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ON SOME GENERALIZATION OF THE BICYCLIC SEMIGROUP: THE TOPOLOGICAL VERSION

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We show that every Hausdorff Baire topology τ on $\mathcal{C}=\langle a, b \mid a^2b = a, ab^2 = b\rangle$ such that $(\mathcal{C},\tau)$ is a semitopological semigroup is discrete and we construct a nondiscrete Hausdorff semigroup topology on $\mathcal{C}$. We also discuss the closure of a semigroup $\mathcal{C}$ in a semitopological semigroup and prove that $\mathcal{C}$ does not embed into a topological semigroup with the countably compact square.

Key words: topological semigroup, semitopological semigroup, bicyclic semigroup, closure, embedding, Baire space.

1. Introduction and preliminaries

In this paper all topological spaces are assumed to be Hausdorff. If Y is a subspace of a topological space X and $A \subseteq Y$, then we shall denote the topological closure of A in Y by $\operatorname{cl}_Y(A)$. Further we shall follow the terminology of [7, 8, 10, 19].

For a topological space X, a family $\{A_s \mid s \in \mathscr{A}\}$ of subsets of X is called *locally finite* if for every point $x \in X$ there exists an open neighbourhood U of x in X such that the set $\{s \in \mathscr{A} \mid U \cap A_s \neq \varnothing\}$ is finite. A subset A of X is said to be

- *co-dense* in X if $X \setminus A$ is dense in X;
- an *F_σ-set* in X if A is a union of a countable family of closed subsets in X.

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We recall that a topological space X is said to be

- *compact* if each open cover of X has a finite subcover;
- *countably compact* if each open countable cover of X has a finite subcover;
- *sequentially compact* if each sequence in X has a convergent subsequence;
- *pseudocompact* if each locally finite open cover of X is finite;
- a *Baire space* if for each sequence $A_1, A_2, \ldots, A_i, \ldots$ of nowhere dense subsets of X the union $\bigcup\limits_{i=1}^{\infty} A_i$ is a co-dense subset of X;
- *Čech complete* if X is Tychonoff and for every compactification cX of X, the remainder $cX \setminus X$ is an F_σ-set in cX;
- *locally compact* if every point of X has an open neighbourhood with a compact closure.

According to Theorem 3.10.22 of [10], a Tychonoff topological space X is pseudocompact if and only if each continuous real-valued function on X is bounded.

If S is a semigroup, then we shall denote the *Green relations* on S by $\mathscr{R}$ and $\mathscr{L}$ (see Section 2.1 of [8]):

$$a\mathscr{R}b \text{ if and only if } aS^1 = bS^1; \qquad \text{and} \qquad a\mathscr{L}b \text{ if and only if } S^1a = S^1b.$$

A semigroup S is called *simple* if S does not contain any proper two-sided ideals.

A *semitopological* (resp. *topological*) *semigroup* is a topological space together with a separately (resp. jointly) continuous semigroup operation.

An important theorem of Andersen [1] (see also [8, Theorem 2.54]) states that in any [0-]simple semigroup which is not completely [0-]simple, each nonzero idempotent (if there are any) is the identity element of a copy of the bicyclic semigroup $\mathcal{B}(a,b) = \langle a, b \mid ab = 1\rangle$. The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every non-annihilating homomorphism h of the bicyclic semigroup is either an isomorphism or the image of $\mathcal{B}(a,b)$ under h is a cyclic group (see Corollary 1.32 in [8]). Eberhart and Selden [9] showed that every Hausdorff semigroup topology on the bicyclic semigroup $\mathcal{B}(a,b)$ is discrete. Bertman and West [6] proved that every Hausdorff topology τ on $\mathcal{B}(a,b)$ such that $(\mathcal{B}(a,b),\tau)$ is a semitopological semigroup is also discrete. Neither stable nor Γ-compact topological semigroups can contain a copy of the bicyclic semigroup [2, 13]. Also, the bicyclic semigroup cannot be embedded into any countably compact topological inverse semigroup [11]. Moreover, the conditions were given in [4] and [5] when a countably compact or pseudocompact topological semigroup cannot contain the bicyclic semigroup, which is topological semigroup with a countably compact square and with a pseudocompact square. However, Banakh, Dimitrova and Gutik [5] have constructed (assuming the Continuum Hypothesis or the Martin Axiom) an example of a Tychonoff countably compact topological semigroup which contains the bicyclic semigroup.

Jones [14] found semigroups $\mathcal{A}$ and $\mathcal{C}$ which play a role similar to the bicyclic semigroup in Andersen's Theorem. Let

$$\mathcal{A} = \langle a, b \mid a^2b = a\rangle$$

and

$$\mathcal{C} = \langle a, b \mid a^2b = a, ab^2 = b\rangle.$$

It is obvious that the semigroup $\mathcal{C}$ is a homomorphic image of $\mathcal{A}$, and the bicyclic semigroup is a homomorphic image of $\mathcal{C}$. Also, every non-injective homomorphic image of the semigroup $\mathcal{C}$ contains an idempotent. Jones [14] showed that every [0-] simple idempotent-free semigroup S on which $\mathscr{R}$ is nontrivial contains (a copy of) $\mathcal{A}$ or $\mathcal{C}$. Moreover, if S is also $\mathscr{L}$-trivial and is not $\mathscr{R}$-trivial then it must contain $\mathcal{A}$ (but not $\mathcal{C}$), and if S is both $\mathscr{R}$- and $\mathscr{L}$-nontrivial then S must contain either $\mathcal{C}$ or both $\mathcal{A}$ and its dual $\mathcal{A}^d$.

In the general case, the countable compactness of topological semigroup S does not guarantee that S contains an idempotent. By Theorem 8 of [4], a topological semigroup S contains an idempotent if S satisfies one of the following conditions: 1) S is doubly countably compact; 2) S is sequentially compact; 3) S is p-compact for some free ultrafilter p on ω; 4) $S^{2^{\mathfrak{c}}}$ is countably compact; 5) $S^{\kappa^{\omega}}$ is countably compact, where κ is the minimal cardinality of a closed subsemigroup of S. This motivates the establishing of the semigroups $\mathcal{A}$ and $\mathcal{C}$ as topological semigroups, in particular their semigroup topologizations and the question of their embeddings into compact-like topological semigroups.

In this paper we study the semigroup $\mathcal{C}$ as a semitopological semigroup. We show that every Hausdorff Baire topology τ on $\mathcal{C}$ such that $(\mathcal{C},\tau)$ is a semitopological semigroup is discrete and we construct a nondiscrete Hausdorff semigroup topology on $\mathcal{C}$. We also discuss the closure of a semigroup $\mathcal{C}$ in a semitopological semigroup and prove that $\mathcal{C}$ does not embed into a topological semigroup with a countably compact square.

2. Algebraic properties of the semigroup $\mathcal{C}$

The semigroup $\mathcal{C}=\langle a,b \mid a^2b=a, ab^2=b\rangle$ was introduced by Rédei [18] and further studied by Megyesi and Pollák [16] and by Rankin and Reis [17]. Its salient properties are summarized here:

Proposition 1. *(i) $\mathcal{C}$ is a 2-generated simple idempotent-free semigroup in which $a\mathscr{R}a^2$ and $b\mathscr{L}b^2$, so that $\mathscr{R}$ and $\mathscr{L}$ are nontrivial; however $\mathscr{H}$ is trivial.*

(ii) Each element of $\mathcal{C}$ is uniquely expressible as $b^k(ab)^la^m$, $k,l,m\geqslant 0$, $k+l+m>0$.

(iii) The product of elements $b^k(ab)^la^m$ and $b^n(ab)^pa^q$ in $\mathcal{C}$ is equal to

$$\begin{cases} b^{k+n-m}(ab)^pa^q, & \text{if } m<n;\\ b^k(ab)^{l+p+1}a^q, & \text{if } m=n\neq 0;\\ b^k(ab)^{l+p}a^q, & \text{if } m=n=0;\\ b^k(ab)^la^{q+m-n}, & \text{if } m>n. \end{cases} \tag{1}$$

(iv) The semigroup $\mathcal{C}$ is minimally idempotent-free (i.e., it is idempotent-free but each of its proper quotients contains an idempotent).

Definition 1 ([15]). A semigroup S is said to be *stable* if the following conditions hold:

(i) $s,t\in S$ and $Ss\subseteq Sst$ implies that $Ss=Sst$; and

(ii) $s,t\in S$ and $sS\subseteq tsS$ implies that $sS=tsS$.

By formula (1) we have that

$$b\cdot b^n(ab)^pa^q=b^{n+1}(ab)^pa^q$$

and

$$a\cdot b\cdot b^n(ab)^pa^q=\begin{cases} (ab)^{p+1}a^q, & \text{if } n=0;\\ b^n(ab)^pa^q, & \text{if } n\geqslant 1, \end{cases}$$

for each $b^n(ab)^pa^q \in \mathcal{C}$. Hence we get that $b \cdot \mathcal{C} \subseteq a \cdot b \cdot \mathcal{C}$, but $b \cdot \mathcal{C} \neq a \cdot b \cdot \mathcal{C}$. This yields the following proposition:

Proposition 2. *The semigroup $\mathcal{C}$ is not stable.*

The following remark follows from formula (1) above:

Remark 1. The semigroup operation in $\mathcal{C}$ implies that the following assertions hold:

- (i) The map $\varphi_{i,j} \colon \mathcal{C} \to \mathcal{C}$ defined by the formula $\varphi_{i,j}(x) = b^i \cdot x \cdot a^j$ is injective for all nonnegative integers i and j (for $i = j = 0$ we put that $\varphi_{0,0}(x) = x$);
- (ii) The subsemigroups $\mathcal{C}_{ab} = \langle ab \rangle$, $\mathcal{C}_a = \langle a \rangle$ and $\mathcal{C}_b = \langle b \rangle$ in $\mathcal{C}$ are infinite cyclic semigroups.

3. On topologizations of the semigroup $\mathcal{C}$

Let X be a topological space. A continuous map $f \colon X \to X$ is called a *retraction* of X if $f \circ f = f$; and the set of all values of a retraction of X is called a *retract* of X (cf. [10]).

Proposition 3. *If τ is a Hausdorff topology on $\mathcal{C}$ such that $(\mathcal{C}, \tau)$ is a semitopological semigroup then for every positive integer k the sets*

$$\mathfrak{R}_k = \{b^n(ab)^pa^q \mid n = k, k+1, k+2, \ldots, p = 0, 1, 2, \ldots, q = 0, 1, 2, \ldots\},$$

and

$$\mathfrak{L}_k = \{b^n(ab)^pa^q \mid q = k, k+1, k+2, \ldots, n = 0, 1, 2, \ldots, p = 0, 1, 2, \ldots\}$$

are retracts in $(\mathcal{C}, \tau)$ and hence closed subsets of $(\mathcal{C}, \tau)$.

Proof. By formula (1) we have that

$$b^m(ab)^la^m \cdot b^n(ab)^pa^q = \begin{cases} b^n(ab)^pa^q, & \text{if } m < n; \\ b^n(ab)^{l+p+1}a^q, & \text{if } m = n \neq 0; \\ (ab)^{l+p}a^q, & \text{if } m = n = 0; \\ b^m(ab)^la^{q+m-n}, & \text{if } m > n, \end{cases} \tag{2}$$

$$b^i(ab)^la^m \cdot b^n(ab)^pa^n = \begin{cases} b^{i+n-m}(ab)^pa^n, & \text{if } m < n; \\ b^i(ab)^{l+p+1}a^n, & \text{if } m = n \neq 0; \\ b^i(ab)^{l+p}, & \text{if } m = n = 0; \\ b^i(ab)^la^m, & \text{if } m > n. \end{cases} \tag{3}$$

Then left and right translations of the element $b^k(ab)^la^k$ of the semigroup $\mathcal{C}$ are retractions of the topological space $(\mathcal{C}, \tau)$ and hence the sets $\mathfrak{R}_k$ and $\mathfrak{L}_k$ are retracts of the topological space $(\mathcal{C}, \tau)$ for every positive integer k. The last statement of the proposition follows from Exercise 1.5.C of [10]. □

Proposition 4. *If τ is a Hausdorff topology on $\mathcal{C}$ such that $(\mathcal{C}, \tau)$ is a semitopological semigroup then $\mathcal{C}_{ab}$ is an open-and-closed subsemigroup of $(\mathcal{C}, \tau)$.*

Proof. We observe that $\mathcal{C}_{ab} = \mathcal{C} \setminus (\mathfrak{R}_1 \cup \mathfrak{L}_1)$ and hence by Proposition 3 we have that $\mathcal{C}_{ab}$ is an open subset of $(\mathcal{C}, \tau)$. Also, formula (1) implies that

$$a \cdot b^n(ab)^p a^q \cdot b = \begin{cases} b^{n-1}(ab)^p a^q \cdot b, & \text{if } n > 1; \\ (ab)^{p+l} a^q \cdot b, & \text{if } n = 1; \\ a^{q+1} \cdot b, & \text{if } n = 0 \end{cases} =$$

$$= \begin{cases} b^n, & \text{if } n > 1 \quad \text{and } q = 0; \\ b^{n-1}(ab)^{p+1} & \text{if } n > 1 \quad \text{and } q = 1; \\ b^{n-1}(ab)^p a^{q-1}, & \text{if } n > 1 \quad \text{and } q > 1; \\ b, & \text{if } n = 1 \quad \text{and } q = 0; \\ (ab)^{p+2}, & \text{if } n = 1 \quad \text{and } q = 1; \\ (ab)^{p+1} a^{q-1}, & \text{if } n = 1 \quad \text{and } q > 1; \\ ab, & \text{if } n = 0 \quad \text{and } q = 0; \\ a, & \text{if } n = 0 \quad \text{and } q = 1; \\ a^q, & \text{if } n = 0 \quad \text{and } q > 1, \end{cases} \tag{4}$$

for nonnegative integers n, p and q. By formula (4),

$$\mathcal{C}_{0,0} = \{(ab)^i \mid i = 1, 2, 3, \ldots\}$$

is the set of solutions of the equation $a \cdot X \cdot b = ab$. Then the Hausdorffness of the space $(\mathcal{C}, \tau)$ and the separate continuity of the semigroup operation in $\mathcal{C}$ imply that $\mathcal{C}_{ab} = \mathcal{C}_{0,0}$ is a closed subset of $(\mathcal{C}, \tau)$. □

We observe that formula (4) implies that

$$b^k(ab)^l a^m \cdot b = \begin{cases} b^{k+1}, & \text{if } m = 0; \\ b^k(ab)^{l+1}, & \text{if } m = 1; \\ b^k(ab)^l a^{m-1}, & \text{if } m > 1, \end{cases} \tag{5}$$

$$a \cdot b^n(ab)^p a^q = \begin{cases} b^{n-1}(ab)^p a^q, & \text{if } n > 1; \\ (ab)^{p+1} a^q, & \text{if } n = 1; \\ a^{q+1}, & \text{if } n = 0, \end{cases} \tag{6}$$

for nonnegative integers k, l, m, n, p and q.

Proposition 5. *If τ is a Hausdorff topology on $\mathcal{C}$ such that $(\mathcal{C}, \tau)$ is a semitopological semigroup then*

$$\mathcal{C}_{0,i} = \{(ab)^p a^i \mid p = 0, 1, 2, 3, \ldots\}$$

and

$$\mathcal{C}_{i,0} = \{b^i(ab)^p \mid p = 0, 1, 2, 3, \ldots\}$$

are open subsets of $(\mathcal{C}, \tau)$ for any positive integer i.

Proof. By Proposition 4, $\mathcal{C}_{0,0}$ is an open subset $(\mathcal{C}, \tau)$ and by Hausdorffness of $(\mathcal{C}, \tau)$ the set $\mathcal{C}_{0,0} \setminus \{ab\}$ is open in $(\mathcal{C}, \tau)$, too. Then formula (5) implies that the equation $X \cdot b = (ab)^{p+2}$, where $p = 0, 1, 2, 3, \ldots$, has a unique solution $X = (ab)^p a$, and hence since all right translations in $(\mathcal{C}, \tau)$ are continuous maps we get that $\mathcal{C}_{0,1}$ is an open subset of the topological space $(\mathcal{C}, \tau)$. Also, formula (4) implies that the equation $a \cdot X = (ab)^{p+2}$, where $p = 0, 1, 2, 3, \ldots$, has a unique solution $X = b(ab)^p$, and hence since all left translations in $(\mathcal{C}, \tau)$ are continuous maps we get that $\mathcal{C}_{1,0}$ is an open subset of the topological space $(\mathcal{C}, \tau)$.

By formula (5), the equation $X \cdot b = (ab)^l a^{m-1}$, where $l-1$ and $m-1$ are positive integers, has a unique solution $X = (ab)^l a^m$. Then the separate continuity of the semigroup operation in $(\mathcal{C}, \tau)$ implies that if $\mathcal{C}_{0,m-1}$ is an open subset of $(\mathcal{C}, \tau)$ then $\mathcal{C}_{0,m}$ is open in $(\mathcal{C}, \tau)$, too. Similarly, formula (6) implies that the equation $a \cdot X = b^{n-1}(ab)^p$, where $n-1$ and $p-1$ are positive integers, has a unique solution $X = b^n(ab)^p$, and hence the separate continuity of the semigroup operation in $(\mathcal{C}, \tau)$ and openess of the set $\mathcal{C}_{n-1,0}$ in $(\mathcal{C}, \tau)$ imply that the set $\mathcal{C}_{n,0}$ is an open subset of the topological space $(\mathcal{C}, \tau)$. Next, we complete the proof of the proposition by induction. □

Proposition 6. *If τ is a Hausdorff topology on $\mathcal{C}$ such that $(\mathcal{C}, \tau)$ is a semitopological semigroup then*

$$\mathcal{C}_{i,j} = \left\{ b^i(ab)^p a^j \mid p = 0, 1, 2, 3, \ldots \right\}$$

is an open subset of $(\mathcal{C}, \tau)$ for all positive integers i and j.

Proof. First we observe that Proposition 5 and Hausdorffness of $(\mathcal{C}, \tau)$ imply that $\mathcal{C}_{k,0} \setminus \left\{ b^k(ab) \right\}$ is an open subset of $(\mathcal{C}, \tau)$ for every positive integer k. Then formula (5) implies that the equation $X \cdot b = b^k(ab)^{p+1}$, where $p = 0, 1, 2, 3, \ldots$, has a unique solution $X = b^k(ab)^p a$, and hence since all right and left translations in $(\mathcal{C}, \tau)$ are continuous maps we get that $\mathcal{C}_{k,1}$ is an open subset of the topological space $(\mathcal{C}, \tau)$.

Also, by formula (5) we have that the equation $X \cdot b = b^k(ab)^p a^l$ has a unique solution $X = b^k(ab)^p a^{l+1}$. Then the openess of the set $\mathcal{C}_{k,l}$ implies that the set $\mathcal{C}_{k,l+1}$ is open in $(\mathcal{C}, \tau)$. Then induction implies the assertion of the proposition. □

Propositions 4, 5 and 6 imply Theorem 1, which describes all Hausdorff topologies τ on $\mathcal{C}$ such that $(\mathcal{C}, \tau)$ is a semitopological semigroup.

Theorem 1. *If τ is a Hausdorff topology on $\mathcal{C}$ such that $(\mathcal{C}, \tau)$ is a semitopological semigroup then $\mathcal{C}_{i,j}$ is an open-and-closed subset of $(\mathcal{C}, \tau)$ for all nonnegative integers i and j.*

Since the bicyclic semigroup $\mathcal{B}(a, b)$ admits only the discrete topology which turns $\mathcal{B}(a, b)$ into a Hausdorff semitopological semigroup [6], Theorem 1 implies the following:

Corollary 1. *If $\mathcal{C}$ is a semitopological semigroup then the homomorphism $h \colon \mathcal{C} \to \mathcal{B}(a, b)$, defined by the formula $h\left(b^k(ab)^l a^m\right) = b^k a^m$, is continuous.*

Later we shall need the following lemma.

Lemma 1. *Every Hausdorff Baire topology on the infinite cyclic semigroup S such that (S, τ) is a semitopological semigroup is discrete.*

Proof. Since every infinite cyclic semigroup is isomorphic to the additive semigroup of positive integers $(\mathbb{N}, +)$ we assume without loss of generality that $S = (\mathbb{N}, +)$.

Fix an arbitrary $n_0 \in \mathbb{N}$. Then Hausdorffness of $(\mathbb{N}, +)$ implies that $\{1, \ldots, n_0\}$ is a closed subset of $(\mathbb{N}, +)$, and hence by Proposition 1.14 of [12] we get that $\mathbb{N}_{n_0} = \mathbb{N} \setminus \{1, \ldots, n_0\}$ with the induced topology from $(\mathbb{N}, \tau)$ is a Baire space.

If no point in $\mathbb{N}_{n_0}$ is isolated, then since $(\mathbb{N}, \tau)$ is Hausdorff, it follows that $\{n\}$ is nowhere dense in $\mathbb{N}_{n_0}$ for all $n > n_0$. But, if this is the case, then since the space $(\mathbb{N}, \tau)$ is countable we conclude that $\mathbb{N}_{n_0}$ cannot be a Baire space. Hence $\mathbb{N}_{n_0}$ contains an isolated point n_1 in $\mathbb{N}_{n_0}$. Then the separate continuity of the semigroup operation in $(\mathbb{N}, +, \tau)$

implies that n_0 is an isolated point in $(\mathbb{N},\tau)$, because $n_1 = n_0 + (\underbrace{1+\ldots+1}_{(n_1-n_0)\text{-times}})$. This completes the proof of the lemma. □

Theorem 2. *Every Hausdorff Baire topology τ on $\mathcal{C}$ such that $(\mathcal{C},\tau)$ is a semitopological semigroup is discrete.*

Proof. By Proposition 4, $\mathcal{C}_{ab}$ is an open-and-closed subsemigroup of $(\mathcal{C},\tau)$. Then by Proposition 1.14 of [12] we have that $\mathcal{C}_{ab}$ is a Baire space and hence Lemma 1 implies that every element of $\mathcal{C}_{ab}$ is an isolated point of the topological space $(\mathcal{C},\tau)$.

Now, by formula (4), the equation $a\cdot X\cdot b=(ab)^{p+2}$ has a unique solution $X=b(ab)^pa$ for every nonnegative integer p, and since the semigroup operation in $(\mathcal{C},\tau)$ is separately continuous we conclude that $b(ab)^pa$ is an isolated point in $(\mathcal{C},\tau)$ for every integer $p\geqslant 0$. Similarly, formula (4) implies that the equation $a\cdot X\cdot b=b^n(ab)^pa^n$ has the unique solution $X=b^{n-1}(ab)^pa^{n-1}$ for every nonnegative integer p and every integer $n>1$. Then by induction we get that the separate continuity of the semigroup operation in $(\mathcal{C},\tau)$ implies that $b^{n+1}(ab)^pa^{n+1}$ is an isolated point in the topological space $(\mathcal{C},\tau)$ for all nonnegative integers n and p.

We fix arbitrary distinct nonnegative integers n and m. We can assume without loss of generality that $n<m$. In the case when $m<n$ the proof is similar. Since by Remark 1(i) we have that the map $\varphi_{m-n,0}\colon\mathcal{C}\to\mathcal{C}$ defined by the formula $\varphi_{m-n,0}(x)=b^{m-n}\cdot x$ is injective and by the previous part of the proof, the point $b^m(ab)^pa^m$ is isolated in $(\mathcal{C},\tau)$ for every nonnegative integer p, we conclude that the separate continuity of the semigroup operation in $(\mathcal{C},\tau)$ implies that $b^n(ab)^pa^m$ is an isolated point in the topological space $(\mathcal{C},\tau)$ for every nonnegative integer p. □

Since every Čech complete space (and hence every locally compact space) is Baire, Theorem 2 implies Corollaries 2 and 3.

Corollary 2. *Every Hausdorff Čech complete (locally compact) topology τ on $\mathcal{C}$ such that $(\mathcal{C},\tau)$ is a Hausdorff semitopological semigroup is discrete.*

Corollary 3. *Every Hausdorff Baire topology (and hence Čech complete or locally compact topology) τ on $\mathcal{C}$ such that $(\mathcal{C},\tau)$ is a Hausdorff topological semigroup is discrete.*

The following example implies that there exists a Tychonoff nondiscrete topology τ_p on the semigroup $\mathcal{C}$ such that $(\mathcal{C},\tau_p)$ is a topological semigroup.

Example 1. Let p be a fixed prime number. We define a topology τ_p on the semigroup $\mathcal{C}$ by the base

$$\mathscr{B}_p\big(b^i(ab)^ka^j\big)=\big\{U_\alpha\big(b^i(ab)^ka^j\big)\mid \alpha=1,2,3,\ldots\big\}$$

at every point $b^i(ab)^ka^j\in\mathcal{C}$, where

$$U_\alpha\big(b^i(ab)^ka^j\big)=\Big\{b^i(ab)^{k+\lambda\cdot p^\alpha}a^j\mid \lambda=1,2,3,\ldots\Big\}.$$

Simple verifications show that the topology τ_p on $\mathcal{C}$ is generated by the following metric:

$$d\left(b^{i_1}(ab)^{k_1}a^{j_1},b^{i_2}(ab)^{k_2}a^{j_2}\right)=\begin{cases} 0, & \text{if } i_1=i_2, k_1=k_2 \text{ and } j_1=j_2;\\ 2^s, & \text{if } i_1=i_2, k_1\neq k_2 \text{ and } j_1=j_2;\\ 1, & \text{otherwise,}\end{cases}$$

where s is the largest of p which divides $|k_1 - k_2|$. This implies that $(\mathcal{C}, \tau_p)$ is a Tychonoff space. Also, it is easy to see that $U_\alpha\big(b^i(ab)^k a^j\big)$ is a closed subset of the topological space $(\mathcal{C}, \tau_p)$, for every $b^i(ab)^k a^j \in \mathcal{C}$ and every positive integer α, and hence $(\mathcal{C}, \tau_p)$ is a 0-dimensional topological space (i.e., $(\mathcal{C}, \tau_p)$ has a base which consists of open-and-closed subsets). We observe that the topological space $(\mathcal{C}, \tau_p)$ doesn't contain any isolated points.

For every positive integer α and arbitrary elements $b^k(ab)^l a^m$ and $b^n(ab)^t a^q$ of the semigroup $\mathcal{C}$, formula (1) implies that the following conditions hold:

- (i) if $m < n$ then $U_\alpha\big(b^k(ab)^l a^m\big) \cdot U_\alpha(b^n(ab)^t a^q) \subseteq U_\alpha\big(b^{k+n-m}(ab)^t a^q\big)$;
- (ii) if $m = n \neq 0$ then $U_\alpha\big(b^k(ab)^l a^m\big) \cdot U_\alpha(b^n(ab)^t a^q) \subseteq U_\alpha\big(b^k(ab)^{l+t+1} a^q\big)$;
- (iii) if $m = n = 0$ then $U_\alpha\big(b^k(ab)^l a^m\big) \cdot U_\alpha(b^n(ab)^t a^q) \subseteq U_\alpha\big(b^k(ab)^{l+t} a^q\big)$; and
- (iv) if $m > n$ then $U_\alpha\big(b^k(ab)^l a^m\big) \cdot U_\alpha(b^n(ab)^t a^q) \subseteq U_\alpha\big(b^k(ab)^l a^{q+m-n}\big)$.

Therefore $(\mathcal{C}, \tau_p)$ is a topological semigroup.

4. On the closure and embedding of the semitopological semigroup $\mathcal{C}$

In the case of the bicyclic semigroup $\mathcal{B}(a,b)$ we have that if a topological semigroup S contains $\mathcal{B}(a,b)$ then the nonempty remainder of $\mathcal{B}(a,b)$ under the closure in S is an ideal in $\operatorname{cl}_S(\mathcal{B}(a,b))$ (see [9]). This immediately follows from that facts that the bicyclic semigroup $\mathcal{B}(a,b)$ admits only the discrete topology which turns $\mathcal{B}(a,b)$ into a Hausdorff semitopological semigroup and that the equations $A \cdot X = B$ and $X \cdot A = B$ have finitely many solutions in $\mathcal{B}(a,b)$ (see [6, Proposition 1] and [9, Lemma I.1]).

The following example shows that the semigroup $\mathcal{C}$ with the discrete topology does not have similar "properties of the closure" as the bicyclic semigroup.

Example 2. It well known that each element of the bicyclic semigroup $\mathcal{B}(a,b)$ is uniquely expressible as $b^i a^j$, where i and j are nonnegative integers. Since all elements of the semigroup have similar expressibility we shall denote later the elements of the bicyclic semigroup by underlining $\underline{b^i a^j}$.

We define a map $\pi \colon \mathcal{C} \to \mathcal{B}(a,b)$ by the formula $\pi(b^i(ab)ka^j) = \underline{b^i a^j}$. Simple verifications and formula (1) show that thus defined map π is a homomorphism. We extend the semigroup operation from the semigroups $\mathcal{C}$ and $\mathcal{B}(a,b)$ on $S = \mathcal{C} \sqcup \mathcal{B}(a,b)$ in the following way:

$$b^k(ab)^l a^m \star \underline{b^n a^q} = \begin{cases} \underline{b^{k+n-m} a^q}, & \text{if } m < n; \\ \underline{b^k a^q}, & \text{if } m = n; \\ b^k(ab)^l a^{q+m-n}, & \text{if } m > n \end{cases}$$

and

$$\underline{b^k a^m} \star b^n(ab)^p a^q = \begin{cases} b^{k+n-m}(ab)^p a^q, & \text{if } m < n; \\ \underline{b^k a^q}, & \text{if } m = n; \\ \underline{b^k a^{q+m-n}}, & \text{if } m > n. \end{cases}$$

A routine check of all 118 cases and their compatibility shows that such a binary operation is associative.

Now, we define the topology τ on the semigroup S in the following way:

- (i) all elements of the semigroup $\mathcal{C}$ are isolated points in (S, τ); and

(ii) the family $\mathscr{B}\left(\underline{b^ia^j}\right)=\left\{U_n\left(\underline{b^ia^j}\right)\mid n=1,2,3,\ldots\right\}$, where

$$U_n\left(\underline{b^ia^j}\right)=\left\{\underline{b^ia^j}\right\}\cup\left\{b^i(ab)^ka^j\in\mathcal{C}\mid k=n,n+1,n+2,\ldots\right\},$$

is a base of the topology τ at the point $\underline{b^ia^j}\in\mathcal{B}(a,b)$.

Simple verifications show that (S,τ) is a Hausdorff 0-dimensional scattered locally compact metrizable space.

Proposition 7. (S,τ) *is a topological semigroup.*

Proof. The definition of the topology τ on S implies that it suffices to show that the semigroup operation in (S,τ) is continuous in the following three cases:

1) $\underline{b^ia^k}\star\underline{b^ma^p}$;
2) $\underline{b^ia^k}\star b^m(ab)^na^p$; and
3) $b^i(ab)^la^k\star\underline{b^ma^p}$.

In case 1) we get that

$$\underline{b^ia^k}\star\underline{b^ma^p}=\begin{cases}\underline{b^{i-k+m}a^p}, & \text{if } k<m;\\ \underline{b^ia^p}, & \text{if } k=m;\\ \underline{b^ia^{k-m+p}}, & \text{if } ,k>m,\end{cases}$$

and for every positive integer u the following statements hold:

$a)$ if $k<m$ then $U_u\left(\underline{b^ia^k}\right)\star U_u\left(\underline{b^ma^p}\right)\subseteq U_u\left(\underline{b^{i-k+m}a^p}\right)$;
$b)$ if $k=m$ then $U_u\left(\underline{b^ia^k}\right)\star U_u\left(\underline{b^ma^p}\right)\subseteq U_u\left(\underline{b^ia^p}\right)$;
$c)$ if $k>m$ then $U_u\left(\underline{b^ia^k}\right)\star U_u\left(\underline{b^ma^p}\right)\subseteq U_u\left(\underline{b^ia^{k-m+p}}\right)$.

In case 2) we have that

$$\underline{b^ia^k}\star b^m(ab)^na^p=\begin{cases}b^{i-k+m}(ab)^na^p, & \text{if } k<m;\\ \underline{b^ia^p}, & \text{if } k=m;\\ \underline{b^ia^{k-m+p}}, & \text{if } ,k>m,\end{cases}$$

and hence for every positive integer u the following statements hold:

$a)$ if $k<m$ then

$$U_u\left(\underline{b^ia^k}\right)\star\{b^m(ab)^na^p\}=\left\{b^{i-k+m}(ab)^na^p\right\};$$

$b)$ if $k=m$ then

$$U_u\left(\underline{b^ia^k}\right)\star\{b^m(ab)^na^p\}\subseteq U_u\left(\underline{b^ia^p}\right);$$

$c)$ if $k>m$ then

$$U_u\left(\underline{b^ia^k}\right)\star\{b^m(ab)^na^p\}\subseteq U_u\left(\underline{b^ia^{k-m+p}}\right).$$

In case 3) we have that

$$b^i(ab)^la^k\star\underline{b^ma^p}=\begin{cases}\underline{b^{i-k+m}a^p}, & \text{if } k<m;\\ \underline{b^ia^p}, & \text{if } k=m;\\ b^i(ab)^la^{k-m+p}, & \text{if } ,k>m.\end{cases}$$

Then for every positive integer u the following statements hold:

$a)$ if $k<m$ then

$$\left\{b^i(ab)^la^k\right\}\star U_u\left(\underline{b^ma^p}\right)\subseteq\left\{b^{i-k+m}(ab)^na^p\right\};$$

b) if $k = m$ then

$$\left\{b^i(ab)^l a^k\right\} \star U_u(\underline{b^m a^p}) \subseteq U_u\left(\underline{b^i a^p}\right);$$

c) if $k > m$ then

$$\left\{b^i(ab)^l a^k\right\} \star U_u(\underline{b^m a^p}) = \left\{b^i(ab)^l a^{k-m+p}\right\}.$$

This completes the proof of the proposition. □

The following example shows that the semigroup $\mathcal{C}$ with the discrete topology may has similar closure in a topological semigroup as the bicyclic semigroup.

Example 3. Let S be the semigroup $\mathcal{C}$ with adjoined zero 0. We determine the topology τ on the semigroup S in the following way:

(i) All elements of the semigroup $\mathcal{C}$ are isolated points in (S,τ); and
(ii) The family $\mathscr{B}(0) = \{U_n(0) \mid n = 1,2,3,\ldots\}$, where

$$U_n(0) = \{0\} \cup \left\{b^i(ab)^k a^j \in \mathcal{C} \mid i,j \geqslant n\right\},$$

is a base of the topology τ at the zero 0.

Simple verifications show that (S,τ) is a Hausdorff 0-dimensional scattered space.

Since all elements of the semigroup $\mathcal{C}$ are isolated points in (S,τ) we conclude that it is sufficient to show that the semigroup operation in (S,τ) is continuous in the following cases:

$$0 \cdot 0, \qquad 0 \cdot b^m(ab)^n a^p, \qquad \text{and} \qquad b^m(ab)^n a^p \cdot 0.$$

Since the following assertions hold for each positive integer i:

(i) $U_i(0) \cdot U_i(0) \subseteq U_i(0)$;
(ii) $U_{i+m}(0) \cdot \{b^m(ab)^n a^p\} \subseteq U_i(0)$;
(iii) $\{b^m(ab)^n a^p\} \cdot U_{i+p}(0) \subseteq U_i(0)$,

we conclude that (S,τ) is a topological semigroup.

Remark 2. We observe that we can show that for the discrete semigroup $\mathcal{C}$ cases of closure of $\mathcal{C}$ in topological semigroups proposed in [9] for the bicyclic semigroup can be realized in the following way: we identify the element $b^i a^j$ of the bicyclic semigroup with the subset $\mathcal{C}_{i,j}$ of the semigroup $\mathcal{C}$.

We don't know the answer to the following question: *Does there exist a topological semigroup S which contains $\mathcal{C}$ as a dense subsemigroup such that $S \setminus \mathcal{C} \neq \varnothing$ and $\mathcal{C}$ is an ideal of S?*

The following proposition describes the closure of the semigroup $\mathcal{C}$ in an arbitrary semitopological semigroup.

Proposition 8. *Let S be a Hausdorff semitopological semigroup which contains $\mathcal{C}$ as a dense subsemigroup. Then there exists a countable family $\mathscr{U} = \left\{U_{\mathcal{C}_{i,j}} \mid i,j = 0,1,2,3,\ldots\right\}$ of open disjunctive subsets of the topological space S such that $\mathcal{C}_{i,j} \subseteq U_{\mathcal{C}_{i,j}}$ for all nonnegative integers i and j.*

Proof. When $S = \mathcal{C}$ the statement of the proposition follows from Theorem 1. Hence we can assume that $S \neq \mathcal{C}$.

First, we observe that formulae (5) and (6) imply that for left and right translations $\lambda_{ab}\colon S\to S\colon x\mapsto ab\cdot x$ and $\rho_{ab}\colon S\to S\colon x\mapsto x\cdot ab$ of the semigroup S their sets of fixed points $\operatorname{Fix}(\lambda_{ab})$ and $\operatorname{Fix}(\rho_{ab})$ are non-empty and moreover

$$\bigcup\{\mathcal{C}_{i,j}\mid i=0,1,2,3,\ldots,\ j=1,2,3,\ldots\}\subseteq \operatorname{Fix}(\rho_{ab});$$

and

$$\bigcup\{\mathcal{C}_{i,j}\mid i=1,2,3,\ldots,\ j=0,1,2,3,\ldots\}\subseteq \operatorname{Fix}(\lambda_{ab}).$$

Also, formulae (2) and (3) imply that for every positive integer n the left and right translations $\lambda_{b^na^n}\colon S\to S\colon x\mapsto b^na^n\cdot x$ and $\rho_{b^na^n}\colon S\to S\colon x\mapsto x\cdot b^na^n$ of the semigroup S have non-empty sets of fixed points $\operatorname{Fix}(\lambda_{b^na^n})$ and $\operatorname{Fix}(\rho_{b^na^n})$, and moreover

$$\bigcup\{\mathcal{C}_{i,j}\mid i=0,1,2,3,\ldots,\ j=n+1,n+2,n+3,\ldots\}\subseteq \operatorname{Fix}(\rho_{b^na^n});$$

and

$$\bigcup\{\mathcal{C}_{i,j}\mid i=n+1,n+2,n+3,\ldots,\ j=0,1,2,3,\ldots\}\subseteq \operatorname{Fix}(\lambda_{b^na^n}).$$

Then the Hausdorffness of S, separate continuity of the semigroup operation on S and Exercise 1.5.C of [10] imply that $\operatorname{Fix}(\lambda_{ab})$, $\operatorname{Fix}(\rho_{ab})$, $\operatorname{Fix}(\lambda_{b^na^n})$ and $\operatorname{Fix}(\rho_{b^na^n})$ are closed non-empty subset of S, for every positive integer n, and hence are retracts of S.

Now, since $\mathcal{C}_{0,0}\subseteq S\setminus(\operatorname{Fix}(\lambda_{ab})\cup\operatorname{Fix}(\rho_{ab}))$ we conclude that there exists an open subset $U_{\mathcal{C}_{0,0}}=S\setminus(\operatorname{Fix}(\lambda_{ab})\cup\operatorname{Fix}(\rho_{ab}))$ which contains the set $\mathcal{C}_{0,0}$ and $\mathcal{C}_{i,j}\cap U_{\mathcal{C}_{0,0}}=\varnothing$ for all nonnegative integers i,j such that $i+j>0$.

Since the semigroup operation in S is separately continuous we conclude that the map $\lambda_a\colon S\to S\colon x\mapsto a\cdot x$ is continuous, and hence

$$U_{\mathcal{C}_{1,0}}=\lambda_a^{-1}\left(U_{\mathcal{C}_{0,0}}\right)\setminus(\operatorname{Fix}(\rho_{ab})\cup\operatorname{Fix}(\lambda_{ba}))$$

is an open subset of S. It is obvious that $\mathcal{C}_{1,0}\subseteq U_{\mathcal{C}_{1,0}}$. We claim that $U_{\mathcal{C}_{1,0}}\cap U_{\mathcal{C}_{0,0}}=\varnothing$. Suppose to the contrary that there exists $x\in S$ such that $x\in U_{\mathcal{C}_{1,0}}\cap U_{\mathcal{C}_{0,0}}$. Since $\operatorname{Fix}(\lambda_{ba})$ and $\operatorname{Fix}(\rho_{ba})$ are closed subsets of S we conclude that there exists $(ab)^i\in U_{\mathcal{C}_{1,0}}\cap U_{\mathcal{C}_{0,0}}$. Then we have that

$$\lambda_a((ab)^i)=a\cdot(ab)^i=a\notin U_{\mathcal{C}_{0,0}},$$

a contradiction. The obtained contradiction implies that $U_{\mathcal{C}_{1,0}}\cap U_{\mathcal{C}_{0,0}}=\varnothing$.

Also, the continuity of the right shift $\rho_b\colon S\to S\colon x\mapsto x\cdot b$ implies that

$$U_{\mathcal{C}_{0,1}}=\rho_b^{-1}\left(U_{\mathcal{C}_{0,0}}\right)\setminus(\operatorname{Fix}(\lambda_{ab})\cup\operatorname{Fix}(\rho_{ba}))$$

is an open neighbourhood of the set $\mathcal{C}_{0,1}$ in S. Similar arguments as in the previous case imply that $U_{\mathcal{C}_{0,1}}\cap U_{\mathcal{C}_{0,0}}=\varnothing$.

Suppose that there exists $x\in S$ such that $x\in U_{\mathcal{C}_{1,0}}\cap U_{\mathcal{C}_{0,1}}$. If $x\in\mathcal{C}$ then $x=b(ab)^p$ for some nonnegative integer p. Then we have that

$$\rho_b(x)=x\cdot b=b(ab)^p\cdot b=b^2\notin U_{\mathcal{C}_{0,0}}.$$

If $x\in U_{\mathcal{C}_{1,0}}\setminus\mathcal{C}$ then every open neighbourhood $V(x)$ of the point x in the topological space S contains infinitely many points of the form $b(ab)^p\in\mathcal{C}$. Then we have that $\rho_b(V(x))\ni b^2$. The obtained contradiction implies that $U_{\mathcal{C}_{1,0}}\cap U_{\mathcal{C}_{0,1}}=\varnothing$.

We put

$$U_{\mathcal{C}_{1,1}}=\left(\rho_b^{-1}\left(U_{\mathcal{C}_{1,0}}\right)\cap\lambda_a^{-1}\left(U_{\mathcal{C}_{0,1}}\right)\right)\setminus(\operatorname{Fix}(\lambda_{ba})\cup\operatorname{Fix}(\rho_{ba})).$$

Then $U_{\mathcal{C}_{1,1}}$ is an open subset of the topological space S such that $\mathcal{C}_{1,1} \subseteq U_{\mathcal{C}_{1,1}}$. Similar arguments as in the previous cases imply that

$$U_{\mathcal{C}_{1,1}} \cap U_{\mathcal{C}_{0,1}} = U_{\mathcal{C}_{1,0}} \cap U_{\mathcal{C}_{1,1}} = U_{\mathcal{C}_{1,1}} \cap U_{\mathcal{C}_{0,0}} = \varnothing.$$

Next, we use induction for constructing the family $\mathscr{U}$. Suppose that for some positive integer $n \geqslant 1$ we have already constructed the family

$$\mathscr{U}_n = \left\{ U'_{\mathcal{C}_{i,j}} \mid i,j = 0,1,\ldots,n \right\}$$

of open disjunctive subsets of the topological space S with the property $\mathcal{C}_{i,j} \subseteq U_{\mathcal{C}_{i,j}}$, for all $i,j = 0,1,\ldots,n$. We shall construct the family

$$\mathscr{U}_{n+1} = \left\{ U_{\mathcal{C}_{i,j}} \mid i,j = 0,1,\ldots,n,n+1 \right\}$$

in the following way. For all $i,j \leqslant n$ we put $U_{\mathcal{C}_{i,j}} = U'_{\mathcal{C}_{i,j}} \in \mathscr{U}_n$ and

$$\begin{aligned}
U_{\mathcal{C}_{0,n+1}} &= \rho_b^{-1}\left(U_{\mathcal{C}_{0,n}}\right) \setminus (\operatorname{Fix}(\lambda_{ab}) \cup \operatorname{Fix}(\rho_{b^{n+1}a^{n+1}}));\\
U_{\mathcal{C}_{1,n+1}} &= \rho_b^{-1}\left(U_{\mathcal{C}_{1,n}}\right) \setminus (\operatorname{Fix}(\lambda_{ba}) \cup \operatorname{Fix}(\rho_{b^{n+1}a^{n+1}}));\\
\cdots &\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
U_{\mathcal{C}_{n,n+1}} &= \rho_b^{-1}\left(U_{\mathcal{C}_{n-1,n}}\right) \setminus (\operatorname{Fix}(\lambda_{b^na^n}) \cup \operatorname{Fix}(\rho_{b^{n+1}a^{n+1}}));\\
U_{\mathcal{C}_{n+1,0}} &= \lambda_a^{-1}\left(U_{\mathcal{C}_{n,0}}\right) \setminus (\operatorname{Fix}(\rho_{ab}) \cup \operatorname{Fix}(\lambda_{b^{n+1}a^{n+1}}));\\
U_{\mathcal{C}_{n+1,1}} &= \lambda_a^{-1}\left(U_{\mathcal{C}_{n,1}}\right) \setminus (\operatorname{Fix}(\rho_{ba}) \cup \operatorname{Fix}(\lambda_{b^{n+1}a^{n+1}}));\\
\cdots &\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
U_{\mathcal{C}_{n+1,n}} &= \lambda_a^{-1}\left(U_{\mathcal{C}_{n,n}}\right) \setminus (\operatorname{Fix}(\rho_{b^na^n}) \cup \operatorname{Fix}(\lambda_{b^{n+1}a^{n+1}}));\\
U_{\mathcal{C}_{n+1,n+1}} &= \left(\rho_b^{-1}\left(U_{\mathcal{C}_{n+1,n}}\right) \cap \lambda_a^{-1}\left(U_{\mathcal{C}_{n,n+1}}\right)\right) \setminus (\operatorname{Fix}(\rho_{b^{n+1}a^{n+1}}) \cup \operatorname{Fix}(\lambda_{b^{n+1}a^{n+1}})).
\end{aligned}$$

Similar arguments as in previous case imply that $\mathscr{U}_{n+1}$ is a family of open disjunctive subsets of the topological space S with the property $\mathcal{C}_{i,j} \subseteq U_{\mathcal{C}_{i,j}}$, for all $i,j = 0,1,\ldots,n+1$.

Next, we put $\mathscr{U} = \bigcup\limits_{n=0}^{\infty} \mathscr{U}_n$. It is easy to see that the family $\mathscr{U}$ is as required. This completes the proof of the proposition. ☐

It well known that if a topological semigroup S is a continuous image of a topological semigroup T such that T is embeddable into a compact topological semigroup, then the semigroup S is not necessarily embeddable into a compact topological semigroup. For example, the bicyclic semigroup $\mathcal{B}(a,b)$ does not embed into any compact topological semigroup, but $\mathcal{B}(a,b)$ admits only discrete semigroup topology and $\mathcal{B}(a,b)$ is a continuous image of the free semigroup F_2 of the rank 2 (i.e., generated by two elements) with the discrete topology. Moreover, the semigroup F_2 with adjoined zero 0 admits a compact Hausdorff semigroup topology τ_c: all elements of F_2 are isolated points and the family $\mathscr{B}_0 = \{U_n \mid n = 1,2,3,\ldots\}$, where the set U_n consists of zero 0 and all words of length $\geqslant n$. Therefore it is natural to ask the following: *Does there exist a Hausdorff compact topological semigroup S which contains the semigroup $\mathcal{C}$?* The following theorem gives a negative answer to this question.

Theorem 3. *There does not exist a Hausdorff topological semigroup S with a countably compact square $S\times S$ such that S contains $\mathcal{C}$ as a subsemigroup.*

Proof. Suppose to the contrary that there exists a Hausdorff topological semigroup S with a countably compact square $S\times S$ which contains $\mathcal{C}$ as a subsemigroup. Then since the closure of a subsemigroup $\mathcal{C}$ in a topological semigroup S is a subsemigroup of S (see [7, Vol. 1, p. 9]) we conclude that Theorem 3.10.4 from [10] implies that without loss of generality we can assume that $\mathcal{C}$ is a dense subsemigroup of the topological semigroup S. We consider the sequence $\{(a^n,b^n)\}_{n=1}^{\infty}$ in $\mathcal{C}\times\mathcal{C}\subseteq S\times S$. Since $S\times S$ is countably compact we conclude that this sequence has an accumulation point $(x;y)\in S\times S$. Since $a^nb^n=ab$, the continuity of the semigroup operation in S implies that $xy=ab$. By Proposition 8 there exists an open neighbourhood $U(ab)$ of the point ab in S such that $U(ab)\cap\mathcal{C}\subseteq\mathcal{C}_{0,0}$. Then the continuity of the semigroup operation in S implies that there exist open neighbourhoods $U(x)$ and $U(y)$ of the points x and y in S such that $U(x)\cdot U(y)\subseteq U(ab)$. Next, by the countable compactness of $S\times S$ we conclude that S is countably compact, too, as a continuous image of $S\times S$ under the projection, and this implies that x and y are accumulation points of the sequences $\{a^n\}_{n=1}^{\infty}$ and $\{b^n\}_{n=1}^{\infty}$ in S, respectively. Then there exist positive integers i and j such that $a^i\in U(x)$, $b^j\in U(y)$ and $j>i$. Therefore we get that

$$a^i\cdot b^j=b^{j-i}\in(U(x)\cdot U(y))\cap\mathcal{C}\subseteq(U(ab))\cap\mathcal{C}\subseteq\mathcal{C}_{0,0},$$

which is a contradiction. The obtained contradiction implies the statement of the theorem. □

Theorem 3 implies the following corollaries:

Corollary 4. *There does not exist a Hausdorff compact topological semigroup which contains $\mathcal{C}$ as a subsemigroup.*

Corollary 5. *There does not exist a Hausdorff sequentially compact topological semigroup which contains $\mathcal{C}$ as a subsemigroup.*

We recall that the Stone-Čech compactification of a Tychonoff space X is a compact Hausdorff space βX containing X as a dense subspace so that each continuous map $f\colon X\to Y$ to a compact Hausdorff space Y extends to a continuous map $\overline{f}\colon\beta X\to Y$ [10].

Theorem 4. *There does not exist a Tychonoff topological semigroup S with the pseudocompact square $S\times S$ which contains $\mathcal{C}$ as subsemigroup.*

Proof. By Theorem 1.3 from [3], for any topological semigroup S with the pseudocompact square $S\times S$ the semigroup operation $\mu\colon S\times S\to S$ extends to a continuous semigroup operation $\beta\mu\colon\beta S\times\beta S\to\beta S$, so S is a subsemigroup of the compact topological semigroup βS. Therefore if S contains the semigroup $\mathcal{C}$ then βS also contains the semigroup $\mathcal{C}$ which contradicts Corollary 4. □

Theorem 5. *The discrete semigroup $\mathcal{C}$ does not embed into a Hausdorff pseudocompact semitopological semigroup S such that $\mathcal{C}$ is a dense subsemigroup of S and $S\setminus\mathcal{C}$ is a left (right, two-sided) ideal of S.*

Proof. Suppose to the contrary that there exists a Hausdorff pseudocompact semitopological semigroup S which contains $\mathcal{C}$ as a dense discrete subsemigroup and $I = S\setminus\mathcal{C}$ is a left ideal of S. Then the set of solutions $\mathscr{S}$ of the equations $x \cdot ba = ba$ in S is a subset of $\mathcal{C}$ and hence by the formula

$$b^k(ab)^l a^m \cdot ba = \begin{cases} b^{k+1}a, & \text{if } m = 0; \\ b^k(ab)^{l+1}a, & \text{if } m = 1; \\ b^k(ab)^l a^m, & \text{if } m > 1, \end{cases}$$

we get that $\mathscr{S} = \mathcal{C}_{0,0}$. Since ba is an isolated point in S and I is a left ideal of S we conclude that the separate continuity of the semigroup operation of S implies that the space S contains a discrete open-and-closed subspace $\mathcal{C}_{0,0}$. This contradicts the pseudocompactness of S. The obtained contradiction implies the statement of the theorem. In the case of a right or a two-sided ideal the proof is similar. □

Theorem 6. *The semigroup $\mathcal{C}$ does not embed into a Hausdorff countably compact semitopological semigroup S such that $\mathcal{C}$ is a dense subsemigroup of S and $S\setminus\mathcal{C}$ is a left (right, two-sided) ideal of S.*

Proof. Suppose to the contrary that there exists a Hausdorff countably compact semitopological semigroup S which contains $\mathcal{C}$ as a dense subsemigroup and $I = S\setminus\mathcal{C}$ is a left ideal of S. Then the arguments presented in the proof of Theorem 5 imply that $\mathcal{C}_{0,0}$ is a closed subset of S, and hence by Theorem 3.10.4 of [10] is countably compact. Since $\mathcal{C}_{0,0}$ is countable we have that the space $\mathcal{C}_{0,0}$ is compact. Since every compact space is Baire, Lemma 1 implies that $\mathcal{C}_{0,0}$ is a discrete subspace of S. Then similar arguments as in the proof of Theorem 2 imply that $\mathcal{C}$, with the topology induced from S, is a discrete semigroup, which contradicts Theorem 5. The obtained contradiction implies the statement of the theorem. □

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ПРО ОДНЕ УЗАГАЛЬНЕННЯ БІЦИКЛІЧНОЇ НАПІВГРУПИ: ТОПОЛОГІЧНА ВІРСІЯ

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Доводимо, що кожна гаусдорфова берівська топологія τ на напівгрупі $\mathcal{C} = \langle a, b \mid a^2b = a, ab^2 = b\rangle$ така, що $(\mathcal{C}, \tau)$ — напівтопологічна напівгрупа є дискретною та будуємо недискретну гаусдорфову напівгрупову топологію на $\mathcal{C}$. Досліджено замикання напівгрупи $\mathcal{C}$ у напівтопологічній напівгрупі та доведено, що $\mathcal{C}$ не занурюється в топологічну напівгрупу зі зліченно компактним квадратом.

Ключові слова: топологічна напівгрупа, напівтопологічна напівгрупа, біциклічна напівгрупа, замикання, занурення, берівський простір.