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REMARKS TO LOWER ESTIMATES FOR CHARACTERISTIC FUNCTIONS OF PROBABILITY LAWS

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For the analytic in $\mathbb{D}_R = \{z : |z| < R\}$ characteristic function φ of a probability law F it is investigated conditions on $W_F(x) = 1 - F(x) + F(-x)$ $(x \geq 0)$ and a positive continuous function h increasing to $+\infty$, under which $h(\ln M(r, \varphi)) \geq (1 + o(1))/(R - r)$ or $\ln M(r, \varphi)) \geq (1 + o(1))h(1/(R - r))$ as $r \uparrow R$, where $M(r, \varphi) = \max\{|\varphi(z)| : |z| = r < R\}$.

Key words: characteristic function, probability law, lower estimate.

1. Introduction

A non-decreasing function F continuous on the left on $(-\infty, +\infty)$ is said [1, p. 10] to be a probability law if $\lim\limits_{x\to+\infty} F(x) = 1$ and $\lim\limits_{x\to-\infty} F(x) = 0$, and the function

$$\varphi(z) = \int\limits_{-\infty}^{+\infty} e^{izx} dF(x)$$

defined for real z is called [1, p. 12] a characteristic function of this law. If φ has an analytic continuation on the disk $\mathbb{D}_R = \{z : |z| < R\}$, $0 < R \leq +\infty$, then we call φ an analytic in $\mathbb{D}_R$ characteristic function of the law F. Further we always assume that $\mathbb{D}_R$ is the maximal disk of the analyticity of φ. It is known [1, p. 37-38] that φ is an analytic in $\mathbb{D}_R$ characteristic function of the law F if and only if

$$W_F(x) =: 1 - F(x) + F(-x) = O(e^{-rx})$$

2020 Mathematics Subject Classification: 30A10, 30J99, 60E10 as $0 \le x \to +\infty$ for every $r \in [0, R)$. Hence it follows that

$$\varliminf_{x\to+\infty} \frac{1}{x} \ln \frac{1}{W_F(x)} = R.$$

If we put

$$M(r, \varphi) = \max\{|\varphi(z)| : |z| = r\} \quad \text{and} \quad \mu(r, \varphi) = \sup\{W_F(x)e^{rx} : x \ge 0\}$$

for $0 \le r < R$ then [1, p. 55] $\mu(r, \varphi) \le 2M(r, \varphi)$. Therefore, the estimates from below for $\ln M(r, \varphi)$ follow from such estimates for $\ln \mu(r, \varphi)$. Further we assume that $\ln \mu(r, \varphi) \uparrow +\infty$ as $r \uparrow R$, i. e.

$$\varlimsup_{x\to+\infty} W_F(x)e^{Rx} = +\infty. \tag{1}$$

By L_{si} we denote a class of positive continuous functions α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0)$ for $x \le x_0$, $0 < \alpha(x) \uparrow +\infty$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \le x \uparrow +\infty$ for each $c \in (0, +\infty)$, i. e. α is a slowly increasing function. In [2] the following statements are proved.

Theorem A. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$, $\dfrac{d\ln \beta^{-1}(\alpha(x))}{d\ln x} \le q < 1$ for all x large enough and $\alpha\left(x/\beta^{-1}(\alpha(x))\right) = (1 + o(1))\alpha(x)$ as $x \to +\infty$, and φ be the analytic in $\mathbb{D}_R$, $0 < R < +\infty$, characteristic function of a probability law F, for which $\beta\left(\dfrac{x_k}{\ln\left(W_F(x_k)e^{Rx_k}\right)}\right) \le \alpha(x_k)$ for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\beta^{-1}(\alpha(x_{k+1})) = O(\beta^{-1}(\alpha(x_k)))$ as $k \to \infty$. Then*

$$\alpha(\ln \mu(r, f)) \ge (1 + o(1))\beta(1/(R - r)), \quad r \uparrow R. \tag{2}$$

Theorem B. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$, $\dfrac{d\ln \alpha^{-1}(\beta(x))}{d\ln x} \le q < 1$ for all x large enough, $\dfrac{d\alpha^{-1}(\beta(x))}{dx} = \frac{1}{f(x)} \downarrow 0$ and $\alpha^{-1}(\beta(f(x))) = O(\alpha^{-1}(\beta(x)))$ as $x \to +\infty$, and φ be an analytic in $\mathbb{D}_R$, $0 < R < +\infty$, characteristic function of a probability law F, for which $\alpha\left(\ln\left(W_F(x_k)e^{Rx_k}\right)\right) \ge \beta(x_k)$ for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\varlimsup_{k\to\infty}(f(x_{k+1})/f(x_k)) < 2$. Then asymptotical inequality (2) holds.*

It is clear that if one of the functions α or β is a power function then the conditions of neither Theorem A nor Theorem B are satisfied. These conditions do not hold also if $\alpha(x) \asymp \beta(x)$ as $x \to +\infty$. Here we examine the cases when one of functions α or β is power. Without loss of generality we can assume that $\alpha(x) \equiv x$ or $\beta(x) \equiv x$ for $x \ge x_0$. Also we examine the case when $\beta(x) = \varrho\alpha(x)$ for all $x \ge x_0$, where $0 < \varrho < +\infty$.

2. Cases of power functions

We use a result from [2]. Let $\Omega(R)$ be a class of positive unbounded on $(0, R)$ function Φ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(0, R)$. For $\Phi \in \Omega(R)$ we denote by ϕ the inverse function to Φ', and let $\Psi(r) = r - \Phi(r)/\Phi'(r)$ be the function associated with Φ in the sense of Newton.

Lemma [2]). *Let $\Phi \in \Omega(R)$, $0 < R < +\infty$, and φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F, for which* (1) *holds and*

$$\ln W_F(x_k) \geq -x_k\Psi(\phi(x_k)) \tag{3}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\phi(x_{k+1})-\phi(x_k) \leq h(x_{k+1})$, where h is positive continuous and non-increasing function on $[x_0, +\infty)$ and $R > \phi(x) - h(x) \to R$ as $x \to +\infty$. Then

$$\ln \mu(r,\varphi) \geq \Phi(r - h(\Phi'(r))), \quad r_0 \leq r < R. \tag{4}$$

At first we consider the case when $\beta(x) \equiv x$ for $x \geq x_0$.

Theorem 1. *Let $\alpha \in L_{si}$, $\alpha(x/\alpha(x)) = (1+o(1))\alpha(x)$ as $x \to +\infty$ and φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F. If*

$$\ln \left(W_F(x_k)e^{Rx_k}\right) \geq x_k/\alpha(x_k) \tag{5}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers such that

$$\alpha(x_{k+1}) = (1+o(1))\alpha(x_k) \qquad as \qquad k \to \infty$$

then

$$\alpha(\ln M(r,\varphi)) \geq (1+o(1))/(R-r), \quad r \uparrow R. \tag{6}$$

Proof. We can assume that the function α is continuously differentiable. Then $\alpha \in L_{si}$ if and only if $x\alpha'(x)/\alpha(x) \to 0$ as $x \to +\infty$. Therefore, using L'Hospital's rule we obtain

$$\varlimsup_{x\to+\infty} \frac{1}{x/\alpha(x)} \int_{x_0}^{x_k} \frac{dt}{\alpha(t)} \leq \varlimsup_{x\to+\infty} \frac{\alpha(x)}{\alpha(x) - x\alpha'(x)} = 1,$$

i.e., $x/\alpha(x) \geq (1+o(1)) \int_{x_0}^{x_k} dt/\alpha(t)$ as $x \to +\infty$ and, therefore, in view of (5) the next result is obtained

$$\ln W_F(x_k) \geq -Rx_k + (1+o(1)) \int_{x_0}^{x_k} \frac{dt}{\alpha(t)}, \quad k \to \infty. \tag{7}$$

We choose a function $\Phi \in \Omega(R)$ so that

$$\Phi(r) = \int_{r_0}^{r} \alpha^{-1}((1-\varepsilon)/(R-x))dx$$

for $r_0 < r_0^* \leq r < R$, where $\varepsilon \in (0, 1)$ is an arbitrary number. Then

$$\Phi'(r) = \alpha^{-1}((1-\varepsilon)/(R-x)), \qquad \phi(x) = R - (1-\varepsilon)/\alpha(x)$$

for $x \geq x^*$ and, since $(x\Psi(\phi(x)))' = \phi(x)$, we have

$$x\Psi(\phi(x)) = Rx - (1-\varepsilon) \int_{x_0}^{x} dt/\alpha(t) + \text{const}.$$

Hence and from (7) the inequality (3) is being followed.

Putting $h(x) = \delta(1-\varepsilon)/\alpha(x)$, where $\delta \in (0, 1)$ is an arbitrary number, we obtain $h(\Phi'(r)) = \delta(R-r)$, $\phi(x) - h(x) \uparrow R$ as $x \to +\infty$ and since $\alpha(x_{k+1}) \le (1+\delta)\alpha(x_k)$ for all $k \ge k_0$ we have

$$\begin{aligned}\phi(x_{k+1}) - \phi(x_k) &= \frac{1-\varepsilon}{\alpha(x_k)} - \frac{1-\varepsilon}{\alpha(x_{k+1})} \le \\ &\le (1-\varepsilon)\left(\frac{1+\delta}{\alpha(x_{k+1})} - \frac{1}{\alpha(x_{k+1})}\right) = \\ &= h(x_{k+1}).\end{aligned}$$

Finally, for every $\eta \in (\delta, 1)$ and all $r \ge (r_0^* + \eta R)/(1+\eta)$ we have

$$\begin{aligned}\Phi(r - h(\Phi'(r))) &= \Phi(r - \delta(R-r)) \ge \\ &\ge \int\limits_{r-\eta(R-r)}^{r-\delta(R-r)} \alpha^{-1}\left(\frac{1-\varepsilon}{R-x}\right)dx \ge \\ &\ge (\eta-\delta)(R-r)\alpha^{-1}\left(\frac{1-\varepsilon}{(1+\eta)(R-r)}\right).\end{aligned}$$

Therefore, by Lemma in view of the conditions $\alpha \in L_{si}$ and $\alpha(x/\alpha(x)) = (1+o(1))\alpha(x)$ as $x \to +\infty$ we obtain

$$\begin{aligned}\alpha(\ln \mu(r,\varphi)) &\ge \alpha\left((\eta-\delta)(R-r)\alpha^{-1}\left(\frac{1-\varepsilon}{(1+\eta)(R-r)}\right)\right) = \\ &= \frac{(1+o(1))(1-\varepsilon)}{(1+\eta)(R-r)}, \quad r \uparrow R.\end{aligned}$$

Since $\ln M(r,\varphi) \ge \ln \mu(r,\varphi) + \ln 2$ and $\alpha \in L_{si}$, hence in view of the arbitrariness of ε and η we obtain (6). Theorem 1 is proved. □

In the case when $\alpha(x) \equiv x$ for $x \ge x_0$ the following theorem is correct.

Theorem 2. *Let $\beta \in L_{si}$, $\beta'(x) \downarrow 0$, $\beta(1/\beta'(x)) = (1+o(1))\beta(x)$ as $x \to +\infty$ and φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F. If*

$$\ln\left(W_F(x_k)e^{Rx_k}\right) \ge \beta(x_k) \tag{8}$$

for some increasing to $+\infty$ sequence (x_k) of positive numbers such that

$$\varlimsup_{k\to\infty} (\beta'(x_k)/\beta'(x_{k+1})) < 2$$

then

$$\ln M(r,\varphi) \ge (1+o(1))\beta(1/(R-r)), \quad r \uparrow R. \tag{9}$$

Proof. We put $x\Psi(\phi(x)) = Rx - \beta(x)$. Then (8) implies (3). Since

$$\phi(x) = (x\Psi(\phi(x)))' = R - \beta'(x),$$

we have $r = R - \beta'(\Phi'(r))$, i.e., $\Phi'(r) = B(R-r)$, where B is the function inverse to β' and $B(t) \uparrow +\infty$ as $t \downarrow 0$. Hence it follows that

$$\Phi(r) = \int\limits_{r_0}^{r} B(R-x)dx + \text{const} =$$
$$= -\int\limits_{t_0}^{B(R-r)} td\beta'(t) + \text{const} =$$
$$= -B(R-r)\beta'(B(R-r)) + \beta(B(R-r)) + \text{const} =$$
$$= (1+o(1))\beta(B(R-r)), \quad r \uparrow R,$$

because $B(R-r) \uparrow +\infty$ as $r \uparrow R$ and $x\beta'(x)/\beta(x) \to 0$ as $x \to +\infty$. But the condition $\beta(1/\beta'(x)) = (1+o(1))\beta(x)$ as $x \to +\infty$ implies the equality $\beta(t) = (1+o(1))\beta(B(1/t))$ as $t \to +\infty$ and, thus, $\beta(B(R-r)) = (1+o(1))\beta(1/(R-r))$ as $r \uparrow R$. Therefore, $\Phi(r) = (1+o(1))\beta(1/(R-r))$ as $r \uparrow R$.

Now if $h(x) = a(R - \phi(x))$, where $a \in (0, 1)$, then $h(\Phi'(r)) = a(R-r)$ and

$$\Phi(r - h(\Phi'(r))) = (1+o(1))\beta\left(\frac{1}{(1+a)(R-r)}\right) = \\ = (1+o(1))\beta\left(\frac{1}{R-r}\right), \quad r \uparrow R. \tag{10}$$

It is easy to see that $\phi(x) - h(x) \uparrow R$ as as $x \to +\infty$ and the condition

$$\phi(x_{k+1}) - \phi(x_k) \le h(x_{k+1})$$

is equivalent to the condition

$$\beta'(x_k) - \beta'(x_{k+1}) \le a\beta'(x_{k+1}),$$

that is $\beta'(x_k) \le (1+a)\beta'(x_{k+1})$. Since $a < 1$ the last condition holds because

$$\varlimsup_{k\to\infty} (\beta'(x_k)/\beta'(x_{k+1})) < 2.$$

Therefore, by Lemma from (10) we obtain (9). Theorem 2 is proved. □

3. The case $\beta(x) = \varrho\alpha(x)$

Using Lemma we prove also the following theorem.

Theorem 3. *Let $\alpha \in L_{si}$ be a continuously differentiable function and φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F. Suppose that one of the following conditions holds:*

1) $1<\varrho<+\infty$, $\varlimsup\limits_{x\to+\infty} \dfrac{d\ln\,\alpha^{-1}(x)}{d\ln\,\alpha^{-1}(\varrho x)} = q(\varrho) < 1$, $\alpha\left(\dfrac{\alpha^{-1}(\varrho\alpha(x))}{x}\right) = (1+o(1))\varrho\alpha(x)$ *as* $x \to +\infty$ *and*

$$\alpha\left(\frac{x_k}{\ln\,(W_F(x_k)e^{Rx_k})}\right) \le \frac{\alpha(x_k)}{\varrho} \tag{11}$$

for some increasing to $+\infty$ *sequence* (x_k) *of positive numbers such that*

$$\alpha^{-1}(\alpha(x_{k+1})/\varrho) = O(\alpha^{-1}(\alpha(x_k)/\varrho)) \qquad as \quad k\to\infty;$$

2) $0<\varrho<1$, $\overline{\lim\limits_{x\to+\infty}}\dfrac{d\ln\,\alpha^{-1}(\varrho x)}{d\ln\,\alpha^{-1}(x)} = q(\varrho)<1$, $\dfrac{d\alpha^{-1}(\varrho\alpha(x))}{dx} = \dfrac{1}{f(x)}\downarrow 0$, $\alpha^{-1}(\varrho\alpha(f(x))) = O(\alpha^{-1}(\varrho\alpha(x)))$ *as* $x\to+\infty$ *and*

$$\alpha(\ln\,\big(W_F(x_k)e^{Rx_k}\big)\geq\varrho\alpha(x_k) \tag{12}$$

for some increasing to $+\infty$ *sequence* (x_k) *of positive numbers such that* $\overline{\lim\limits_{k\to\infty}}\dfrac{f(x_{k+1})}{f(x_k)}<2.$

Then

$$\alpha(\ln\,M(r,\varphi))\geq(1+o(1))\varrho\alpha(1/(R-r)),\quad r\uparrow R. \tag{13}$$

Proof. At first let $1<\varrho<+\infty$. Then from (11) it follows that

$$\ln\,\big(W_F(x_k)e^{Rx_k}\big)\geq -Rx_k+\frac{x_k}{\alpha^{-1}(\alpha(x_k)/\varrho)}.$$

Since $\overline{\lim\limits_{x\to+\infty}}\dfrac{d\ln\,\alpha^{-1}(x)}{d\ln\,\alpha^{-1}(\varrho x)} = q(\varrho)<1$, we have $\dfrac{d\ln\,\alpha^{-1}(\alpha(x)/\varrho)}{d\ln x}\leq(1+o(1))q(\varrho)$ and $\dfrac{x}{\alpha^{-1}(\alpha(x)/\varrho)}\uparrow+\infty$ as $x_0\leq x\to+\infty$. Therefore, using L'Hospital's rule we get

$$\frac{x}{\alpha^{-1}(\alpha(x)/\varrho)}\geq(1+o(1))(1-q(\varrho))\int\limits_{x_0}^{x}\frac{dt}{\alpha^{-1}(\alpha(t)/\varrho)},\quad x\to+\infty,$$

and, thus,

$$\ln\,\big(W_F(x_k)e^{Rx_k}\big)\geq -Rx_k+(1-q_1)\int\limits_{x_0}^{x_k}\frac{dt}{\alpha^{-1}(\alpha(t)/\varrho)} \tag{14}$$

for each $q_1\in(q(\varrho),1)$ and for all x large enough. We choose a function $\Phi\in\Omega(R)$ so that for $r_0\leq r<R$

$$\Phi(r)=\int\limits_{r_0}^{r}\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{R-x}\right)\right)dx,\quad q_1<q_2<1. \tag{15}$$

Then

$$\Phi'(r)=\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{R-r}\right)\right),\quad \phi(x)=R-\frac{1-q_2}{\alpha^{-1}(\alpha(x)/\varrho))}$$

and

$$x\Psi(\phi(x))=\int\limits_{x_0}^{x}\phi(t)dt+\text{const}=Rx-(1-q_2)\int\limits_{x_0}^{x}\frac{dt}{\alpha^{-1}(\alpha(t)/\varrho)}+\text{const},$$

i.e., in view of (14) and the inequality $q_1<q_2$ we get (3).

Since

$$\alpha^{-1}(\alpha(x_{k+1})/\varrho)\leq K\alpha^{-1}(\alpha(x_k)/\varrho)\quad(K>1)$$

for all $k \geq 1$, we have

$$\frac{1}{\alpha^{-1}(\alpha(x_k)/\varrho)} - \frac{1}{\alpha^{-1}(\alpha(x_{k+1})/\varrho)} \leq \frac{K-1}{\alpha^{-1}(\alpha(x_k)/\varrho)} \leq$$
$$\leq \frac{K(K-1)}{\alpha^{-1}(\alpha(x_{k+1})/\varrho)}.$$

Therefore, putting

$$h(x) = \frac{K(K-1)(1-q_2)}{\alpha^{-1}(\alpha(x)/\varrho)},$$

we get

$$\phi(x) - h(x) = R - \frac{(K^2+1-K)(1-q_2)}{\alpha^{-1}(\alpha(x)/\varrho)} \uparrow R, \quad x \to +\infty,$$

$$h(\Phi'(r)) = K(K-1)(R-r) \qquad \text{and} \qquad \phi(x_{k+1}) - \phi(x_k) \leq h(x_{k+1})$$

for all $k \geq 1$.

Finally, for all $r \in [R/2, R)$ from (15) follows that

$$\Phi(r) \geq \int\limits_{2r-R}^{r} \alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{R-x}\right)\right) dx \geq$$
$$\geq (R-r)\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{2(R-r)}\right)\right).$$

Therefore, by Lemma

$$\ln\, \mu(r,\varphi) \geq (R - r + h(\Phi'(r)))\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{2(R-r+h(\Phi'(r)))}\right)\right) =$$
$$= (K^2+1-K)(R-r))\alpha^{-1}\left(\varrho\alpha\left(\frac{1-q_2}{2(K^2+1-K)(R-r))}\right)\right),$$

whence in view of conditions $\alpha \in L_{si}$ and

$$\alpha\left(\frac{\alpha^{-1}(\varrho\alpha(x))}{x}\right) = (1+o(1))\varrho\alpha(x)$$

as $x \to +\infty$ we get

$$\alpha(\ln\, M(r,\varphi)) \geq (1+o(1))\alpha(\ln\, \mu(r,\varphi)) \geq (1+o(1))\varrho\alpha(1/(R-r))$$

as $r \uparrow R$, i.e., (13) is proved.

Now let $0 < \varrho < 1$. If we put

$$x\Psi(\phi(x)) = Rx - \alpha^{-1}(\varrho\alpha(x))$$

then (12) implies (3),

$$\phi(x) = R - 1/f(x), \qquad \Phi'(r) = f^{-1}(1/(R-r))$$

and since

$$\frac{d\ln\, \alpha^{-1}(\varrho x)}{d\ln\, x} \leq (1+o(1))q(\varrho)$$

as $x\to+\infty$, we have

$$\begin{aligned}\Phi(r) &= \int\limits_{r_0}^{r} f^{-1}\left(\frac{1}{R-x}\right)dx+\text{const}=\\ &= \int\limits_{f^{-1}(1/(R-r_0))}^{f^{-1}(1/(R-r))} t d\left(\frac{-1}{f(t)}\right)+\text{const}=\\ &= -(R-r)f^{-1}\left(\frac{1}{R-r}\right)+\alpha^{-1}\left(\varrho\alpha\left(f^{-1}\left(\frac{1}{R-r}\right)\right)\right)+\text{const}=\\ &= \alpha^{-1}\left(\varrho\alpha\left(f^{-1}\left(\frac{1}{R-r}\right)\right)\right)\left\{1-\frac{(R-r)f^{-1}\left(\frac{1}{R-r}\right)+\text{const}}{\alpha^{-1}\left(\varrho\alpha\left(f^{-1}\left(\frac{1}{R-r}\right)\right)\right)}\right\}\ge\\ &\ge (1-q)\alpha^{-1}\left(\varrho\alpha\left(f^{-1}\left(\frac{1}{R-r}\right)\right)\right)\end{aligned}$$

for each $q\in(q(\varrho),1)$ and all $r\in[r_0(q),R)$. Since the condition

$$\alpha^{-1}(\varrho\alpha(f(x)))=O(\alpha^{-1}(\varrho\alpha(x)))\quad\text{as}\quad x\to+\infty$$

implies the inequality

$$\alpha^{-1}\left(\varrho\alpha\left(\frac{1}{R-r}\right)\right)\le K\alpha^{-1}\left(\varrho\alpha\left(f^{-1}\left(\frac{1}{R-r}\right)\right)\right),\quad K=\text{const}>0,$$

we have $\Phi(r)\ge\dfrac{1-q}{K}\alpha^{-1}\left(\varrho\alpha\left(\dfrac{1}{R-r}\right)\right)$. Therefore, if we put $h(x)=a(R-\phi(x))$, $0<a<1$, then

$$\Phi(r-h(\Phi'(r))\ge\frac{1-q}{K}\alpha^{-1}\left(\varrho\alpha\left(\frac{1}{(1+a)(R-r)}\right)\right). \tag{16}$$

It is clear that $\phi(x_{k+1})-\phi(x_k)\le h(x_{k+1})$ if and only if $f(x_{k+1})\le(1+a)f(x_k)$ and the last condition follows from the condition $\overline{\lim\limits_{k\to\infty}}\dfrac{f(x_{k+1})}{f(x_k)}<2$. Therefore, by Lemma from (16) in view of conditions $\alpha\in L_{si}$ we get

$$\alpha(\ln M(r,\varphi))\ge(1+o(1))\alpha(\ln\mu(r,\varphi))\ge(1+o(1))\varrho\alpha(1/(R-r))\quad\text{as}\quad r\uparrow R,$$

i.e., (13) is proved again. The proof of Theorem 3 is completed. □

Choosing $\alpha(x)=\ln x$ for $x\ge e$ from Theorem 3 we obtain the following statement.

Corollary 1. *Let φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F. Suppose that one of the following conditions holds:*

1) $1<\varrho<+\infty$ *and* $\ln\left(W_F(x_k)e^{Rx_k}\right)\ge x_k^{(\varrho-1)/\varrho}$ *for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $x_{k+1}=O(x_k)$ as $k\to\infty$;*
2) $0<\varrho<1$ *and* $\ln\left(W_F(x_k)e^{Rx_k}\right)\ge x_k^{\varrho}$ *for some increasing to $+\infty$ sequence (x_k) of positive numbers such that* $\overline{\lim\limits_{k\to\infty}}\left(\dfrac{x_{k+1}}{x_k}\right)^{1-\varrho}<2$.

Then $\ln\ln M(r,\varphi) \geq (1+o(1))\varrho \ln(1/(R-r))$ *as* $r \uparrow R$.

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ЗАУВАЖЕННЯ ДО ОЦІНОК ЗНИЗУ ХАРАКТЕРИСТИЧНИХ ФУНКЦІЙ З ЙМОВІРНІСНОГО РОЗПОДІЛУ

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Для аналітичної в $\mathbb{D}_R = \{z : |z| < R\}$ характеристичної функції φ ймовірнісного розподілу F досліджені умови на $W_F(x) = 1 - F(x) + F(-x)$ $(x \geq 0)$ і додатної неперервної функції h зростаючої до $+\infty$ такої, що $h(\ln M(r,\varphi)) \geq (1+o(1))/(R-r)$ або $\ln M(r,\varphi)) \geq (1+o(1))h(1/(R-r))$ при $r \uparrow R$, де $M(r,\varphi) = \max\{|\varphi(z)| : |z| = r < R\}$.

Ключові слова: характерестична функція, ймовірнісний закон, оцінка знизу.