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ON REGULARLY CONVERGING SERIES ON SYSTEMS OF FUNCTIONS IN A DISK

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Let $f(z)=\sum\limits_{k=0}^{\infty} f_k z^k$ be an entire transcendental function, (λ_n) be a sequence of positive numbers increasing to $+\infty$ and the series $A(z)=\sum\limits_{n=1}^{\infty} a_n f(\lambda_n z)$ be regularly convergent in $\mathbb{D}=\{z: |z|<1\}$, i.e., $\sum\limits_{n=1}^{\infty} |a_n| M_f(r\lambda_n)<+\infty$ for all $r\in[0,1)$, where $M_f(r)=\max\{|f(z)|: |z|=r\}$. Suppose that α and β are slowly increasing such that $x/\beta^{-1}(c\alpha(x))\uparrow+\infty$, $\alpha(x/\beta^{-1}(c\alpha(x)))=(1+o(1))\alpha(x)$ and $\alpha(\ln x)=o(\beta(x))$ as $x_0(c)\le x\to+\infty$ for every $c\in(0,+\infty)$. It is proved, for example, that if $a_n>0$ for all n and $\alpha(\ln n)=o(\beta(\Gamma_f(c\lambda_n)/\ln n))$ as $n\to\infty$, where $\Gamma_f(r)=\dfrac{d\ln M_f(r)}{d\ln r}$, then

$$\varlimsup_{r\uparrow 1}\frac{\alpha(\ln M_A(r))}{\beta(1/(1-r))}=\varlimsup_{k\to\infty}\frac{\alpha(\ln^+(|f_k|\mu_D(k))}{\beta(k)},$$

where $\mu_D(\sigma):=\max\big\{|a_n|\exp\{\sigma\ln\lambda_n\}: n\geqslant 0\big\}$.

Key words: series in systems of functions, regularly convergent series, generalized order.

1. Introduction

Let

$$f(z)=\sum_{k=0}^{\infty} f_k z^k \tag{1}$$

2020 Mathematics Subject Classification: 30B50, 30D10 be an entire transcendental function,

$$M_f(r)=\max\{|f(z)|\colon |z|=r\}$$

and (λ_n) be a sequence of positive numbers increasing to $+\infty$. At first we suppose that the series

$$A(z)=\sum_{n=1}^{\infty} a_n f(\lambda_n z) \tag{2}$$

in the system $\{f(\lambda_n z)\}$ regularly converges in $\mathbb{C}$, i.e. for all $r\in[0,+\infty)$

$$\sum_{n=1}^{\infty}|a_n|M_f(r\lambda_n)<+\infty. \tag{3}$$

Many authors have studied the representation of analytic functions by series in the system $f(\lambda_n z)$. We will specify here only on the monographs of A. F. Leont'ev [1] and B. V. Vinnitskyi [2], where references are to other works.

Since series (2) regularly convergent in $\mathbb{C}$, the function A is entire. To study its growth, generalized orders are used. For this purpose, as in [3] by L we denote a class of continuous non-negative on $(-\infty,+\infty)$ functions α such that $\alpha(x)=\alpha(x_0)\ge 0$ for $x\le x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\le x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_{si}$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each $c\in(0,+\infty)$, i.e., α is a slowly increasing function. Clearly, $L_{si}\subset L^0$. For $\alpha\in L$ and $\beta\in L$ the quantity

$$\varrho_{\alpha,\beta}[f]=\varlimsup_{r\to+\infty}\frac{\alpha(\ln M_f(r))}{\beta(\ln r)}$$

is called [3] the generalized (α,β)-order of the entire function f.

In the papers [4–5] the relationship between the growth of functions $M_f(r)$, $M_A(r)$ and $M_f^{-1}(M_A(r))$ was studied. The logarithmic convexity of the function $\ln M_f(r)$ implies

$$\Gamma_f(r):=\frac{d\ln M_f(r)}{d\ln r}\nearrow+\infty,\quad r\to+\infty,$$

(in points where the derivative does not exist, $\dfrac{d\ln M_f(r)}{d\ln r}$ means the right-hand derivative). For example, in [5] the following theorem was proved.

Theorem A. *Let $\alpha\in Lsi$, $\beta\in L^0$ and $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ for each $c\in(0,+\infty)$. If $a_n\ge 0$ for all $n\ge 1$, the series (2) is regularly convergent in $\mathbb{C}$, $\ln n=O(\Gamma_f(\lambda_n))$ and $\ln\lambda_n=o\left(\beta^{-1}\left(c\alpha\left(\dfrac{1}{\ln\lambda_n}\ln\dfrac{1}{a_n}\right)\right)\right)$ as $n\to\infty$ each $c\in(0,+\infty)$, then $\varrho_{\alpha,\beta}[A]=\varrho_{\alpha,\beta}[f]$.*

Here we consider the case when the series (2) regularly converges in a finite disk.

2. Radius of regular convergence

Let $R[A]$ be the radius of regular convergence of the series (2), that is (3) holds for $r < R[A]$ and does not hold for $r > R[A]$. The following statement is true.

Proposition 1. *Let*

$$\alpha_0 := \varliminf_{n\to\infty} \frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right).$$

If $\Gamma_f(cr) \asymp \Gamma_f(r)$ *as* $r \to +\infty$ *for each* $c \in (0, +\infty)$ *and* $\ln n = o(\Gamma_f(\lambda_n))$ *as* $n \to \infty$, *then* $R[A] = \alpha_0$.

Proof. Suppose that $R[A] > 0$. For every $r < R[A]$ from (3) we have $|a_n|M_f(r\lambda_n) \to 0$ as $n \to \infty$, i. e. $|a_n|M_f(r\lambda_n) \le 1$ for all $n \ge n_0(r)$, whence $\frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right) \ge r$ for all $n \ge n_0(r)$ and, therefore,

$$\alpha_0 := \varliminf_{n\to\infty} \frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right) \ge r.$$

In view of the arbitrariness of $r < R[A]$ we get $R[A] \le \alpha_0$. If $R[A] = 0$, then this inequality is trivial.

Now suppose on the contrary that the equality $R[A] = \alpha_0$ not holds. Then $R[A] < \alpha_0$. We choose $R[A] < \alpha_1 < \alpha_2 < \alpha_0$. Then $\frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right) > \alpha_2$ for all $n \ge n_0(\alpha_2)$, i.e., $|a_n| < \frac{1}{M_f(\alpha_2\lambda_n)}$ for $n \ge n_0(\alpha_2)$. Therefore,

$$\begin{aligned}
|a_n|M_f(\alpha_1\lambda_n) &< \frac{M_f(\alpha_1\lambda_n)}{M_f(\alpha_2\lambda_n)} = \\
&= \exp\left\{-\int\limits_{\alpha_1\lambda_n}^{\alpha_2\lambda_n} \frac{d\ln\, M_f(r)}{d\ln\, r} d\ln\, r\right\} = \\
&= \exp\left\{-\int\limits_{\alpha_1\lambda_n}^{\alpha_2\lambda_n} \Gamma_f(r) d\ln\, r\right\} \le \\
&\le \exp\left\{-\Gamma_f(\alpha_1\lambda_n)\ln\frac{\alpha_2}{\alpha_1}\right\} = \\
&= \exp\left\{-\frac{\Gamma_f(\alpha_1\lambda_n)}{\Gamma_f(\lambda_n)}\ln\frac{\alpha_2}{\alpha_1}\Gamma_f(\lambda_n)\right\}.
\end{aligned}$$

Since $\Gamma_f(cr) \asymp \Gamma_f(r)$ as $r \to +\infty$ for each $c \in (0, +\infty)$ and $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$, we have

$$\frac{\Gamma_f(\alpha_1\lambda_n)}{\Gamma_f(\lambda_n)} \ge K(\alpha_1) = \text{const} > 0$$

and $\Gamma_f(\lambda_n) > \dfrac{\ln n}{\varepsilon}$ for each $\varepsilon > 0$ and all $n \geq n_1(\varepsilon)$, and thus,

$$\sum_{n=n_1}^{\infty} |a_n| M_f(\alpha_1\lambda_n) \leq \sum_{n=n_1}^{\infty} \exp\left\{-\frac{K(\alpha_1)}{\varepsilon} \ln \frac{\alpha_2}{\alpha_1} \ln n\right\} < +\infty.$$

Hence we get $R[A] \geq \alpha_1$ what is impossible. Proposition 1 is proved. □

3. Analytic in the unit disk functions of the finite order

For an analytic in $\mathbb{D} = \{z : |z| < 1\}$ function

$$\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^k \tag{4}$$

the order is defined as

$$\varrho[\varphi] := \varlimsup_{r\uparrow 1} \frac{\ln^+ \ln^+ M_\varphi(r)}{\ln(1/(1-r))}.$$

It is known [6–7] that

$$\frac{\varrho[\varphi]}{\varrho[\varphi]+1} = \gamma[\varphi] := \varlimsup_{k\to\infty} \frac{\ln^+ \ln^+ |\varphi_k|}{\ln k}. \tag{5}$$

Using formula (5) we prove the following theorem.

Theorem 1. *Let* $R[A] = 1$ *and* $a_n > 0$ *for all* n. *If* $\Gamma_f(c\lambda_n) \geq \omega(n)$ *for all* n *and some* $0 < c < 1$, *where* ω *is a positive function on* $[1, +\infty)$ *increasing to* $+\infty$ *such that* $\ln \ln x = o(\ln \omega(x))$ *as* $x \to +\infty$ *then*

$$\varrho[A] = \frac{\gamma}{1-\gamma}, \quad \gamma = \varlimsup_{k\to\infty} \frac{\ln^+ \ln^+ (|f_k|\mu_D(k))}{\ln k}, \tag{6}$$

where $\mu_D(\sigma) = \max\{|a_n| \exp\{\sigma \ln \lambda_n\} : n \geq 1\}$ *is the maximal term of entire Dirichlet series*

$$D(\sigma) = \sum_{n=1}^{\infty} |a_n| \exp\{\sigma \ln \lambda_n\}. \tag{7}$$

Proof. Since $a_n \geq 0$ for all $n \geq 1$ and

$$A(z) = \sum_{n=1}^{\infty} a_n \sum_{k=0}^{\infty} f_k (z\lambda_n)^k = \sum_{k=0}^{\infty} f_k \left(\sum_{n=1}^{\infty} a_n \lambda_n^k\right) z^k,$$

in view of Cauchy inequality we have

$$M_A(r) \geq |f_k| \left(\sum_{n=1}^{\infty} a_n \lambda_n^k\right) r^k \geq a_n |f_k| (\lambda_n r)^k \tag{8}$$

for all $n \geq 1$, $k \geq 0$ and $r \in [0, 1)$. Hence it follows that $M_A(r) \geq |f_k| \mu_D(k) r^k$, where $\mu_D(\sigma) = \max\{|a_n| \exp\{\sigma \ln \lambda_n\}$ is the maximal term of entire Dirichlet series

$$D(\sigma) = \sum_{n=1}^{\infty} |a_n| \exp\{\sigma \ln \lambda_n\}.$$

Therefore, $M_A(r) \geq \mu_G(r)$ for $r \in [0, 1)$, where

$$\mu_G(r) = \max\{|f_k|\mu_D(k)r^k : k \geq 0\}$$

is the maximal term of the series

$$G(r) = \sum_{k=0}^{\infty} |f_k|\mu_D(k)r^k, \quad 0 < r < 1. \tag{9}$$

On the other hand, since the series (2) is regularly convergent in $\mathbb{D}$, for every $r \in [0, 1)$ we have

$$M_A(r) \leq \sum_{n=1}^{\infty} |a_n| M_f(r\lambda_n) \leq \mu_A\left(\frac{1+r}{2}\right) \sum_{n=1}^{\infty} \frac{M_f(r\lambda_n)}{M_f\left(\frac{r+1}{2}\lambda_n\right)}, \tag{10}$$

where $\mu_A(r) = \max\{|a_n| M_f(r\lambda_n) : n \geq 1\}$. For $r \in [c, 1)$ we have

$$\begin{aligned} \ln M_f\left(\frac{r+1}{2}\lambda_n\right) - \ln M_f(r\lambda_n) &= \int\limits_{r\lambda_n}^{(r+1)\lambda_n/2} \Gamma_f(x) d \ln x \geq \\ &\geq \Gamma_f(r\lambda_n) \ln \frac{1+r}{2r} = \\ &= \Gamma_f(r\lambda_n) \ln \left(1 + \frac{1-r}{2r}\right) \geq \\ &\geq \frac{(1-r)\Gamma_f(c\lambda_n)}{4}. \end{aligned}$$

Since $\Gamma_f(c\lambda_n) \geq \omega(n)$ for all n and ω is a positive function on $[1, +\infty)$ increasing to $+\infty$ such that $\ln \ln x = o(\ln \omega(x))$ as $x \to +\infty$, we get $\omega(n) \geq (\ln n)^{1/\varepsilon}$ for every $\varepsilon \in (0, 1)$ and all $n \geq n^*(\varepsilon)$ and, therefore, $\Gamma_f(c\lambda_n) \geq (\ln n)^{1/\varepsilon - 1} \ln n$.

We put

$$n_0(r) = \left[\exp\left\{\left(\frac{8}{1-r}\right)^{\varepsilon/(1-\varepsilon)}\right\}\right] + 1.$$

Then $n_0(r) \geq n^*(\varepsilon)$ for $r \in [r_0(\varepsilon), 1)$ and for $n \geq n_0(r)$ we get

$$\begin{aligned} \frac{(1-r)\Gamma_f(c\lambda_n)}{4} &\geq \frac{(1-r)(\ln n)^{1/\varepsilon-1} \ln n}{4} \geq \\ &\geq \frac{(1-r)(\ln n_0(r))^{1/\varepsilon-1} \ln n}{4} = \\ &= \frac{8}{1-r} \frac{(1-r) \ln n}{4} = \\ &= 2 \ln n. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{M_f(r\lambda_n)}{M_f\left(\frac{r+1}{2}\lambda_n\right)} \le \left(\sum_{n=1}^{n_0(r)-1} + \sum_{n_0(r)}^{\infty}\right) \exp\left\{-\frac{(1-r)\Gamma_f(r\lambda_n)}{4}\right\} \le$$
$$\le \sum_{n=1}^{n_0(r)-1} 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} \le$$
$$\le n_0(r) + 2 \le$$
$$\le \exp\left\{\left(\frac{8}{1-r}\right)^{\varepsilon/(1-\varepsilon)}\right\} + 3$$

Thus, (10) implies

$$M_A(r) \le \mu_A\left(\frac{1+r}{2}\right)\left(\exp\left\{\left(\frac{8}{1-r}\right)^{\varepsilon/(1-\varepsilon)}\right\} + 3\right), \quad (r \in [r_0(\varepsilon), 1)). \tag{11}$$

Also we have

$$\mu_A(r) \le \max\left\{|a_n| \sum_{k=0}^{\infty} |f_k|(r\lambda_n)^k : n \ge 1\right\} \le$$
$$\le \sum_{k=0}^{\infty} \max\{|a_n|\lambda_n^k : n \ge 1\}|f_k|r^k =$$
$$= G(r) =$$
$$= \sum_{k=0}^{\infty} \mu_D(k)|f_k| \left(\frac{1+r}{2}\right)^k \left(\frac{2r}{1+r}\right)^k \le$$
$$\le \mu_G\left(\frac{1+r}{2}\right)\frac{1+r}{1-r} \le$$
$$\le \frac{2}{1-r}\mu_G\left(\frac{1+r}{2}\right).$$

Therefore, in view of (11)

$$\ln\, \mu_G(r) \le \ln\, M_A(r) \le \ln\, \mu_G\left(\frac{3+r}{4}\right) + \ln\, \frac{4}{1-r} + \left(\frac{8}{1-r}\right)^{\varepsilon/(1-\varepsilon)} + \ln\, 6. \tag{12}$$

Since (12) implies

$$\ln^+\, \ln^+\, \mu_G(r) \le \ln^+\, \ln^+\, M_A(r) \le$$
$$\le \ln^+\, \ln^+\, \mu_G\left(\frac{3+r}{4}\right) + \ln^+\, \ln^+\, \frac{4}{1-r} + + \frac{\varepsilon}{1-\varepsilon}\ln^+\frac{8}{1-r} + \ln\, 24,$$

we get

$$\varrho[G] \le \varrho[A] \le \varrho[G] + \frac{\varepsilon}{1-\varepsilon},$$

i.e., in view of the arbitrariness of ε we obtain $\varrho[A] = \varrho[G]$. By formula (5) we have $\varrho[G] = \dfrac{\gamma[G]}{1-\gamma[G]}$ where

$$\gamma[G] = \varlimsup_{k\to\infty} \frac{\ln^+ \ln^+ (|f_k|\mu_D(k))}{\ln k},$$

i.e., (6) holds with $\gamma = \gamma[G]$. Theorem 1 is proved. □

Let

$$\varrho_l[f] = \varlimsup_{r\to+\infty} \frac{\ln \ln M_f(r)}{\ln \ln r} > 1$$

be the logarithmic order of the entire function (1) and

$$p_R[D] = \varlimsup_{\sigma\to+\infty} \frac{\ln \ln D(\sigma)}{\ln \sigma}$$

be the logarithmic R-order of the entire Dirichlet series (7). Then [8]

$$\varrho_l[f] = \varlimsup_{k\to\infty} \frac{\ln k}{\ln\left(\dfrac{1}{k}\ln\dfrac{1}{|f_k|}\right)} + 1$$

and [9] $p_R[D] = q_R[D] + 1$, where

$$q_R[D] = \varlimsup_{n\to\infty} \frac{\ln \ln \lambda_n}{\ln\left(\dfrac{1}{\ln \lambda_n}\ln\dfrac{1}{|a_n|}\right)},$$

provided $\varlimsup\limits_{n\to\infty} \dfrac{\ln \ln n}{\ln \lambda_n} < 1$.

If $\varrho_l[f] < +\infty$ and $p_R[D] < +\infty$, then we obtain $D(\sigma) \le \exp\{\sigma^{p_R[D]+\varepsilon}\}$ for every $\varepsilon > 0$ and all $\sigma \ge \sigma_0(\varepsilon)$, and

$$|f_k| \le \exp\{-k^{(\varrho_l[f]+\varepsilon)/(\varrho_l[f]+\varepsilon-1)}\}$$

for every $\varepsilon \in (0,\, \varrho_l[f]-1)$ and all $k \ge k_0(\varepsilon)$. Therefore, if

$$p_R[D] + \varepsilon \le (\varrho_l[f]+\varepsilon)/(\varrho_l[f]+\varepsilon-1)$$

then

$$|f_k|\mu_D(k)) \le \exp\{k^{p_R[D]+\varepsilon} - k^{(\varrho_l[f]+\varepsilon)/(\varrho_l[f]+\varepsilon-1)}\} \le 1$$

and $\gamma[G] = 0$. We remark that if $p_R[D] < \varrho_l[f]/(\varrho_l[f]-1)$ then ε can be chosen so that

$$p_R[D] + \varepsilon \le (\varrho_l[f]+\varepsilon)/(\varrho_l[f]+\varepsilon-1).$$

Therefore, Theorem 1 implies the following corollary.

Corollary 1. *Let the conditions of Theorem 1 hold. If* $p_R[D] < \varrho_l[f]/(\varrho_l[f]-1)$ *(i.e.* $q_R[D] < 1/(\varrho_l[f]-1)$*) then* $\varrho[A] = 0$.

4. Generalized orders

For $\alpha \in L$ and $\beta \in L$, the quantity

$$\varrho_{\alpha,\beta}[\varphi] = \overline{\lim_{r\uparrow 1}} \frac{\alpha(\ln M_\varphi(r))}{\beta(1/(1-r))}$$

is called [10] the *generalized* (α, β)*-order* of an analytic in $\mathbb{D}$ function (4). From the results proved in [11], it is easy to obtain the following statement.

Lemma 1. *Let* $\alpha \in L_{si}$, $\beta \in L_{si}$ *and* $\alpha(\ln x) = o(\beta(x))$ *as* $x \to +\infty$. *Then:*

1) *if* $\dfrac{x}{\beta^{-1}(c\alpha(x))} \uparrow +\infty$ *and* $\alpha\left(\dfrac{x}{\beta^{-1}(c\alpha(x))}\right) = (1+o(1))\alpha(x)$ *as* $x_0(c) \le x \to +\infty$ *for every* $c \in (0, +\infty)$, *then* $\varrho_{\alpha,\beta}[\varphi] = \overline{\lim\limits_{k\to\infty}} \dfrac{\alpha(k)}{\beta\left(k/\ln^+|\varphi_k|\right)}$.
2) *if* $\dfrac{x}{\alpha^{-1}(c\beta(x))} \uparrow +\infty$ *and* $\beta\left(\dfrac{x}{\alpha^{-1}(c\beta(x))}\right) = (1+o(1))\beta(x)$ *as* $x_0(c) \le x \to +\infty$ *for every* $c \in (0, +\infty)$, *then* $\varrho_{\alpha,\beta}[\varphi] = \overline{\lim\limits_{k\to\infty}} \dfrac{\alpha(\ln^+|\varphi_k|)}{\beta(k)}$.

Using Lemma 1 we prove the following theorem.

Теорема 1. *Let* $R[A] = 1$ *and* $a_n > 0$ *for all* n, $\alpha \in L_{si}$, $\beta \in L_{si}$, $\alpha(\ln x) = o(\beta(x))$ *as* $x \to +\infty$ *and*

$$\alpha(\ln n) = o\left(\beta\left(\frac{\Gamma_f(c\lambda_n)}{\ln n}\right)\right), \quad n \to \infty. \tag{13}$$

If the conditions of the assertion 1) of Lemma 1 hold then

$$\varrho_{\alpha,\beta}[A] = \overline{\lim_{k\to\infty}} \frac{\alpha(k)}{\beta(k/\ln^+(|f_k|\mu_D(k)))}, \tag{14}$$

and if the conditions of the assertion 2) of Lemma 1 hold then

$$\varrho_{\alpha,\beta}[A] = \overline{\lim_{k\to\infty}} \frac{\alpha(\ln^+(|f_k|\mu_D(k))}{\beta(k)}. \tag{15}$$

Proof. From (13) it follows that $\Gamma_f(c\lambda_n) \ge \beta^{-1}\left(\dfrac{\alpha(\ln n)}{\varepsilon}\right)\ln n$ every $\varepsilon \in (0, 1)$ and all $n \ge n^*(\varepsilon)$. We put

$$n_0(r) = \left[\exp\left\{\alpha^{-1}\left(\varepsilon\beta\left(\frac{8}{1-r}\right)\right)\right\}\right] + 1.$$

Then $n_0(r) \ge n^*(\varepsilon)$ for $r \in [r_0(\varepsilon), 1)$ and for $n \ge n_0(r)$ we get

$$\frac{(1-r)\Gamma_f(c\lambda_n)}{4} \ge \frac{(1-r)}{4}\beta^{-1}\left(\frac{\alpha(\ln n_0(r))}{\varepsilon}\right)\ln n = \frac{8}{1-r}\frac{(1-r)\ln n}{4} = 2\ln n.$$

Therefore, as above

$$\sum_{n=1}^{\infty} \frac{M_f(r\lambda_n)}{M_f\left(\dfrac{r+1}{2}\lambda_n\right)} \le n_0(r) + 2 \le \exp\left\{\alpha^{-1}\left(\varepsilon\beta\left(\frac{8}{1-r}\right)\right)\right\} + 3$$

and (10) implies

$$M_A(r) \le \mu_A\left(\frac{1+r}{2}\right)\left(\exp\left\{\alpha^{-1}\left(\varepsilon\beta\left(\frac{8}{1-r}\right)\right)\right\}+3\right), \quad (r\in[r_0(\varepsilon),\,1)),$$

i.e., in view of (12) we get

$$\ln\,\mu_G(r) \le \ln\,M_A(r) \le \ln\,\mu_G\left(\frac{3+r}{4}\right)+\ln\,\frac{4}{1-r}+\alpha^{-1}\left(\varepsilon\beta\left(\frac{8}{1-r}\right)\right)+\ln\,6.$$

Since $\alpha\in L_{si}$, from hence we have

$$\alpha(\ln\,\mu_G(r)) \le \alpha(\ln\,M_A(r)) \le$$
$$\le \alpha\left(4\max\left\{\ln\,\mu_G\left(\frac{3+r}{4}\right),\ln\,\frac{4}{1-r},\alpha^{-1}\left(\varepsilon\beta\left(\frac{8}{1-r}\right)\right),\ln\,6\right\}\right)=$$
$$=(1+o(1))\alpha\left(\max\left\{\ln\,\mu_G\left(\frac{3+r}{4}\right),\ln\,\frac{4}{1-r},\alpha^{-1}\left(\varepsilon\beta\left(\frac{8}{1-r}\right)\right),\ln\,6\right\}\right)=$$
$$=(1+o(1))\max\left\{\alpha\left(\ln\,\mu_G\left(\frac{3+r}{4}\right)\right),\alpha\left(\ln\,\frac{4}{1-r}\right),\varepsilon\beta\left(\frac{8}{1-r}\right),\alpha(\ln\,6)\right\}\le$$
$$\le(1+o(1))\left(\alpha\left(\ln\,\mu_G\left(\frac{3+r}{4}\right)\right)+\alpha\left(\ln\,\frac{4}{1-r}\right)+\varepsilon\beta\left(\frac{8}{1-r}\right)+\alpha(\ln\,6)\right),\quad r\uparrow 1.$$

Therefore, if $\alpha\in L_{si}$ and $\alpha(\ln\,x)=o(\beta(x))$ as $x\to+\infty$, we obtain

$$\overline{\lim_{r\uparrow 1}}\,\frac{\alpha(\ln\,\mu_G(r))}{\beta(1/(1-r))}\le\overline{\lim_{r\uparrow 1}}\,\frac{\alpha(\ln\,M_A(r))}{\beta(1/(1-r))}\le\overline{\lim_{r\uparrow 1}}\,\frac{\alpha(\ln\,\mu_G(r))}{\beta(1/(1-r))}+\varepsilon,$$

whence in view of the arbitrariness of ε we get $\varrho_{\alpha,\beta}[A]=\varrho_{\alpha,\beta}[G]$. Since by Lemma 1

$$\varrho_{\alpha,\beta}[G]=\overline{\lim_{k\to\infty}}\,\frac{\alpha(k)}{\beta(k/\ln^+(|f_k|\mu_D(k)))}$$

provided the conditions of the assertion 1) of Lemma 1 hold and

$$\varrho_{\alpha,\beta}[A]=\overline{\lim_{k\to\infty}}\,\frac{\alpha(\ln^+(|f_k|\mu_D(k))}{\beta(k)}$$

provided the conditions of the assertion 2) of Lemma 1 hold, Theorem 2 is proved. □

For entire function (1) the generalized (α,β)-order is defined [3] by

$$\varrho^*_{\alpha,\beta}[f]=\overline{\lim_{r\to+\infty}}\,\frac{\alpha(\ln\,M_f(r))}{\beta(\ln\,r)}.$$

It is known [3] that if $\alpha\in L_{si}$, $\beta\in L^0$ and $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln\,x}=O(1)$ as $x\to+\infty$ for every $c\in(0,\,+\infty)$ then

$$\varrho^*_{\alpha,\beta}[f]=\overline{\lim_{k\to\infty}}\,\frac{\alpha(k)}{\beta\left(\dfrac{1}{k}\ln\,\dfrac{1}{|f_k|}\right)},$$

i.e., $|f_k| \le \exp\{-k\beta^{-1}(\alpha(k)/\varrho_1)\}$ for every $\varrho_1 > \varrho^*_{\alpha,\beta}[f]$ and all $k \ge k_0(\varrho)$. On the other hand, for entire Dirichlet series (7) the modified generalized (β,α)-order is defined [12] as

$$\varrho^M_{\beta,\alpha}[D] = \varlimsup_{\sigma\to+\infty} \frac{1}{\alpha(\sigma)}\beta\left(\frac{\ln D(\sigma)}{\sigma}\right).$$

If $\varrho^M_{\beta,\alpha}[D] < +\infty$ then

$$\ln \mu_D(\sigma) \le \ln D(\sigma) \le \sigma\beta^{-1}(\varrho_2\alpha(\sigma))$$

for every $\varrho_2 > \varrho^M_{\beta,\alpha}[D]$ and all $\sigma \ge \sigma_0(\varrho_2)$.

Therefore, if $\varrho_2 \le 1/\varrho_1$ then

$$|f_k|\mu_D(k) \le \exp\{-k\beta^{-1}(\alpha(k)/\varrho_1) + k\beta^{-1}(\varrho_2\alpha(k))\} \le 1$$

for all sufficiently large k, i. e. $\ln^+(|f_k|\mu_D(k)) = 0$ for all sufficiently large k. Hence in view of formulas (14) and (15) it follows that $\varrho_{\alpha,\beta}[A] = 0$.

We remark that if $\varrho^*_{\alpha,\beta}[f]\varrho^M_{\beta,\alpha}[D] < 1$ then we can choose $\varrho_1 > \varrho^*_{\alpha,\beta}[f]$ and $\varrho_2 > \varrho^M_{\beta,\alpha}[D]$ so that $\varrho_1\varrho_2 \le 1$. Therefore, Theorem 2 implies the following corollary.

Corollary 2. *Let the conditions of Theorem 2 hold and* $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ *as* $x \to +\infty$ *for every* $c \in (0, +\infty)$. *If* $\varrho^*_{\alpha,\beta}[f]\varrho^M_{\beta,\alpha}[D] < 1$ *then* $\varrho[A] = 0$.

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ПРО РЕГУЛЯРНО ЗБІЖНІ В КРУЗІ РЯДИ ЗА СИСТЕМОЮ ФУНКЦІЙ

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Нехай $f(z) = \sum_{k=0}^{\infty} f_k z^k$ – ціла трансцендентна функція, (λ_n) – зростаюча до $+\infty$ послідовність додатних чисел, а ряд $A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)$ регулярно збіжний в $\mathbb{D} = \{z : |z| < 1\}$, тобто $\sum_{n=1}^{\infty} |a_n| M_f(r\lambda_n) < +\infty$ для всіх $r \in [0, 1)$, де $M_f(r) = \max\{|f(z)| : |z| = r\}$. Припустимо, що α і β – такі повільно зростаючі функції, що $x/\beta^{-1}(c\alpha(x)) \uparrow +\infty$, $\alpha(x/\beta^{-1}(c\alpha(x))) = (1 + o(1))\alpha(x)$ і $\alpha(\ln x) = o(\beta(x))$ при $x_0(c) \le x \to +\infty$ для кожного $c \in (0, +\infty)$. Доведено, наприклад, що якщо $a_n > 0$ для всіх n і $\alpha(\ln n) = o(\beta(\Gamma_f(c\lambda_n)/\ln n))$ при $n \to \infty$, де $\Gamma_f(r) = \dfrac{d \ln M_f(r)}{d \ln r}$, то

$$\varlimsup_{r\uparrow 1} \frac{\alpha(\ln M_A(r))}{\beta(1/(1-r))} = \varlimsup_{k\to\infty} \frac{\alpha(\ln^+(|f_k|\mu_D(k))}{\beta(k)},$$

де $\mu_D(\sigma) := \max\big\{|a_n| \exp\{\sigma \ln \lambda_n\} \colon n \geqslant 0\big\}$.

Ключові слова: ряд за системою функцій, регулярно збіжний ряд, узагальнений порядок.