ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 93. С. 97–107
Visnyk of the Lviv Univ. Series Mech. Math. 2022. Issue 93. P. 97–107
http://publications.lnu.edu.ua/bulletins/index.php/mmf
doi: http://dx.doi.org/10.30970/vmm.2022.93.097-107

УДК 517.926.4+517.546.1

ON THE SOLUTIONS OF SOME GENERALIZATION OF THE SHAH-TYPE DIFFERENTIAL EQUATION

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A Dirichlet series $F(s) = e^{hs} + \sum_{k=2}^{\infty} f_k e^{s\lambda_k}$ with the exponents $0 < h < \lambda_k \uparrow +\infty$ and the abscissa of absolute convergence $\sigma_a[F] \geq 0$ is said to be pseudostarlike of order $\alpha \in [0, h)$ in $\Pi_0 = \{s : \operatorname{Re} s < 0\}$ if $\operatorname{Re}\{F'(s)/F(s)\} > \alpha$ for all $s \in \Pi_0$. Similarly, the function F is said to be pseudoconvex of order $\alpha \in [0, h)$ if $\operatorname{Re}\{F''(s)/F'(s)\} > \alpha$ for all $s \in \Pi_0$.
The equation $\dfrac{d^n w}{ds^n} + \left(\sum_{j=0}^{n} \gamma_j e^{hjs}\right) w = ae^{hs}$ is considered, where $n \geq 3$, $h > 0$ and $a \neq 0$. It is proved that if $h^n + \gamma_0 > 0$ then this equation has a solution $F(s) = \dfrac{a}{h^n + \gamma_0} e^{sh} + \sum_{k=2}^{\infty} f_k e^{skh}$, where $f_k = -\dfrac{1}{(hk)^n + \gamma_0} \sum_{j=1}^{\min\{k-1,n\}} \gamma_j f_{k-j}$ for $k \geq 2$ (Lemma 1). For $n = 3$, $a = h^3 + \gamma_0$ conditions on parameters γ_j, under which the function F is pseudostarlike (Theorem 1) or pseudoconvex of order $\alpha \in [0, h)$ (Theorem 2) are found.

Key words: differential equation, Dirichlet series, pseudostarlikeness, pseudoconvexity.

1. Introduction

Let S be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} f_n z^n \tag{1}$$

analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$. A function $f \in S$ is said to be convex if $f(\mathbb{D})$ is a convex domain and is said to be starlike if $f(\mathbb{D})$ is starlike domain regarding

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2020 Mathematics Subject Classification: 35B08, 30B50, 30D45

the origin. It is well known [1, p. 203] that the condition $\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the convexity of f, and [1, p. 202] that the condition $\mathrm{Re}\,\{zf'(z)/f(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the starlikeness of $f \in S$. By W. Kaplan [2] a function $f \in S$ is said to be close-to-convex (see also [1, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0\,(z \in \mathbb{D})$. We remark that every starlike function is close-to-convex. A. W. Goodman [3] (see also [4, p. 9]) proved that if $\sum\limits_{n=2}^{\infty} n|f_n| \le 1$ then function $f \in S$ is starlike, and if $\sum\limits_{n=2}^{\infty} n^2|f_n| \le 1$ then function $f \in S$ is convex in $\mathbb{D}$. The concept of the starlikeness of function $f \in S$ got the series of generalizations. I. S. Jack [5] studied starlike functions of order $\alpha \in [0,\,1)$, i. e. such functions $f \in S$, for which $\mathrm{Re}\,\{zf'(z)/f(z)\} > \alpha$ $(z \in \mathbb{D})$. It is proved [5], [4, p. 13] that if $\sum\limits_{n=2}^{\infty}(n-\alpha)|f_n| \le 1-\alpha$ then function $f \in S$ is starlike function of order α.

Let $h > 0$, (λ_k) be an increasing to $+\infty$ sequence of positive numbers $(\lambda_2 > h)$ and Dirichlet series

$$F(s) = e^{sh} + \sum_{k=2}^{\infty} f_k \exp\{s\lambda_k\} \ (s = \sigma + it) \tag{2}$$

absolutely convergent in a half-plane $\Pi_0 = \{s : \mathrm{Re}\, s < 0\}$. It is known [6], [4, p. 135] that each such function F is non-univalent in Π_0, but if $\sum\limits_{k=2}^{\infty} \lambda_k|f_k| \le h$ then function F is conformal at every point of Π_0. A conformal at every point of Π_0 function F is said to be pseudostarlike if $\mathrm{Re}\{F'(s)/F(s)\} > 0\,(s \in \Pi_0)$. In [6] (see also [4, p. 139]) it is proved that if $\sum\limits_{k=2}^{\infty} \lambda_k|f_k| \le h$ then function F is pseudostarlike. A conformal at every point of Π_0 function F is said to be pseudostarlike of the order $\alpha \in [0,\,h)$ if $\mathrm{Re}\{F'(s)/F(s)\} > \alpha$ for all $s \in \Pi_0$. In [7] it is proved that if

$$\sum_{k=2}^{\infty}(\lambda_k - \alpha)|f_k| \le h - \alpha \tag{3}$$

for some $\alpha \in [0,\,h)$, then F is pseudostarlike of the order α.

We remark that if in the definition of the pseudostarlikeness instead of F'/F we put F''/F' then we get ([6], [4, p. 139]) the definition of the pseudoconvexity of F.

S. M. Shah [8] indicated conditions on real parameters γ_0, γ_1, γ_2 of the differential equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0,$$

under which there exists an entire transcendental solution (1) such that f and all its derivatives are close-to-convex in $\mathbb{D}$. The convexity of solutions of the Shah equation has been studied in [9, 10]. Substituting $z = e^s$ we obtain the differential equation

$$\frac{d^2w}{ds^2} + (\gamma_0 e^{2s} + \gamma_1 e^s + \gamma_2)w = 0.$$

In [4, p. 147-153] the pseudoconvexity and the pseudostarlikeness of solutions has been studied for the equation

$$\frac{d^2w}{ds^2} + (\gamma_0 e^{2hs} + \gamma_1 e^{hs} + \gamma_2)w = 0.$$

Here we consider a more general differential equation

$$\frac{d^n w}{ds^n} + \left(\sum_{j=0}^{n} \gamma_j e^{hjs}\right) w = ae^{hs}, \tag{4}$$

where $n \geq 3$, $h > 0$ and $a \neq 0$. We will find the solution of the equation (4) in the form

$$w = F(s) = \sum_{k=1}^{\infty} f_k e^{s\lambda_k}, \quad s = \sigma + it, \tag{5}$$

where $0 < \lambda_k \uparrow +\infty$ as $k \to \infty$, we find recurrent formulas for f_k and λ_k, and in the case of $n = 3$ we study the conditions under which this solution is pseudostarlike or pseudoconvex.

2. Recurrent formulas

Putting (5) into (4) we have

$$\sum_{k=1}^{\infty}(\lambda_k^n+\gamma_0)f_k e^{s\lambda_k}+\gamma_1\sum_{k=1}^{\infty} f_k e^{s(\lambda_k+h)}+\gamma_2\sum_{k=1}^{\infty} f_k e^{s(\lambda_k+2h)}+\ldots+\gamma_n\sum_{k=1}^{\infty} f_k e^{s(\lambda_k+nh)} \equiv ae^{hs},$$

i.e.

$$(\lambda_1^n+\gamma_0)f_1 e^{s\lambda_1}+\sum_{k=2}^{\infty}(\lambda_k^n+\gamma_0)f_k e^{s\lambda_k}+\gamma_1\sum_{k=1}^{\infty} f_k e^{s(\lambda_k+h)}+\ldots+\gamma_n\sum_{k=1}^{\infty} f_k e^{s(\lambda_k+nh)} \equiv ae^{hs},$$

whence as $\sigma \to -\infty$ we get

$$(\lambda_1^n + \gamma_0)f_1 e^{s\lambda_1}(1 + o(1)) = ae^{hs}(1 + o(1))$$

and, thus, since $a \neq 0$, we have $\lambda_1^n + \gamma_0 \neq 0$ and

$$\lambda_1 = h, \quad f_1 = \frac{a}{\lambda_1^n + \gamma_0} = \frac{a}{h^n + \gamma_0}$$

and

$$\begin{aligned} &\sum_{k=2}^{\infty}(\lambda_k^n + \gamma_0)f_k e^{s\lambda_k} + \gamma_1\sum_{k=1}^{\infty} f_k e^{s(\lambda_k+h)} + \gamma_2\sum_{k=1}^{\infty} f_k e^{s(\lambda_k+2h)} + \\ &\qquad + \ldots + \gamma_n\sum_{k=1}^{\infty} f_k e^{s(\lambda_k+nh)} \equiv 0. \end{aligned} \tag{6}$$

In the future, we will assume that $h^n + \gamma_0 = \lambda_1^n + \gamma_0 > 0$, whence $(kh)^n + \gamma_0 > 0$ for all $k \geq 1$.

Writing the identity (6) in the form

$$(\lambda_2^n+\gamma_0)f_2e^{s\lambda_2}+\sum_{k=3}^{\infty}(\lambda_k^n+\gamma_0)f_ke^{s\lambda_k}+\gamma_1f_1e^{2hs}+\gamma_1\sum_{k=2}^{\infty}f_ke^{s(\lambda_k+h)}+$$
$$+\gamma_2\sum_{k=1}^{\infty}f_ke^{s(\lambda_k+2h)}+\ldots+\gamma_n\sum_{k=1}^{\infty}f_ke^{s(\lambda_k+nh)}\equiv 0,$$

we obtain, as above,

$$\lambda_2=2h,\quad f_2=-\frac{\gamma_1f_1}{\lambda_2^n+\gamma_0}=-\frac{\gamma_1f_1}{(2h)^2+\gamma_0}$$

and

$$\sum_{k=3}^{\infty}(\lambda_k^n+\gamma_0)f_ke^{s\lambda_k}+\gamma_1\sum_{k=2}^{\infty}f_ke^{s(\lambda_k+h)}+\gamma_2\sum_{k=1}^{\infty}f_ke^{s(\lambda_k+2h)}+\ldots+\gamma_n\sum_{k=1}^{\infty}f_ke^{s(\lambda_k+nh)}\equiv 0,$$

i.e.,

$$(\lambda_3^n+\gamma_0)f_3e^{s\lambda_3}+\sum_{k=4}^{\infty}(\lambda_k^n+\gamma_0)f_ke^{s\lambda_k}+\gamma_1f_2e^{3hs}+\gamma_1\sum_{k=3}^{\infty}f_ke^{s(\lambda_k+h)}+$$
$$+\gamma_2f_1e^{3hs}+\gamma_2\sum_{k=2}^{\infty}f_ke^{s(\lambda_k+2h)}+\ldots+\gamma_n\sum_{k=1}^{\infty}f_ke^{s(\lambda_k+nh)}\equiv 0,$$

whence, as above, we obtain

$$\lambda_3=3h,\quad f_3=-\frac{\gamma_1f_2+\gamma_2f_1}{(3h)^n+\gamma_0}$$

and

$$\sum_{k=4}^{\infty}(\lambda_k^n+\gamma_0)f_ke^{s\lambda_k}+\gamma_1\sum_{k=3}^{\infty}f_ke^{s(\lambda_k+h)}+\gamma_2\sum_{k=2}^{\infty}f_ke^{4hs}+\gamma_3\sum_{k=1}^{\infty}f_ke^{s(\lambda_k+3h)}+$$
$$+\gamma_4\sum_{k=1}^{\infty}f_ke^{s(\lambda_k+4h)}+\ldots+\gamma_n\sum_{k=1}^{\infty}f_ke^{s(\lambda_k+nh)}\equiv 0.$$

Continuing this process, we have for $4\le m\le n-1$

$$\lambda_m=mh,\quad f_m=-\frac{\gamma_1f_{m-1}+\gamma_2f_{m-2}+\ldots+\gamma_{m-1}f_1}{(mh)^n+\gamma_0}$$

and

$$\sum_{k=m+1}^{\infty}(\lambda_k^n+\gamma_0)f_ke^{s\lambda_k}+\gamma_1\sum_{k=m}^{\infty}f_ke^{s(\lambda_k+h)}+\ldots+\gamma_j\sum_{k=m+1-j}^{\infty}f_ke^{(\lambda_k+jh)}+\ldots+$$
$$+\gamma_m\sum_{k=1}^{\infty}f_ke^{(\lambda_k+mh)}+\ldots+\gamma_n\sum_{k=1}^{\infty}f_ke^{s(\lambda_k+nh)}\equiv 0.$$

Choosing $m=n-1$, hence we get

$$\lambda_{n-1}=(n-1)h,\quad f_{n-1}=-\frac{\gamma_1f_{n-2}+\gamma_2f_{n-3}+\ldots+\gamma_{n-2}f_1}{((n-1)h)^n+\gamma_0}$$

and

$$\sum_{k=n}^{\infty}(\lambda_k^n+\gamma_0)f_ke^{s\lambda_k}+\gamma_1\sum_{k=n-1}^{\infty}f_ke^{s(\lambda_k+h)}+\ldots+$$
$$+\gamma_{n-1}\sum_{k=1}^{\infty}f_ke^{(\lambda_k+(n-1)h)}+\gamma_n\sum_{k=1}^{\infty}f_ke^{s(\lambda_k+nh)}\equiv 0,$$

i.e.,

$$(\lambda_n^n+\gamma_0)f_ne^{s\lambda_n}+\sum_{k=n+1}^{\infty}(\lambda_k^n+\gamma_0)f_ke^{s\lambda_k}+\gamma_1f_{n-1}e^{s(\lambda_{n-1}+h)}+\gamma_1\sum_{k=n}^{\infty}f_ke^{s(\lambda_k+h)}+\ldots+$$
$$+\gamma_{n-1}f_1e^{(\lambda_1+(n-1)h)}+\gamma_{n-1}\sum_{k=2}^{\infty}f_ke^{(\lambda_k+(n-1)h)}+\gamma_n\sum_{k=1}^{\infty}f_ke^{s(\lambda_k+nh)}\equiv 0,$$

whence

$$\lambda_n=nh,\quad f_n=-\frac{\gamma_1f_{n-1}+\gamma_2f_{n-2}+\ldots+\gamma_{n-1}f_1}{(nh)^n+\gamma_0}$$

and

$$\sum_{k=n+1}^{\infty}(\lambda_k^n+\gamma_0)f_ke^{s\lambda_k}+\gamma_1\sum_{k=n}^{\infty}f_ke^{s(\lambda_k+h)}+\ldots+ \tag{7}$$
$$+\gamma_{n-1}\sum_{k=2}^{\infty}f_ke^{(\lambda_k+(n-1)h)}+\gamma_n\sum_{k=1}^{\infty}f_ke^{s(\lambda_k+nh)}\equiv 0.$$

Since (7) implies

$$(\lambda_{n+1}^n+\gamma_0)f_{n+1}e^{s\lambda_{n+1}}+\sum_{k=n+2}^{\infty}(\lambda_k^n+\gamma_0)f_ke^{s\lambda_k}+\gamma_1f_ne^{s(n+1)h}+\gamma_1\sum_{k=n+1}^{\infty}f_ke^{s(\lambda_k+h)}+$$
$$+\ldots+\gamma_{n-1}f_2e^{s(n+1)h}+\gamma_{n-1}\sum_{k=3}^{\infty}f_ke^{(\lambda_k+(n-1)h)}+\gamma_nf_1e^{s(n+1)h}+$$
$$+\gamma_n\sum_{k=2}^{\infty}f_ke^{s(\lambda_k+nh)}\equiv 0,$$

we have

$$\lambda_{n+1}=(n+1)h,\quad f_{n+1}=-\frac{\gamma_1f_n+\gamma_2f_{n-1}+\ldots+\gamma_{n-1}f_2+\gamma_nf_1}{((n+1)h)^n+\gamma_0}$$

and

$$\sum_{k=n+2}^{\infty}(\lambda_k^n+\gamma_0)f_ke^{s\lambda_k}+\gamma_1\sum_{k=n+1}^{\infty}f_ke^{s(\lambda_k+h)}+\ldots+$$
$$+\gamma_{n-1}\sum_{k=3}^{\infty}f_ke^{(\lambda_k+(n-1)h)}+\gamma_n\sum_{k=2}^{\infty}f_ke^{s(\lambda_k+nh)}\equiv 0,$$

whence, as above, we get

$$\lambda_{n+2} = (n+2)h, \quad f_{n+2} = -\frac{\gamma_1 f_{n+1} + \gamma_2 f_n + \ldots + \gamma_{n-1} f_3 + \gamma_n f_2}{((n+2)h)^n + \gamma_0}$$

and

$$\sum_{k=n+3}^{\infty} (\lambda_k^n + \gamma_0) f_k e^{s\lambda_k} + \gamma_1 \sum_{k=n+2}^{\infty} f_k e^{s(\lambda_k+h)} + \ldots +$$
$$+\gamma_{n-1} \sum_{k=4}^{\infty} + f_k e^{s(\lambda_k+(n-1)h)} + \gamma_n \sum_{k=3}^{\infty} f_k e^{s(\lambda_k+nh)} \equiv 0.$$

Continuing the process, we will come to the formulas

$$\lambda_{n+j} = (n+j)h, \quad f_{n+j} = -\frac{\gamma_1 f_{n+j-1} + \gamma_2 f_{n+j-2} + \ldots + \gamma_{n-1} f_{j+1} + \gamma_n f_j}{((n+j)h)^n + \gamma_0}.$$

So, we proved that $\lambda_k = kh$ for all $k \geq 1$, $f_1 = \dfrac{a}{h^n + \gamma_0}$,

$$f_k = -\frac{1}{(hk)^n + \gamma_0} \sum_{j=1}^{k-1} \gamma_j f_{k-j}$$

for $2 \leq k \leq n$ and

$$f_k = -\frac{1}{(hk)^n + \gamma_0} \sum_{j=1}^{n} \gamma_j f_{k-j}$$

for $k > n$.

Thus, the following statement is true.

Lemma 1. *If $h^n + \gamma_0 > 0$ then differential equation (4) has a solution*

$$F(s) = \frac{a}{h^n + \gamma_0} e^{sh} + \sum_{k=2}^{\infty} f_k e^{skh}, \tag{8}$$

where $f_k = -\dfrac{1}{(hk)^n + \gamma_0} \displaystyle\sum_{j=1}^{\min\{k-1,n\}} \gamma_j f_{k-j}$ for $k \geq 2$.

3. Pseudostarlikeness

Using Lemma 1, we can find the conditions under which solution (5) of the equation (4) will be pseudostarlike or pseudoconvex. We will limit ourselves to considering the case $n = 3$. We assume also that $a = h^3 + \gamma_0 > 0$. Then solution (5) of equation (4) has the form (8),

$$f_1 = 1, \quad f_2 = -\frac{\gamma_1 f_1}{(2h)^3 + \gamma_0}, \quad f_3 = -\frac{\gamma_1 f_2 + \gamma_2 f_1}{(3h)^3 + \gamma_0}$$

and for $k \geq 4$

$$f_k = -\frac{\gamma_1 f_{k-1} + \gamma_2 f_{k-2} + \gamma_3 f_{k-3}}{(hk)^3 + \gamma_0}.$$

Therefore,

$$
\begin{aligned}
&\sum_{k=2}^{\infty}(kh-\alpha)|f_k| = (2h-\alpha)|f_2| + (3h-\alpha)|f_3| + \\
&+\sum_{k=4}^{\infty}(kh-\alpha)|f_k| \le (2h-\alpha)|f_2| + (3h-\alpha)|f_3| + \\
&+\sum_{k=3}^{\infty}\frac{h(k+1)-\alpha}{(h(k+1))^3+\gamma_0}|\gamma_1||f_k| + \sum_{k=2}^{\infty}\frac{(k+2)h-\alpha}{(h(k+2))^3+\gamma_0}|\gamma_2||f_k| + \\
&+\sum_{k=1}^{\infty}\frac{(k+3)h-\alpha}{(h(k+3))^3+\gamma_0}|\gamma_3||f_k| = \\
&= (2h-\alpha)|f_2| + (3h-\alpha)|f_3| + \frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_1||f_3| + \\
&+\sum_{k=2}^{3}\frac{(k+2)h-\alpha}{(h(k+2))^3+\gamma_0}|\gamma_2||f_k| + \sum_{k=1}^{3}\frac{(k+3)h-\alpha}{(h(k+3))^3+\gamma_0}|\gamma_3||f_k| + \\
&+\sum_{k=4}^{\infty}\left(\frac{h(k+1)-\alpha}{(h(k+1))^3+\gamma_0}|\gamma_1| + \frac{(k+2)h-\alpha}{(h(k+2))^3+\gamma_0}|\gamma_2| + \frac{(k+3)h-\alpha}{(h(k+3))^3+\gamma_0}|\gamma_3|\right)|f_k| = \\
&= \sum_{k=2}^{\infty}\left(\frac{h(k+1)-\alpha}{(h(k+1))^3+\gamma_0}|\gamma_1| + \frac{(k+2)h-\alpha}{(h(k+2))^3+\gamma_0}|\gamma_2| + \frac{(k+3)h-\alpha}{(h(k+3))^3+\gamma_0}|\gamma_3|\right)|f_k| - \\
&-\left(\frac{3h-\alpha}{(3h)^3+\gamma_0}|\gamma_1| + \frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_2| + \frac{5h-\alpha}{(5h)^3+\gamma_0}|\gamma_3|\right)|f_2| - \\
&-\left(\frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_1| + \frac{5h-\alpha}{(5h)^3+\gamma_0}|\gamma_2| + \frac{6h-\alpha}{(6h)^3+\gamma_0}|\gamma_3|\right)|f_3| + \\
&+(2h-\alpha)|f_2| + (3h-\alpha)|f_3| + \frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_1||f_3| + \\
&+\frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_2||f_2| + \frac{5h-\alpha}{(5h)^3+\gamma_0}|\gamma_2||f_3| + \frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_3||f_1| + \\
&+\frac{5h-\alpha}{(5h)^3+\gamma_0}|\gamma_3||f_2| + \frac{6h-\alpha}{(6h)^3+\gamma_0}|\gamma_3||f_3| = \\
&= \sum_{k=2}^{\infty}A_k(kh-\alpha)|f_k| + B,
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
A_k = &\frac{(h(k+1)-\alpha)|\gamma_1|}{(hk-\alpha)((h(k+1))^3+\gamma_0)} + \\
&+\frac{((k+2)h-\alpha)|\gamma_2|}{(hk-\alpha)((h(k+2))^3+\gamma_0)} + \\
&+\frac{((k+3)h-\alpha)|\gamma_3|}{(hk-\alpha)((h(k+3))^3+\gamma_0)}
\end{aligned}
$$

and

$$\begin{gathered}B=\frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_3||f_1|+(2h-\alpha)|f_2|+\frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_2||f_2|+\frac{5h-\alpha}{(5h)^3+\gamma_0}|\gamma_3||f_2|-\\ -\left(\frac{3h-\alpha}{(3h)^3+\gamma_0}|\gamma_1|+\frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_2|+\frac{5h-\alpha}{(5h)^3+\gamma_0}|\gamma_3|\right)|f_2|+(3h-\alpha)|f_3|+\\ +\frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_1||f_3|+\frac{5h-\alpha}{(5h)^3+\gamma_0}|\gamma_2||f_3|+\frac{6h-\alpha}{(6h)^3+\gamma_0}|\gamma_3||f_3|-\\ -\left(\frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_1|+\frac{5h-\alpha}{(5h)^3+\gamma_0}|\gamma_2|+\frac{6h-\alpha}{(6h)^3+\gamma_0}|\gamma_3|\right)|f_3|=\\ \frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_3||f_1|+(2h-\alpha)|f_2|-\frac{3h-\alpha}{(3h)^3+\gamma_0}|\gamma_1||f_2|+(3h-\alpha)|f_3|\le\\ \le\frac{4h-\alpha}{(4h)^3+\gamma_0}|\gamma_3||f_1|+(2h-\alpha)|f_2|-\frac{3h-\alpha}{(3h)^3+\gamma_0}|\gamma_1||f_2|+\\ +(3h-\alpha)\left(\frac{|\gamma_1||f_2|}{(3h)^3+\gamma_0}+\frac{|\gamma_2||f_1|}{(3h)^3+\gamma_0}\right)=\\ =Q:=\frac{(4h-\alpha)|\gamma_3|}{(4h)^3+\gamma_0}+\frac{(3h-\alpha)|\gamma_2|}{(3h)^3+\gamma_0}+\frac{(2h-\alpha)|\gamma_1|}{(2h)^3+\gamma_0}.\end{gathered}$$

Since $\alpha<h$, for $k\ge 2$ and $1\le j\le 3$ we have $\dfrac{h(k+j)-\alpha}{hk-\alpha}\le 1+j$. Therefore,

$$A_k\le A:=\frac{2|\gamma_1|}{(3h)^3+\gamma_0}|+\frac{3|\gamma_2|}{(4h)^3+\gamma_0}+\frac{4|\gamma_3|}{(5h)^3+\gamma_0}.$$

Suppose that $A<1$. Then (9) implies

$$\sum_{k=2}^{\infty}(kh-\alpha)|f_k|\le A\sum_{k=2}^{\infty}(kh-\alpha)|f_k|+Q,$$

whence

$$\sum_{k=2}^{\infty}(kh-\alpha)|f_k|\le\frac{Q}{1-A}\tag{10}$$

In view of (3) and (10) function (8) is pseudostarlike if $Q\le(1-A)(h-\alpha)$. Therefore, the following theorem is true.

Theorem 1. *Let $n=3$, $a=h^3+\gamma_0$, $h+\gamma_0>0$ and*

$$\begin{gathered}\frac{(4h-\alpha)|\gamma_3|}{(4h)^3+\gamma_0}+\frac{(3h-\alpha)|\gamma_2|}{(3h)^3+\gamma_0}+\frac{(2h-\alpha)|\gamma_1|}{(2h)^3+\gamma_0}\le\\ \le\left(1-\frac{2|\gamma_1|}{(3h)^3+\gamma_0}+\frac{3|\gamma_2|}{(4h)^3+\gamma_0}+\frac{4|\gamma_3|}{(5h)^3+\gamma_0}\right)(h-\alpha).\end{gathered}$$

Then differential equation (4) *has solution* (8) *pseudostarlike of the order $\alpha\in[0,h)$.*

4. Pseudoconvexity

In [7] it is proved that if

$$\sum_{k=2}^{\infty}\lambda_k(\lambda_k-\alpha)|f_k|\le h(h-\alpha)$$

then function (8) is pseudoconvex of the order $\alpha\in[0,\,h)$. Now, as above, we have

$$\sum_{k=2}^{\infty}\lambda_k(\lambda_k-\alpha)|f_k|=\sum_{k=2}^{\infty}kh(kh-\alpha)|f_k|=$$

$$=2h(2h-\alpha)|f_2|+3h(3h-\alpha)|f_3|+\sum_{k=4}^{\infty}kh(kh-\alpha)|f_k|\le$$

$$\le\sum_{k=2}^{\infty}\left(\frac{h(k+1)(h(k+1)-\alpha)}{(h(k+1))^3+\gamma_0}|\gamma_1|+\frac{h(k+2)((k+2)h-\alpha)}{(h(k+2))^3+\gamma_0}|\gamma_2|+\right.$$

$$(11)\qquad \left.+\frac{h(k+3)((k+3)h-\alpha)}{(h(k+3))^3+\gamma_0}|\gamma_3|\right)|f_k|+\frac{4h(4h-\alpha)|\gamma_3|}{(4h)^3+\gamma_0}+$$

$$+\frac{3h(3h-\alpha)|\gamma_2|}{(3h)^3+\gamma_0}+\frac{2h(2h-\alpha)|\gamma_1|}{(2h)^3+\gamma_0}=$$

$$=\sum_{k=2}^{\infty}A_k^*\lambda_k(\lambda_k-\alpha)|f_k|+Q^*,$$

where

$$A_k^*=\frac{h(k+1)(h(k+1)-\alpha)}{(hk-\alpha)((h(k+1))^3+\gamma_0)}|\gamma_1|+$$
$$+\frac{h(k+2)((k+2)h-\alpha)}{(hk-\alpha)((h(k+2))^3+\gamma_0)}|\gamma_2|+$$
$$+\frac{h(k+3)((k+3)h-\alpha)}{(hk-\alpha)((h(k+3))^3+\gamma_0)}|\gamma_3|$$

and

$$Q^*=\frac{4h(4h-\alpha)|\gamma_3|}{(4h)^3+\gamma_0}+\frac{3h(3h-\alpha)|\gamma_2|}{(3h)^3+\gamma_0}+\frac{2h(2h-\alpha)|\gamma_1|}{(2h)^3+\gamma_0}.$$

Since $\dfrac{h(k+j)-\alpha}{hk-\alpha}\le 1+j$ and

$$\frac{h(k+j)}{(h(k+j))^3+\gamma_0}\le\frac{h(k+j)}{(h(k+j))^3-h}\le\frac{h(2+j)}{(h(2+j))^3-h},$$

we get

$$A_k^*\le A^*=\frac{6h}{(3h)^3-h}|\gamma_1|+\frac{12h}{(4h)^3-h}|\gamma_2|+\frac{20h}{(5h)^3-h}|\gamma_3|$$

If $A^*<1$ then (11) implies

$$(1-A^*)\sum_{k=2}^{\infty}kh(kh-\alpha)|f_k|\le Q^*,$$

i.e., if $Q^* \le (1 - A^*)h(h-\alpha)$ then function (2) is pseudoconvex of the order $\alpha \in [0, h)$. Therefore, the following theorem is true.

Theorem 2. *Let $n = 3$, $a = h^3 + \gamma_0$, $h + \gamma_0 > 0$ and*

$$\frac{4h(4h-\alpha)|\gamma_3|}{(4h)^3+\gamma_0} + \frac{3h(3h-\alpha)|\gamma_2|}{(3h)^3+\gamma_0} + \frac{2h(2h-\alpha)|\gamma_1|}{(2h)^3+\gamma_0} \le$$
$$\le \left(1 - \frac{6h}{(3h)^3-h}|\gamma_1| + \frac{12h}{(4h)^3-h}|\gamma_2| + \frac{20h}{(5h)^3-h}|\gamma_3|\right) h(h-\alpha).$$

Then differential equation (4) *has solution* (8) *pseudoconvex of the order $\alpha \in [0, h)$.*

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Стаття: надійшла до редколегії 07.04.2022
доопрацьована 31.05.2022
прийнята до друку 22.06.2022

ПРО РОЗВ'ЯЗКИ ОДНОГО УЗАГАЛЬНЕННЯ ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ ТИПУ ШАХА

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Ряд Діріхле $F(s) = e^{hs} + \sum_{k=2}^{\infty} f_k e^{s\lambda_k}$ з показниками $0 < h < \lambda_k \uparrow +\infty$ і абсцисою абсолютної збіжності $\sigma_a[F] \geq 0$ називається псевдозірковим порядку $\alpha \in [0, h)$ в $\Pi_0 = \{s : \operatorname{Re} s < 0\}$, якщо $\operatorname{Re}\{F'(s)/F(s)\} > \alpha$ для всіх $s \in \Pi_0$. Подібно, функція F називається псевдоопуклою порядку $\alpha \in [0, h)$, якщо $\operatorname{Re}\{F''(s)/F'(s)\} > \alpha$ для всіх $s \in \Pi_0$.
Розглядається рівняння $\frac{d^n w}{ds^n} + \left(\sum_{j=0}^{n} \gamma_j e^{hjs}\right) w = ae^{hs}$, де $n \geq 3$, $h > 0$ і $a \neq 0$. Доведено таке: якщо $h^n + \gamma_0 > 0$, то це рівняння має розв'язок $F(s) = \frac{a}{h^n + \gamma_0} e^{sh} + \sum_{k=2}^{\infty} f_k e^{skh}$, де $f_k = -\frac{1}{(hk)^n + \gamma_0} \sum_{j=1}^{\min\{k-1,n\}} \gamma_j f_{k-j}$ для $k \geq 2$ (Lemma 1). Для $n = 3$, $a = h^3 + \gamma_0$ знайдено умови на параметри γ_j, за яких функція F є псевдозірковою (Theorem 1) або псевдоопуклою порядку $\alpha \in [0, h)$ (Theorem 2).

Ключові слова: диференціальне рівняння, ряд Діріхле, псевдозірковість, псевдоопуклість.