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ON SPACES OF IDEMPOTENT MEASURES WITH FINITE SUPPORTS

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The main result states that the space of idempotent measures $\bar{I}_\omega(X)$ with finite support on a space X is homeomorphic to the pre-Hilbert space l^2_f if and only if X is a σ-compact, strongly countable-dimensional infinite metric space.

Key words: idempotent measure, space of finite sequences, strong universality.

1. Introduction

The aim of this note is to show connection between idempotent measures and some subspaces of Hilbert cube. The idempotent measures are analogs of probability measures in the idempotent mathematics, i.e. a part of mathematics in which operations of addition and multiplication in $\mathbb{R}$ are replaced by idempotent ones (e.g., max or min). The spaces of idempotent measures (also called Maslov measures) are considered in numerous publications; see, e.g., the survey paper [6] and references therein.

Most of publications on spaces of idempotent measures concerns the compact case. In the present note we consider the σ-compact case. Methods of infinite-dimensional topology are used for identification of spaces arising in different areas of mathematics.

As an example, we consider the following result [1]. The hyperspace $\mathcal{F}(X)$ of all nonempty finite subsets of a metric space X, with the Hausdorff metric, is homeomorphic to the $\aleph_0$-dimensional linear metric space l^2_f if and only if X is nondegenerate, connected, locally pathconnected, σ-compact, and strongly countable-dimensional.

The aim of this note is to prove the following result. The space $\bar{I}_\omega(X)$ of idempotent measures with finite supports is homeomorphic to l^2_f if and only if X is a σ-compact, strongly countable-dimensional infinite metrizable space.

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We use the technique of towers of subsets elaborated in [1] and apply it to spaces of idempotent measures.

2. Preliminaries

Let X be a metric space. By $\mathbb{I}$ we denote the segment $[0,1]$. By $\bar{I}(X)$, we denote the family of all closed subsets $A \subseteq X \times \mathbb{I}$ satisfying:

(1) $X \times \{0\} \subseteq A$;
(2) A is saturated, i.e. for every $(x,t) \in A$ and every $t' \in [0,t]$, $(x,t') \in A$;
(3) there is $x \in X$ such that $(x,1) \in A$.

The set $\bar{I}(X)$ is endowed with the Hausdorff metric d_H,

$$\mathrm{d}_H(A,B) = \inf\{r > 0 \mid A \subseteq O_r(B),\ B \subseteq O_r(A)\}$$

(here $O_r(Y)$ denotes the r-neighborhood of a subset Y in a metric space).

It is easy to check that d_H is well defined. Every $A \in \bar{I}(X)$ determines a functional $h_A : C(X,\mathbb{I}) \to \mathbb{I}$ defined by the formula:

$$h_A(\varphi) = \max\{\varphi(x)t \mid (x,t) \in A,\ \varphi \in C(X,\mathbb{I})\}.$$

This functional satisfies the properties from the definition of idempotent measure (see [2]).

We also define the product $\circ : [0,1] \times \bar{I}(X) \to \exp(X \times [0,1])$ by the formula:

$$\alpha \circ A = \{(x, \alpha t) \mid (x,t) \in A\}.$$

The union operation is defined on $\bar{I}(X)$. Given $A \in \bar{I}(X)$, we define the support of A as follows:

$$\mathrm{supp}(A) = \mathrm{cl}(\{x \in X \mid \exists t > 0 \text{ such that } (x,t) \in A\}).$$

For $n \in \mathbb{N}$, we let

$$\bar{I}_n(X) = \{A \in \bar{I}(X) \mid |\mathrm{supp}(A)| \le n\}$$

and let $\bar{I}_\omega(X) = \bigcup_{n=1}^{\infty} \bar{I}_n(X)$. Given a map $f : X \to Y$, define a map $\bar{I}(f)(A) : \bar{I}(X) \to \bar{I}(Y)$ as follows:

$$\bar{I}(f)(A) = \{(f(x),t) \mid (x,t) \in A\} \cup (Y \times \{0\}).$$

By $\bar{I}_\omega(f) : \bar{I}_\omega(X) \to \bar{I}_\omega(Y)$ we denote the restriction of $\bar{I}(f)$ to $\bar{I}_\omega(X)$.

Proposition 1. *The map $\bar{I}_\omega(f)$ is continuous.*

Proof. Let $A \in \bar{I}_\omega(X)$, $B = \bar{I}(f)(A)$, and $\varepsilon > 0$. Let $\mathrm{supp}(A) = \{x_1, x_2, \ldots, x_n\}$. There exists $\delta > 0$ such that for every x_i' with $\mathrm{d}(x_i, x_i') < \delta$ we have $\mathrm{d}(f(x_i), f(x_i')) < \varepsilon$. We additionally assume that $\delta \leqslant \varepsilon$.

Suppose now that $C \in \bar{I}_\omega(X)$ and $\mathrm{d}_H(A,C) < \delta$. Let $\mathrm{supp}(C) = \{c_1, c_2, ..., c_m\}$. We are going to show that $\mathrm{d}_H(\bar{I}_\omega(f)(A), \bar{I}_\omega(f)(C)) < \varepsilon$. Suppose that (c_j, t) is an arbitrary point from $\bar{I}_\omega(f)(C)$. We consider two cases:

a) There exists i such that $\mathrm{d}(a_i, c_i) < \delta$. Then $\mathrm{d}(f(a_i), f(c_i)) < \varepsilon$ and

$$\mathrm{d}((f(a_i),t),(f(c_i),t)) < \varepsilon.$$

b) $\mathrm{d}(a_i, c_i) \geqslant \delta$, for every $i = 1, \dots, n$. Then $t = \mathrm{d}((y_j, t), (y_j, 0)) < \delta$ and therefore

$$\mathrm{d}((f(y_j), t), (f(y_j), 0)) < \delta \leqslant \varepsilon.$$

It follows that $\bar{I}_\omega(X)(C)$ lies in the ε-neighbourhood of $\bar{I}_\omega(f)(A)$. The symmetric proof could be done for showing that $\bar{I}_\omega(X)(A)$ lies in the ε-neighbourhood of $\bar{I}_\omega(f)(C)$. Therefore, $\mathrm{d}_H(\bar{I}_\omega(f)(A), \bar{I}_\omega(f)(C)) < \varepsilon$. □

Lemma 1. *If X is σ-compact, then so is $\bar{I}_\omega(X)$.*

Proof. Suppose that $X = \bigcup_{i=1}^{\infty} X_i$, where $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ are compact sets. Then, clearly, $\bar{I}_\omega(X) = \bigcup_{n=1}^{\infty} \bar{I}_n(X_n)$, and we are going to show that every subspace $I_n(X_n)$ is compact. To this end, define the map $\Phi : \mathbb{I}^n \times X_n^n \to \bar{I}_n(X_n)$ as follows:

$$\Phi(t_1, \dots, t_n, x_1, \dots, x_n) = (X \times \{0\}) \cup \bigcup_{i=1}^{n} (\{x_i\} \times [0, t_i]) .$$

It is easy to see that Φ is continuous and onto. Then the compactness of $\bar{I}_n(X_n)$ is a consequence of the compactness of $\mathbb{I}^n \times X_n^n$. □

Recall that a space X is strongly countable-dimensional if $X = \bigcup_{i=1}^{\infty} X_n$, where $X_n,\ n \in \mathbb{N}$, are closed finite-dimensional subspaces.

Proposition 2. *If X is σ-compact, strongly countable-dimensional space, then so is $\bar{I}_\omega(X)$.*

Proof. We represent X as $X = \bigcup_{n=1}^{\infty} X_n$, where $X_1 \subseteq X_2 \subseteq \dots$ and $X_n,\ n \in \mathbb{N}$, are finite-dimensional compact spaces. Then $\bar{I}_\omega(X) = \bigcup_{n=1}^{\infty} \bar{I}_n(X_n)$.

That the spaces $\bar{I}_n(X_n)$ are finite-dimensional can be immediately deduced from the general results of the theory of functors of finite degree [5]. Actually, $\bar{I}_n$ is such a functor. □

3. Results

3.1. **Spaces of idempotent measures with finite support.** Let $\alpha : Y \to (0, \infty)$ be a function, where (Y, ρ) is a metric space. We say that maps $f, g : X \to Y$, are α-close, if $\rho(f(x), g(x)) < \alpha(f(x))$, for all $x \in X$.

A closed set $A \subset X$ is called a Z-set (respectively a strong Z-set) if, for every function $\alpha : X \to (0, \infty)$, there exists a map $f : X \to X$ which is α-close to 1_X and $f(X) \cap A = \varnothing$ (respectively $\mathrm{cl}(f(X)) \cap A = \varnothing$).

Proposition 3. *For every $n \in \mathbb{N}$, the set $\bar{I}_n(X_n)$ is a strong Z-set in $\bar{I}_\omega(X)$.*

Proof. Let $\alpha : X \to (0, \infty)$ be a function. Since X is infinite, there are pairwise distinct points $x_1, x_2, \ldots, x_{n+1} \in X$. Define $f : \bar{I}_\omega(X) \to \bar{I}_\omega(X)$ by the formula:

$$f(A) = A \cup (\{x_1, \ldots, x_{n+1}\}) \times [0, \alpha(A)/2], \qquad \text{where } A \in \bar{I}_\omega(X).$$

We are going to prove that $\mathrm{cl}(f(\bar{I}_\omega)) \cap \bar{I}_n(X_n) = \varnothing$. Suppose the contrary. Then there is $A_0 \in \bar{I}_n(X_n)$ and a sequence $(A_i)_{i=1}^\infty$ in $\bar{I}_\omega(X)$ such that $\lim_{i\to\infty} f(A_i) = A_0$.

Let $c = \inf\{\alpha(A) \mid A \in \bar{I}_n(X_n)\}$. There exists a neighborhood U of $\bar{I}_n(X_n)$ such that $\inf\{\alpha(A) \mid A \in U\} \geqslant 2c/3$. There exists $r > 0$ such that

$$\mathrm{d}_H(\mathrm{supp}(A), \{x_1, \ldots, x_{n+1}\}) \geqslant r, \qquad \text{for all } A \in \bar{I}_n(X_n).$$

Then, clearly, $\mathrm{d}_H(f(A), C) \geqslant \min\{r, \alpha(f(A))/3\}$, for all $A \in U$ and $C \in \bar{I}_n(X_n)$. This contradicts to the fact that $A_i \to A_0$, when $i \to \infty$. □

Proposition 4. *Every compact subset A of $\bar{I}_\omega(X)$ is a strong Z-set in $\bar{I}_\omega(X)$.*

Proof. We will need the following result of Curtis [3]: if A is a topologically complete closed subset of an ANR-space M, and A is a countable union of strong Z-sets, then A is a strong Z-set. In our case, A is compact and $A = \bigcup_{n=1}^{\infty}(A \cap \bar{I}_i(X_i))$. □

The following notion is introduced in [3].

Definition 1. A tower of subsets $Y_1 \subset Y_2 \subset \ldots$ in a metric space Y is called a *strongly universal tower* for finite-dimensional (f.-d.) compacta if, for every map $f : A \to Y$ of a finite-dimensional compactum into Y, for every closed subset B of A such that $f|_B : B \to Y_m$ is an embedding into some Y_m and for every $\varepsilon > 0$, there exists an embedding $h : A \to Y_n$, for some $n > m$, such that $h|_B = f|_B$ and $\mathrm{d}(h, f) < \varepsilon$.

We will need the following result from [4]. It needs a modification, namely Z-sets should be replaced by strong Z-sets.

Proposition 5. *Let Y be a strongly countable dimensional compact metric AR in which every compact subset is a strong Z-set and suppose that Y contains a strongly universal tower for f.-d.-compacta. Then $Y \cong l^2_f$.*

Proposition 6. *Let X be a σ-compact infinite space. Then there exists a strongly universal tower for f.-d.-compacta in $\bar{I}(X)$.*

Proof. Let $X = \bigcup_{i=1}^{\infty} X_i$, where $X_1 \subset X_2 \subset X_3 \subset \ldots$ are compact and finite-dimensional sets. We are going to show that $\{\bar{I}_n(X_n)\}_{n=1}^\infty$ is a strongly universal tower for f.-d.-compacta.

Let (A, B) be a compact f.-d. pair and let $f : A \to \bar{I}_\omega(X)$ be a map such that $f|_B : B \to Y_m$ is an embedding into some Y_m.

Let $\varepsilon > 0$. Since $\dim(A\backslash B) = k$, for some $k \in \mathbb{N}$, there is a countable cover $\mathcal{U} = \{U_i\}$ of $A \setminus B$ of multiplicity not greater than $k+1$ such that diameters of elements of $\mathcal{U}$ tend to 0 as they approach to B.

We consider a partition of unity $\{\phi_i\}$ subordinated to $\{U_i\}$. Also define $\psi_i : A\setminus B \to [0,1]$ as follows:

$$\psi_i(x) = \frac{\phi_i(x)}{\max\{\phi_i(x) \mid U_i \in \mathcal{U}\}}$$

Then, clearly, $\max\{\psi_i(x) \mid U_i \in \mathcal{U}\} = 1$. For every $i \in \mathbb{N}$, pick up $a_i \in U_i$. Define $g : A \to \bar{I}_\omega(X)$ as follows:

$$g(a) = \begin{cases} f(a), & \text{if } a \in B, \\ \bigcup\{\psi_i(a) \circ f(a_i) \mid i \in \mathbb{N}\}, & \text{if } a \notin B. \end{cases}$$

Note that there is $p \in \mathbb{N}$ such that $g(A) \subset \bar{I}_p(X_p)$. Also, if the diameters of U_i are small enough, then $\mathrm{d}(f, g) < \varepsilon/2$. We apply the standard calculations that confirm this.

Our next step is to modify g in order to obtain an embedding $h : A \to \bar{I}_m(X_m)$, for some $m \in \mathbb{N}$, $m \geqslant p$.

Let $\lambda : A \to [0, \frac{\varepsilon}{3}]$ be a function such that $\lambda^{-1}(0) = B$. Let

$$\Gamma^{q-1} = \{(x_1, \dots, x_q) \in [0,1]^q \mid \max\{x_1, \dots, x_q\} = 1\}.$$

Since Γ^{q-1} is homeomorphic to $[0,1]^{q-1}$, there is an embedding $\alpha : A \to \Gamma^{q-1} \subset [0,1]^q$, for some q. We write $\alpha(a) = (\alpha_1(a)), \dots, \alpha_q(a))$, $a \in A$. Choose a set

$$\{x_{ij} \mid 1 \leqslant i \leqslant q,\ 1 \leqslant j \leqslant 2n+1\}$$

of cardinality $q(2n+1)$ in X. Then there exists $N \geqslant q(2n+1)$ such that $x_{ij} \in X_N$ for all i, j.

Define $h : A \to \bar{I}_N(X_N)$ by the formula:

$$h(a) = g(a) \cup \bigcup_{i=1}^{q} \bigcup_{j=1}^{2n+1} \{x_{ij}\} \times [0, \lambda(a)\alpha_i(a)].$$

We are going to show that h is an embedding that extends $f|_B$. If $a \in B$, then $\lambda(a) = 0$ and $h(a) = g(a) = f(b)$. In order to show that h is an embedding, we have to consider the following cases.

1) $a \in A \setminus B,\ b \in B$. Then, by the definition of h, there exists $i \in \{1, \dots, q\}$ such that $\alpha_i \neq 0$. Therefore, $\{x_{i,j} \mid j = 1, \dots, 2n+1\} \subset \mathrm{supp}(h(a))$. We conclude that

$$h(a) \notin h(B) = f(B) \subset \bar{I}_m(X_m).$$

2) $a, b \in A \setminus B,\ a \neq b$. By the definition of h, there exists j such that

$$(\mathrm{supp}(g(a)) \cup \mathrm{supp}(g(b)) \cap \{x_{i,j} \mid i = 1, \dots, q\}) = \varnothing.$$

Assume that $h(a) = h(b)$. Then

$$\lambda(a) = \sup\{\lambda(a)\alpha_i(a) \mid i = 1, \dots, q\} = \sup\{\lambda(b)\alpha_i(b) \mid i = 1, \dots, q\} = \lambda(b).$$

Since $\lambda(a) > 0$, we conclude that $\alpha_i(a) = \alpha_i(b)$, for all $i = 1, \dots, q$. Since α is in embedding, we see that $a = b$, a contradiction. □

Theorem 1. *The space $\bar{I}_\omega(X)$ of idempotent measures with finite supports is homeomorphic to l^2_f if and only if X is σ-compact, strongly countable-dimensional infinite metrizable space.*

Proof. That the space $\bar{I}_\omega(X)$ is AR is proved in [9]. Next, applying Proposition 3.5 we see that there exists a strongly universal tower in $\bar{I}_\omega(X)$. By Proposition 3.2, every compact subsets A of $\bar{I}_\omega(X)$ is a strong Z-set in $\bar{I}_\omega(X)$, applying Proposition 2.3 $\bar{I}_\omega(X)$ is strongly countable-dimensional. Thus, by Proposition 3.4, $\bar{I}_\omega(X) \cong l^2_f$.

Conversely, if $\bar{I}_\omega(X) \cong l^2_f$, then one can use the closed embedding $x \mapsto \delta_x : X \to \bar{I}_\omega(X)$ to deduce that X is a strongly countable-dimensional σ-compact space. □

3.2. **Pairs of spaces of idempotent measures.** By Q we denote the Hilbert cube $[0,1]^\omega$. Let $B_f(Q)$ denote the pseudoboundary of Q,

$$B_f(Q) = \{(x_i)_{i=1}^\infty \in Q \mid \text{all but finitely many } x_i \text{ are equal to } 0\}.$$

A characterization of the pair $(Q, B_f(Q))$ can be given in terms of finite-dimensional cap-sets (compact absorption property) [8]. A subset M of X is called an *f.-d. cap set* if,

(1) $M = \bigcup\limits_{i=1}^{\infty} M_i$, where M_i are finite-dimensional compact Z-sets in X such that $M_i \subset M_{i+1}$, where $i \in \mathbb{N}$;
(2) for every $\varepsilon > 0$, every natural $m \in \mathbb{N}$, and every finite-dimensional compact subset $K \subset X$, there is $n \in \mathbb{N}$ and an embedding $h : K \to M_n$ such that $h|_K \cap M_m = \mathrm{id}$ and $\mathrm{d}(h, \mathrm{id}) < \varepsilon$.

Note that $B_f(Q)$ is homeomorphic to l^2_f.

Proposition 7. *Let X be a compact metric space $Y \subset X$ its σ-f.-d.sional σ-compact infinite dense subset. Then the pair $(\bar{I}(X), \bar{I}_\omega(Y))$ is homeomorphic to the pair $(Q, B_f(Q))$.*

Proof. It is known that, under this conditions, $\bar{I}(X)$ is homeomorphic to the Hilbert cube (see [7]). That $I_\omega(Y)$ is a f.-d.-cap set of $\bar{I}(X)$ is actually a part of the proof of Proposition 6. From result of Anderson [8], the pair $(\bar{I}(X), \bar{I}_\omega(Y))$ is homeomorphic to $(Q, B_f(Q))$. □

4. Remark

Let $\Sigma = \left\{(x_i)_{i=1}^\infty \in l^2 \mid \sum_{i=1}^\infty (ix_i)^2 < \infty\right\}$. It is well-known that Σ is homeomorphic to the space

$$B(Q) = \{(x_i)_{i=1}^\infty \in Q \mid \text{such that } x_i = 0, \text{ for all but finitely many } i\}.$$

The set $B(Q)$ is called the pseudoboundary of the Hilbert cube Q.

Question 1. *Characterize the subspaces of $\bar{I}(X)$ homeomorphic to Σ.*

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ПРО ПРОСТІР ІДЕМПОТЕНТНИХ МІР ЗІ СКІНЧЕННИМИ НОСІЯМИ

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Основний результат: простір ідемпотентних мір $\bar{I}_\omega(X)$ зі скінченними носіями на просторі X гомеоморфний передгільбертовому просторові l_f^2 тоді і тільки тоді, коли X є σ-компактним, σ-скінченновимірним нескінченним метричним простором.

Ключові слова: ідемпотентна міра, простір скінченних послідовностей, сильна універсальність.