefore,

$$|(a,b)_\triangle = |b_\triangle| + (|a_\triangle| - 1)|M_2|.$$

The statement of the theorem follows from Proposition 4. □

**Corollary 1.** *For the lexicographic product*

$$L_{n_1} \circ L_{n_2} \circ \cdots \circ L_{n_k}$$

*of linearly ordered sets $L_{n_1}$, $L_{n_2}$, …, $L_{n_k}$*

$$|\mathrm{Decr}\,(L_{n_1} \circ L_{n_2} \circ \cdots \circ L_{n_k})| = (n_1 n_2 \cdots n_k)!.$$

**Theorem 6.** *For the direct product $M_1 \times M_2$ of partially ordered sets $M_1$ and $M_2$*

$$|\mathrm{Decr}\,(M_1 \times M_2)| = |\mathrm{Decr}\, M_1|^{|M_2|} |\mathrm{Decr}\, M_2|^{|M_1|}.$$

*Proof.* Since $|(a,b)_\triangle| = |a_\triangle| \cdot |b_\triangle|$ for $(a,b) \in M_1 \times M_2$, according to the Proposition 4 we get that

$$|\mathrm{Decr}\,(M_1 \times M_2)| = \prod_{(a,b)\in M_1\times M_2} |(a,b)_\triangle| = \prod_{a\in M_1,\ b\in M_2} |a_\triangle| \cdot |b_\triangle| =$$

$$= \prod_{a\in M_1,\ b\in M_2} |a_\triangle| \prod_{a\in M_1,\ b\in M_2} |b_\triangle| = |\ \mathrm{Decr}\, M_1|^{|M_2|} \cdot |\mathrm{Decr}\, M_2|^{|M_1|}.$$

□

**Corollary 2.** *For the direct product $M_1 \times \cdots \times M_k$ of partially ordered sets $M_1, \ldots, M_k$*

$$|\mathrm{Decr}\,(M_1 \times \cdots \times M_k| = \prod_{i=1}^{k} |\mathrm{Decr}\, M_i|^{\prod\limits_{j\neq i} |M_j|}.$$

**Corollary 3.** *For the direct product*

$$L_{n_1} \times L_{n_2} \times \cdots \times L_{n_k}$$

*of linearly ordered sets $L_{n_1}, L_{n_2}, \ldots, L_{n_k}$*

$$|\mathrm{Decr}\,(L_{n_2} \times \cdots \times L_{n_k})| = (n_1!)^{n_2 \cdots n_k} \cdot (n_2!)^{n_1 n_3 \cdots n_k} \cdot (n_k!)^{n_1 \cdots n_{k-1}}.$$

**Corollary 4.** *If the group $G$ decomposes into the direct product $G = H_1 \times \cdots \times H_k$ of subgroups $H_1, \ldots, H_k$ of pairwise coprime orders, then*

$$|\mathrm{Decr}\,\mathcal{L}(G)| = \prod_{i=1}^{k} |\mathrm{Decr}\,\mathcal{L}(H_i)|^{\prod\limits_{j\neq i} |\mathcal{L}(H_j)|}.$$

*Доведення. [Proof]* From the Fricke-Klein theorem, it follows that when the orders of subgroups $A$ and $B$ are coprime, each subgroup $H$ from $A \times B$ decomposes into the direct product $H = A_1 \times B_1$ of subgroup $A_1$ from $A$ and subgroup $B_1$ from $B$. Therefore, the lattice $\mathcal{L}(G)$ decomposes into the direct product $\mathcal{L}(G) = \mathcal{L}(H_1) \times \cdots \times \mathcal{L}(H_k)$ of lattices $\mathcal{L}(H_1), \ldots, \mathcal{L}(H_k)$. □

For any prime number $p$ and a group $G$, $\mathrm{Syl}_p(G)$ denotes the Sylow $p$-subgroup of the group $G$.

**Corollary 5.** *If for each prime number $p$ the group $G$ contains a unique Sylow $p$-subgroup (in particular, if $G$ is abelian), then*

$$|\mathrm{Decr}\,\mathcal{L}(G)| = \prod_{p||G|} |\mathrm{Decr}\,\mathcal{L}(\mathrm{Syl}_p(G))|^{\prod\limits_{q\neq p} |\mathrm{Syl}_q(G)|}.$$

*Proof.* This follows from Corollary 4, since such a group decomposes into the direct product of its Sylow subgroups. □

**Corollary 6.** *For a cyclic group*

$$C_{p_1^{n_1} \cdots p_k^{n_k}},$$

*we have*

$$|\mathrm{Decr}\,\mathcal{L}(G)| = (n_1+1)!^{n_2 \cdots n_k} \cdot (n_2+1)!^{n_1 n_3 \cdots n_k} \cdot (n_k+1)!^{n_1 \cdots n_{k-1}}.$$

*Proof.* This follows from Proposition 2 and Corollary 5, since the Sylow $p_i$-subgroup of the group

$$C_{p_1^{n_1} \cdots p_k^{n_k}}$$

is the group $C_{p_i^{n_i}}$, the lattice of subgroups of which is linearly ordered and has $(n_i + 1)$ elements. □

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# СКІНЧЕННІ НАПІВГРУПИ СТИСКУЮЧИХ ПЕРЕТВОРЕНЬ

**Дмитро БЕЗУЩАК**

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Вивчаємо напівгрупи частково визначених стискуючих перетворень частково впорядкованих множин і напівгрупи частково визначених стискуючих перетворень лексикографічного та прямого добутку частково впорядкованих множин.

*Ключові слова:* напівгрупа, частково визначене перетворення, стискуюче перетворення, лексикографічний добуток, прямий добуток.