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ON QUASI-PRIME SUBSEMIMODULES

Ivanna MELNYK

Ivan Franko National University of Lviv,
Universytetska Str., 1, 79000, Lviv, UKRAINE
e-mail: ivannamelnyk@yahoo.com

The paper is devoted to the investigation of the notion of a quasi-prime subsemimodule of the differential semimodule, which generalizes the notion of quasi-prime ideal of a ring. Some natural properties of quasi-prime sebsemimodules are investigated. The interrelation between quasi-prime subsemimodules and different types of differential subsemimodules of differential semimodules is studied.

Key words: Semimodule, semiring, semimodule derivation, semiring derivation, differential semiring, differential ideal, prime subsemimodule, quasi-prime subsemimodule.

1. Introduction

The notion of a derivation for semirings is defined in [3] as an additive map satisfying the Leibnitz rule. Recently in [2], [11] the authors investigated some natural properties of semiring and semimodule derivations, differential semirings, i.e. semirings considered together with a derivation, and differential semimodules.

Keigher [6], [7] introduced and studied the notion of a quasi-prime ideal of differential rings. Its generalizations to differential modules, semirings and semimodules were investigated by different authors, e.g. [14], [13], [12], [11], [10].

A subsemimodule P of a subsemimodule M is called prime if for any ideal I of R and any submodule N of M the inclusion $IN \subseteq P$ implies $N \subseteq P$ or $I \subseteq (P : M)$. Prime subsemimodules of semimodules over semirings were introduced and extensively investigated in [1].

The concept of a differentially prime ideal of a differential ring was introduced in [8]. Differentially prime submodules of modules over associative rings were studied in [10].

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The development of semiring and semimodule theory over the years motivates a further study into properties of differential semirings, differential semimodules, semiring ideals and subsemimodules defined by similar conditions. The aim of this paper is to investigate quasi-prime subsemimodules of differential semimodules, and their interrelation with other types of subsemimodules.

For the sake of completeness some definitions and properties used in the paper will be given here. For more information see [3], [4], [9], [5].

Throughout the paper, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ the set of non-negative integers.

Let R be a nonempty set and let $+$ and $\cdot$ be binary operations on R. An algebraic system $(R, +, \cdot)$ is called a *semiring* if $(R, +, 0)$ is a commutative monoid, $(R, \cdot)$ is a semigroup and multiplication distributes over addition from either side. A semiring $(R, +, \cdot)$ is said to be *commutative* if $\cdot$ is commutative on R. A semiring which is not a ring is called a *proper semiring*.

Zero $0_R \in R$ is called (*multiplicatively*) *absorbing* if

$$a \cdot 0_R = 0_R \cdot a = 0$$

for all $a \in R$. Note that $0_R \in R$ cannot be additively absorbing when R contains more than one element. An element $1_R \in R$ is called *identity* if $a \cdot 1_R = 1_R \cdot a = a$ for all $a \in R$. Suppose $1_R \neq 0_R$, otherwise $R = \{0\}$ if zero is absorbing. If 1=0, then

$$a = a \cdot 1 = a = \cdot 0 = 0$$

for any $a \in R$.

A subset S of R closed under addition and multiplication is called a *subsemiring* of R. The *center* of a semiring R is a set

$$Z(R) = \{r \in R \mid rs = sr, \forall s \in R\}.$$

It is a subsemiring of R. Since $0 \in Z(R)$, $Z(R) \neq \varnothing$. An element $r \in Z(R)$ is called *central*. A semiring R is commutative if $Z(R) = R$.

A *left ideal* of a semiring R is a nonempty set $I \neq R$ which is closed under $+$ and satisfying the following conditions $ra \in I$ for all $a \in I$, $r \in R$. Similarly we can define right ideal and two-sided ideal of a semiring. An ideal I of a semiring R is called *subtractive* (or *k-ideal*) if $a \in I$ and $a + b \in I$ implies $b \in I$.

An ideal I of the semiring R is called *strong* if $a + b \in I$ implies $a \in I$ and $b \in I$ for every $a, b \in R$. Every strong ideal is subtractive.

Let R be a semiring with $1_R \neq 0_R$. A *left semimodule over a semiring* R (or *left R-semimodule*) is a nonempty set M together with two operations $+\colon M \times M \to M$ and $\cdot\colon R \times M \to M$ such that the following conditions hold:

(1) $(M, +)$ is a commutative monoid with 0_M;
(2) $(M, \cdot)$ is a semigroup;
(3) $(r + s)m = rm + sm$ for all $r, s \in R$, $m \in M$;
(4) $r(m_1 + m_2) = rm_1 + rm_2$ for all $r \in R$, $m_1, m_2 \in M$;
(5) $0_R \cdot m = r \cdot 0_M = 0_M$ for all $r \in R$ and $m \in M$;
(6) $1_R \cdot m = m$ for all $m \in M$.

A subset N of an R-semimodule M is called a *subsemimodule* of M if N itself is a semimodule with respect to the operations for M, i. e. if $m + n \in N$ and $rm \in N$ for any

$m, n \in N$, and $r \in R$. A subsemimodule N of an R-semimodule M is called *subtractive* or *k-subsemimodule* if $m_1 \in N$ and $m_1 + m_2 \in N$ follow $m_2 \in N$. So $\{0_M\}$ is a subtractive subsemimodule of M.

A subsemimodule N of the semimodule M is called *strong* if $m_1 + m_2 \in N$ implies $m_1 \in N$ and $m_2 \in N$ for every $m_1, m_2 \in N$. Every strong subsemimodule is clearly subtractive.

Let R be a semiring. A map $\delta\colon R \to R$ is called a *derivation on* R [3] if

$$\delta(a+b) = \delta(a) + \delta(b) \quad \text{and} \quad \delta(ab) = \delta(a)\, b + a\delta(b)$$

for any $a, b \in R$.

A semiring R equipped with a derivation δ is called a *differential semiring* with respect to the derivation δ (or *δ-semiring*), and denoted by (R, δ) [2]. A derivation of a semiring R is called *trivial* if it sends all a in R to 0_R. A semiring is called *differentially trivial* if it has no non-trivial derivation.

For an element $r \in R$ denote by $r^{(0)} = r$, $r' = \delta(r)$, $r'' = \delta(r')$, $r^{(n)} = \delta\left(r^{(n-1)}\right)$, for any $n \in \mathbb{N}_0$. An ideal I of the semiring R is called *differential* if the set I is differentially closed under δ, i.e. $\delta(r) \in I$ for any $r \in I$. The set $r^{(\infty)} = \{r^{(n)} | n = 0, 1, 2, 3 \ldots\}$ of all derivations of an element $r \in R$ is differentially closed. The ideal $[r] = \left(r^{(\infty)}\right) = (r, r', r'', \ldots)$ of R, generated by the set $r^{(\infty)}$, is differentially generated by $r \in R$; it is the smallest differential ideal containing the element $r \in R$ [11].

Let M be a left semimodule over the semiring R. A map $d\colon M \to M$ is called a *derivation* of the semimodule M, associated with the semiring derivation $\delta\colon R \to R$ (or a δ-derivation) if the following conditions hold:

(1) $d(m+n) = d(m) + d(n)$ for any $m, n \in M$;
(2) $d(rm) = \delta(r)\, m + r d(m)$ for any $m \in M$, $r \in R$.

A left R-semimodule M together with a derivation $d\colon M \to M$ is called a *differential semimodule* (or *d-δ-semimodule*) and denoted by (M, d).

A subsemimodule N of the R-semimodule M is called *differential* if $d(N) \subseteq N$. Any differential semimodule has two trivial differential subsemimodules: $\{0_M\}$ and itself.

For an element $m \in M$ denote by $m^{(0)} = m$, $m' = d(m)$, $m'' = d(m')$, $m^{(n)} = d\left(m^{(n-1)}\right)$, for any $n \in \mathbb{N}_0$. Moreover, let $m^{(\infty)} = \{m^{(n)} | n \in \mathbb{N}_0\}$. It is easy to see that the set $m^{(\infty)}$ is differentially closed. The subsemimodule $[m] = \left(m^{(\infty)}\right) = (m, m', m'', \ldots)$ is the smallest differential subsemimodule of M containing $m \in M$.

2. Quasi-prime differential semimodules

For a subset X of M its *differential* $X_\#$ is defined to be the set

$$X_\# = \left\{ x \in M \;\middle|\; x^{(n)} \in X \text{ for all } n \in \mathbb{N}_0 \right\}.$$

Proposition 1. *Let X, Y, X_i, $i \in I$, be subsets of M, let A be a subset of R. The following properties hold:*

(1) $X_\# \subseteq X$;
(2) $(X_\#)_\# = X_\#$;
(3) $X_\# = X$ *if and only if* $\forall x \in X\ d(x) \in X$;
(4) *If* $X \subseteq Y$ *then* $X_\# \subseteq Y_\#$;

(5) $\left(\bigcap_{i\in I} X_i\right)_{\#} = \bigcap_{i\in I} (X_i)_{\#}$;
(6) $\bigcup_{i\in I} (X_i)_{\#} \subseteq \left(\bigcup_{i\in I} X_i\right)_{\#}$;
(7) $X_{\#} + Y_{\#} \subseteq (X+Y)_{\#}$;
(8) $A_{\#} \cdot X_{\#} \subseteq (AX)_{\#}$.

Proposition 2. *Let M_1 and M_2 be differential semimodules, let $f\colon M_1 \to M_2$ be a differential semimodule homomorphism, and let X be a subset of M_1, Y be a subset of M_2. The following properties are true:*
(1) *$f(X_{\#}) \subseteq (f(X))_{\#}$;*
(2) *If f is a monomorphism, then $f(X_{\#}) = (f(X))_{\#}$;*
(3) *If f is an epimorphism, then $f^{-1}(Y_{\#}) = \left(f^{-1}(Y)\right)_{\#}$.*

The proofs of Proposotions 1 and 2 are straightforward when using standard set-theoretic technique and corresponding definitions, therefore are omitted.

Proposition 3. (1) *If N is an subsemimodule of M, then $N_{\#}$ is a differential subsemimodule of M.*
(2) *If N is a strong subsemimodule of M, then $N_{\#}$ is a differential strong semimodule of M.*
(3) *If N is a subtractive subsemimodule of M, then $N_{\#}$ is a differential subtractive subsemimodule of M.*
(4) *$N_{\#} = N$ if and only if N is a differential subsemimodule of M.*

Proof. (1) Let $x, y \in N_{\#}$. Then $x^{(n)} \in N$ and $y^{(n)} \in N$ for any $n \in \mathbb{N}_0$, thus $(x+y)^{(n)} = x^{(n)} + y^{(n)} \in N$. Hence $x + y \in N_{\#}$. If $x \in N_{\#}$ and $r \in R$ then $x^{(k)} \in N$ for any $k \in \mathbb{N}_0$. By the Leibnitz rule $(rx)^{(n)} = \sum_{k=0}^{n} C_n^k r^{(n-k)} x^{(k)} \in N$, which follows $rx \in N_{\#}$. Hence $N_{\#}$ is a subsemimodule of M. The subsemimodule $N_{\#}$ is differential since $N_{\#}$ is differentially closed for any subset N of M.

(2) Suppose $x + y \in N_{\#}$. Then $(x+y)^{(n)} = x^{(n)} + y^{(n)} \in N$ for any $n \in \mathbb{N}_0$. The subsemimodule N being strong follows that $x^{(n)} \in N$ and $y^{(n)} \in N$. Thus $x \in N_{\#}$ and $y \in N_{\#}$, so $N_{\#}$ is strong.

(3) Follows from (2) since every strong subsemimodule is subtractive. (4) Follows from (1). (5) follows from Proposition 1. □

Proposition 4. *Let N be an arbitrary subtractive subsemimodule of M and let K be a differential subsemimodule of M. Then the following equality holds:*

$$(N : K)_{\#} = (N_{\#} : K).$$

Proof. Suppose $r \in (N : K)_{\#}$. Then $r^{(n)} \in (N : K)$ for all $n \in \mathbb{N}_0$, so $r^{(n)} m \in N$ for all $m \in K$. Since K is differentially closed, then $rm' \in N$. Therefore $(rm)' = r'm + rm' \in N$. By induction we obtain that $(rm)^{(n)} \in N$ for all $n \in \mathbb{N}_0$. Hence $r \in (N_{\#} : K)$.

Conversely, let $r \in (N_{\#} : K)$. Then $(rm)^{(n)} \in N$ for all $m \in K$, $n \in \mathbb{N}_0$, i.e., $rm \in N$, $(rm)' = r'm + rm' \in N$, $(rm)'' = r''m + 2r'm' + rm'' \in N$, …, $(rm)^{(n)} =$

$\sum_{k=0}^{n} C_n^k r^{(n-k)} m^{(k)} \in N$. Since K is differentially closed, by subtractiveness of N, $(rm)' \in N$ and $rm' \in N$ follow $r'm \in N$. We may infer by induction that $r^{(n)}m \in N$ for all $m \in K$, $n \in \mathbb{N}_0$. It follows that $r^{(n)} \in (N : K)$, i.e., $r \in (N : K)_{\#}$. □

Proposition 5. *Let N be an arbitrary subtractive subsmimodule of M and let I be a differential ideal of R. Then the following equality holds:*

$$(N : I)_{\#} = (N_{\#} : I).$$

Proof. Take $m \in (N : I)_{\#}$. Then $m^{(n)} \in (N : I)$ for all $n \in \mathbb{N}_0$, so $am^{(n)} \in N$ for all $a \in I$. We obtain that $(am)^{(n)} \in N$ for all $n \in \mathbb{N}_0$. Hence $m \in (N_{\#} : I)$.

If $m \in (N_{\#} : I)$, then $(am)^{(n)} \in N$ for all $a \in I$, $n \in \mathbb{N}_0$. We conclude that $am^{(n)} \in N$ for all $a \in I$, $n \in \mathbb{N}_0$. Therefore, $m^{(n)} \in (N : I)$, i. e. $m \in (N : I)_{\#}$. □

Corollary 1. *If N is a subtractive subsemimodule of M and A is a differentially closed subset of R, then $(N_{\#} : A)$ is a differential subtractive subsemimodule of M.*

Corollary 2. *Let N be an arbitrary subtractive subsemimodule of M and $a \in R$. Then $\left(N : a^{(\infty)}\right)_{\#} = \left(N_{\#} : a^{(\infty)}\right)$.*

A non-empty subset S of the semiring R is called an *m-system* of R if for every $s, t \in S$ there exists $r \in R$ such that $srt \in S$.

Let S be an m-system in R. A non-empty subset T of the semimodule M is called an *Sm-system* of M if for every $s \in S$ and $t \in T$ there exists $r \in R$ such that $srt \in T$. A non-empty subset T of the semimodule M is called an *Smd-system* in M if for every $s \in S$ and $k \in T$ there exist $r \in R$ and $n \in \mathbb{N}_0$ such that $srt^{(n)} \in T$.

A differential subsemimodule N of the left differential semimodule M is called *quasi-prime* if there exists an Sm-system T of M such that N is maximal differential subsemimodule satisfying $N \cap T = \varnothing$.

For instance, every prime differential subsemimodule is quasi-prime, since the complement of the prime subsemimodule is an Sm-system, where the role of S is played by the set $\{1\}$.

In the case of a regular semimodule, we obtain the notion of a quasi-prime semiring ideal. For differential semiring ideals it is known that every maximal among differential ideals not meeting some m-system is quasi-prime. The following lemma establishes the analogue of this fact for differential semimodules:

Proposition 6. *Let M be a differential semimodule. If Q is a maximal differential subsemimodule of M, then Q is quasi-prime.*

Proof. Let Q be a maximal amongst differential subsemimodules of M, $S = U(R)$ be the group of units of R and $T = M \backslash Q$. Then T is an Sm-system and Q is a maximal amongst differential submodules disjoint from T. Hence Q is a quasi-prime submodule. □

Corollary 3. *Let M be a differential semimodule. If P is a prime subsemimodule of M then the differential subsemimodule $P_{\#}$ is quasi-prime.*

A differential subtractive subsemimodule P of M is called *differentially prime* if for any differential subtractive ideal I of R and any differential subtractive subsemimodule N of M, $IN \subseteq P$ follows $N \subseteq P$ or $I \subseteq (P : M)$.

Theorem 1. *Let R be a differential semiring, S be and m-system in R, and let M be a differential semimodule over R, let T be an Sdm-system of M, and let N be a differential submodule of M such that $N \cap T = \varnothing$. Then the maximal differential subsemimodule P among differential subsemimodules of M not meeting T and containing N exists and is differentially prime.*

Proof. The existence follows from Zorn's lemma. Let I be a differential ideal of R, K be a differential subsemimodule of M such that $IK \subseteq P$. Then $K \subseteq P$ or $I \subseteq (P : M)$. P being maximal follows the existence of $s \in S$ and $x \in M$ such that $s \in (I + (P : M)) \cap S$ and $x \in (K+P) \cap T$. Then there exist $r \in R$ and $n \in \mathbb{N}_0$ such that $srx^{(n)} \in T$. Moreover, $s = a + b$ for some $a \in I$, $b \in (P : M)$, and $x = k + l$ for some $k \in K$, $l \in P$. Then $srx^{(n)} = (a + b)r(k + l)^{(n)} \in N$. Contradiction. □

Dually we can obtain the following

Theorem 2. *Let N be a differential submodule of M such that $N \cap T = \varnothing$ for some Sdm-system $T$of M. Then T is contained in some maximal Sdm-system T' such that $N \cap T' = \varnothing$.*

Let P be a differentially prime subsemimodule, K be any differential subsemimodule. P will be called *minimal over* K if $K \subseteq N \subseteq P$ follows $N = P$ for any differentially prime subsemimodule N of M.

Theorem 3. *Let K be any differential subsemimodule pf M. A subset $P \neq \varnothing$ is minimal over K if and only if $M \subseteq P$ is a maximal Sdm-system not meeting K.*

Proof. Follows from Theorems 1 and 2. □

Theorem 4. *Let M be a differential semimodule satisfying the ascending chain condition for differential subsemimodules. For every differential substractive subsemimodule N of M the following conditions are equivalent:*

(1) *N is a differentially prime subtractive subsemimodule;*
(2) *N is a quasi-prime subtractive subsemimodule;*
(3) *$N = P_{\#}$ for some prime subtractive subsemimodule P of M.*

Proof. (1) $\Longrightarrow$ (2) Let N be some differentially prime subtractive subsemimodule of M. Then the set $M \backslash N$ is a Sdm-system for some dm-system S of the semiring R. Since N is maximal differential subtractive subsemimodule disjoint from $T = M \backslash N$, then it is quasi-prime.

(2) $\Longrightarrow$ (3) Let N be a subsemimodule of M, maximal among differential subsemimodules disjoint from the Sm-system X, and let K be maximal subsemimodule disjoint from X and containing N. Then K is a prime subsemimodule in M. Since N is a differential subsemimodule of M, then $N \subseteq K_{\#}$. The converse inclusion implies due to maximality of the differential subsemimodule N among those disjoint from X. Therefore, $N = K_{\#}$.

(3) $\Longrightarrow$ (1) Let $N = P_{\#}$ for some prime subtractive subsemimodule R of M. Then N is maximal amongst differential subsemimodules of M contained in P. Let $T = M \backslash P$. Clearly, T is an Sm-system for some m-system of the semiring R. Denote by K the intersection of all Sdm-systems of the semimodule M, which contain T. Then K is the least Sdm-system of those containing T. Hence N is a differentially prime subsemimodule

of M. It remains to verify that $N = M\backslash K$. Since $M\backslash K$ is disjoint from T, then $M\backslash K \subseteq P$, and due to the fact that $M\backslash K$ is a differential subsemimodule of M, we have the inclusion $M\backslash K \subseteq N$. Taking into consideration the minimality of the set K, we obtain that the set $M\backslash K$ is a maximal subsemimodule among the differential subsemimodules of N. Thus, $M\backslash K = N$. □

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ПРО КВАЗІПЕРВИННІ ПІДНАПІВМОДУЛІ

Іванна Мельник

Львівський національний університет імені Івана Франка,
вул. Університетська 1, 79000, м. Львів
e-mail: ivannamelnyk@yahoo.com

Досліджено поняття квазіпервинного піднапівмодуля диференціального напівмодуля, яке є узагальненням поняття квазіпервинного ідеала кільця. Досліджено деякі властивості таких піднапівмодулів. Вивчено взаємозв'язки між квазіпервинними та різними типами диференціальних піднапівмодулів диференціальних піднапівмодулів.

Ключові слова: напівмодуль, напівкільце, диференціювання напівмодуля, диференціювання напівкільця, диференціальне напівкільце, диференціальний ідеал, первинний піднапівмодуль, квазіпервинний піднапівмодуль.