ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 93. С. 34–41
Visnyk of the Lviv Univ. Series Mech. Math. 2022. Issue 93. P. 34–41
http://publications.lnu.edu.ua/bulletins/index.php/mmf
doi: http://dx.doi.org/10.30970/vmm.2022.93.034-041

УДК 512.534.5

ON ENDOMORPHISMS OF THE INVERSE SEMIGROUP OF CONVEX ORDER ISOMORPHISMS OF THE SET ω OF A BOUNDED RANK WHICH ARE GENERATED BY REES CONGRUENCES

Olha POPADIUK

Ivan Franko National University of Lviv,
Universytetska Str., 1, Lviv, 79000, Ukraine
e-mail: olha.popadiuk@lnu.edu.ua

Let $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ be the inverse semigroup of convex order isomorphisms of $(\omega,\leqslant)$ of the rank $\leqslant n$. Let $\mathfrak{End}^1(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ be a subsemigroup of $\mathfrak{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ which consists of $\mathfrak{a} \in \mathfrak{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ such that the image $(\alpha)\mathfrak{a}$ is isomorphic to a subsemigroup of the semigroup of $\omega\times\omega$-matrix units for all $\alpha \in \mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$. We describe the semigroup $\mathfrak{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ of all endomorphisms of the monoid $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ up to its ideal $\mathfrak{End}^1(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$.

Key words: bicyclic extension, inverse semigroup, endomorphism, automorphism, the semigroup of $\omega\times\omega$-matrix units.

We shall follow the terminology of [1, 2, 9, 10]. By $\mathbb{N}$ and ω we denote the set of all positive integers and the set of all non-negative integers, respectively.

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F \in \mathscr{P}(\omega)$ and $n, m \in \omega$ we put $n-m+F = \{n-m+k\colon k \in F\}$ if $F \neq \varnothing$ and $n-m+\varnothing = \varnothing$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called *ω-closed* if $F_1 \cap (-n+F_2) \in \mathscr{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathscr{F}$.

We denote $[0;0] = \{0\}$ and $[0;k] = \{0,\ldots,k\}$ for any positive integer k. The set $[0;k]$, $k \in \omega$, is called an *initial interval* of ω.

A nonempty set S with a binary associative operation is called a *semigroup*. By $(\omega,+)$ we denote the set ω with the usual addition $(x,y) \mapsto x+y$. We consider the following ideal $I_n = \{x \in \omega \mid x \geqslant n\}$ of $(\omega,+)$. Define $(\omega_n, \dot{+}) = (\omega,+)/I_n$.

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the mapping $\mathrm{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

2020 Mathematics Subject Classification: 20M15, 20M18, 20M20

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents.

For semigroups S and T a map $\mathfrak{h}\colon S \to T$ is called:

- a *homomorphism* if $\mathfrak{h}(s_1 \cdot s_2) = \mathfrak{h}(s_1) \cdot \mathfrak{h}(s_2)$ for all $s_1, s_2 \in S$;
- an *annihilating homomorphism* if $\mathfrak{h}$ is a homomorphism and $\mathfrak{h}(s_1) = \mathfrak{h}(s_2)$ for all $s_1, s_2 \in S$;
- an *isomorphism* if $\mathfrak{h}\colon S \to T$ is a bijective homomorphism.

For a semigroup S a homomorphism (an isomorphism) $\mathfrak{h}\colon S \to S$ is called an *endomorphism* (*automorphism*) of S. For simplicity of calculation, the image of $s \in S$ under an endomorphism $\mathfrak{e}$ of a semigroup S we shall denote it by $(s)\mathfrak{e}$.

A *congruence* on a semigroup S is an equivalence relation $\mathfrak{C}$ on S such that $(s,t) \in \mathfrak{C}$ implies $(as, at), (sb, tb) \in \mathfrak{C}$ for all $a, b \in S$. Every congruence $\mathfrak{C}$ on a semigroup S generates the *associated natural homomorphism* $\mathfrak{C}^\natural\colon S \to S/\mathfrak{C}$ which assigns to each element s of S its congruence class $[s]_{\mathfrak{C}}$ in the quotient semigroup $S/\mathfrak{C}$. Also every homomorphism $\mathfrak{h}\colon S \to T$ of semigroups S and T generates the congruence $\mathfrak{C}_{\mathfrak{h}}$ on S: $(s_1, s_2) \in \mathfrak{C}_{\mathfrak{h}}$ if and only if $(s_1)\mathfrak{h} = (s_2)\mathfrak{h}$.

A nonempty subset I of a semigroup S is called an *ideal* of S if

$$SIS = \{asb\colon s \in I,\ a, b \in S\} \subseteq I.$$

Every ideal I of a semigroup S generates the congruence $\mathfrak{C}_I = (I \times I) \cup \Delta_S$ on S, which is called the *Rees congruence* on S. An endomorphism $\mathfrak{r}$ of a semigroup S is said to be *Rees* if $\mathfrak{r}$ generates a Rees congruence $\mathfrak{C}_{\mathfrak{r}}$ on S.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of λ together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \quad \text{if} \quad x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha\colon y\alpha \in \operatorname{dom}\beta\}, \qquad \text{for} \quad \alpha, \beta \in \mathscr{I}_\lambda.$$

The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the cardinal λ (see [1]). For any $\alpha \in \mathscr{I}_\lambda$ the cardinality of $\operatorname{dom}\alpha$ is called the *rank* of α and it is denoted by $\operatorname{rank}\alpha$. The symmetric inverse semigroup was introduced by V. V. Wagner [11] and it plays a major role in the theory of semigroups.

Put $\mathscr{I}_\lambda^n = \{\alpha \in \mathscr{I}_\lambda\colon \operatorname{rank}\alpha \leqslant n\}$, for $n = 1, 2, 3, \ldots$. Obviously, $\mathscr{I}_\lambda^n$ $(n = 1, 2, 3, \ldots)$ is an inverse semigroup, $\mathscr{I}_\lambda^n$ is an ideal of $\mathscr{I}_\lambda$, for each $n = 1, 2, 3, \ldots$. The semigroup $\mathscr{I}_\lambda^n$ is called the *symmetric inverse semigroup of finite transformations of the rank* $\leqslant n$ [7]. By

$$\left(\begin{smallmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{smallmatrix}\right)$$

we denote a partial one-to-one transformation which maps x_1 onto y_1, x_2 onto y_2, $\ldots$, and x_n onto y_n. Obviously, in such case we have $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$ $(i, j = 1, 2, 3, \ldots, n)$. The empty partial map $\varnothing\colon \lambda \rightharpoonup \lambda$ is denoted by $\mathbf{0}$. It is obvious that $\mathbf{0}$ is zero of the semigroup $\mathscr{I}_\lambda^n$.

For a partially ordered set $(P, \leqq)$, a subset X of P is called *order-convex*, if $x \leqq z \leqq y$ and $\{x, y\} \subseteq X$ implies that $z \in X$, for all $x, y, z \in P$ [8]. It is obvious that the set of all partial order isomorphisms between convex subsets of $(\omega,\leqslant)$ under the composition of partial self-maps forms an inverse subsemigroup of the symmetric inverse semigroup $\mathscr{I}_\omega$ over the set ω. We denote this semigroup by $\mathscr{I}_\omega(\overrightarrow{\mathrm{conv}})$. We put $\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})=\mathscr{I}_\omega(\overrightarrow{\mathrm{conv}})\cap\mathscr{I}_\omega^n$ and it is obvious that $\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$ is closed under the semigroup operation of $\mathscr{I}_\omega^n$. The semigroup $\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$ is called the *inverse semigroup of convex order isomorphisms of* $(\omega,\leqslant)$ *of the rank* $\leqslant n$. Obviously that every non-zero element of the semigroup $\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$ of the rank $k\leqslant n$ has a form

$$\begin{pmatrix} i & i+1 & \cdots & i+k-1\\ j & j+1 & \cdots & j+k-1\end{pmatrix}$$

for some $i,j\in\omega$.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^kp^l\cdot q^mp^n=q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [1].

On the set $\boldsymbol{B}_\omega=\omega\times\omega$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1,j_1)\cdot(i_2,j_2)=\begin{cases}(i_1-j_1+i_2,j_2), & \text{if } j_1\leqslant i_2;\\ (i_1,j_1-i_2+j_2), & \text{if } j_1\geqslant i_2.\end{cases}$$

It is well known that the semigroup $\boldsymbol{B}_\omega$ is isomorphic to the bicyclic monoid by the mapping $\mathfrak{h}\colon\mathscr{C}(p,q)\to\boldsymbol{B}_\omega$, $q^kp^l\mapsto(k,l)$ (see: [1, Section 1.12] or [10, Exercise IV.1.11(ii)]).

Next we shall describe the construction which is introduced in [4].

Let $\boldsymbol{B}_\omega$ be the bicyclic monoid and $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_\omega\times\mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1,j_1,F_1)\cdot(i_2,j_2,F_2)=\begin{cases}(i_1-j_1+i_2,j_2,(j_1-i_2+F_1)\cap F_2), & \text{if } j_1\leqslant i_2;\\ (i_1,j_1-i_2+j_2,F_1\cap(i_2-j_1+F_2)), & \text{if } j_1\geqslant i_2.\end{cases}$$

In [4] is proved that if the family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ is ω-closed then $(\boldsymbol{B}_\omega\times\mathscr{F},\cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I}=\{(i,j,\varnothing)\colon i,j\in\omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_\omega\times\mathscr{F},\cdot)$. For any ω-closed family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_\omega^{\mathscr{F}}=\begin{cases}(\boldsymbol{B}_\omega\times\mathscr{F},\cdot)/\boldsymbol{I}, & \text{if } \varnothing\in\mathscr{F};\\ (\boldsymbol{B}_\omega\times\mathscr{F},\cdot), & \text{if } \varnothing\notin\mathscr{F}\end{cases}$$

is defined in [4]. The semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in [4] that $\boldsymbol{B}_\omega^{\mathscr{F}}$ is combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_\omega^{\mathscr{F}}$ and its set of idempotents are described. The criteria of simplicity, 0-simplicity, bisimplicity, 0-bisimplicity of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ and when $\boldsymbol{B}_\omega^{\mathscr{F}}$ has the identity, is isomorphic to the bicyclic semigroup or the countable semigroup of matrix units are given. In particularly in [4] is proved that the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega\times\omega$-matrix units if and only if $\mathscr{F}$ consists of a singleton set and the empty set.

The semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ in the case when the family $\mathscr{F}$ consists of the empty set and some singleton subsets of ω is studied in [3]. It is proved that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the subsemigroup $\mathscr{B}_{\omega}^{\mathfrak{r}}(\boldsymbol{F}_{\min})$ of the Brandt ω-extension of the subsemilattice $(\boldsymbol{F},\min)$ of $(\omega,\min)$, where $\boldsymbol{F}=\bigcup\mathscr{F}$. Also topologizations of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its closure in semitopological semigroups are studied.

For any $n\in\omega$ we put $\mathscr{F}_n=\{[0;k]\colon k=0,\ldots,n\}$. It is obvious that $\mathscr{F}_n$ is an ω-closed family of ω.

In the paper [5] we study the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. It is shown that the Green relations $\mathscr{D}$ and $\mathscr{J}$ coincide in $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is isomorphic to the semigroup $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$, and $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ admits only Rees congruences. Also in [5], we study shift-continuous topologies of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. In particular, we prove that for any shift-continuous T_1-topology τ on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, every non-zero element of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is an isolated point of $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau)$, $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ admits the unique compact shift-continuous T_1-topology, and every $\omega_{\mathfrak{d}}$-compact shift-continuous T_1-topology is compact, where $\omega_{\mathfrak{d}}$ is the discrete infinite countable space. We describe the closure of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ in a Hausdorff semitopological semigroup and prove the criterium when a topological inverse semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is H-closed in the class of Hausdorff topological semigroups.

In the paper [6] injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ for a positive integer $n\geqslant 2$ are desribed. In particular, it is proved that for $n\geqslant 1$, the semigroup of injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is isomorphic to $(\omega,+)$. Also, there the structure of the semigroup $\mathfrak{End}(\mathscr{B}_{\lambda})$ of all endomorphisms of the semigroup of $\lambda\times\lambda$-matrix units $\mathscr{B}_{\lambda}$ is described.

This paper is a continuations of the investigation which are presented in [5, 6]. Let $\mathfrak{End}^1(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ be a subsemigroup of $\mathfrak{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ which consists of $\mathfrak{a}\in\mathfrak{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ such that the image $(\alpha)\mathfrak{a}$ is isomorphic to a subsemigroup of the semigroup of $\omega\times\omega$-matrix units for all $\alpha\in\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$. We describe the semigroup $\mathfrak{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ of all endomorphisms of the monoid $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ up to its ideal $\mathfrak{End}^1(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$.

By Theorem 1 of [5], for any $n\in\omega$ the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is isomorphic to the semigroup $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$ by the mapping $\mathfrak{I}\colon\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\to\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$, defined by the formulae $(\boldsymbol{0})\mathfrak{I}=\boldsymbol{0}$ and

$$(i,j,[0;k])\mathfrak{I}=\left(\begin{smallmatrix} i & i+1 & \cdots & i+k\\ j & j+1 & \cdots & j+k\end{smallmatrix}\right).$$

Later we study endomorphisms of the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$.

By Theorem 2 of [5] for an arbitrary $n\in\omega$ the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ (and hence the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$) admits only Rees congruences. Moreover, by Theorem 3 of [5] for any homomorphism $\mathfrak{h}$ from $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ into a semigroup S the image $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n})\mathfrak{h}$ is either isomorphic to $\boldsymbol{B}_{\omega}^{\mathscr{F}_k}$ for some $k=0,1,\ldots,n$, or is a singleton. Also, Lemma 1 of [6] states that if n is any positive integer and $\mathfrak{a}$ is an arbitrary non-annihilating endomorphism of the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ then $(\boldsymbol{0})\mathfrak{a}=\boldsymbol{0}$.

By Proposition 3 of [5] for any non-negative integer n the map $\mathfrak{h}_0\colon\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\to\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ defined by the formulae $(\boldsymbol{0})\mathfrak{h}_0=\boldsymbol{0}$ and

$$(i,j,[0;k])\mathfrak{h}_0=\begin{cases}\boldsymbol{0}, & \text{if } k=0;\\ (i,j,[0;k-1]), & \text{if } k=1,\ldots,n,\end{cases}$$

is an endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. Using the isomorphism $\mathfrak{I}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \to \mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$ we get that the endomorphism $\mathfrak{h}_0$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ generates the following endomorphism $\mathfrak{r}_1\colon \mathscr{I}_{\omega}^{m+1}(\overrightarrow{\mathrm{conv}}) \to \mathscr{I}_{\omega}^{m}(\overrightarrow{\mathrm{conv}})$ $(m \in \mathbb{N})$ which is defined by the formulae

$$(\mathbf{0})\mathfrak{r}_1 = \mathbf{0}, \quad \left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)\mathfrak{r}_1 = \mathbf{0}, \quad \left(\begin{smallmatrix} i & i+1 \\ j & j+1 \end{smallmatrix}\right)\mathfrak{r}_1 = \left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right), \quad \ldots, \quad \left(\begin{smallmatrix} i & \cdots & i+k-1 & i+k \\ j & \cdots & j+k-1 & j+k \end{smallmatrix}\right)\mathfrak{r}_1 = \left(\begin{smallmatrix} i & \cdots & i+k-1 \\ j & \cdots & j+k-1 \end{smallmatrix}\right)$$

for all $i, j \in \omega$ and $k = 1, \ldots, m$. It is obvious that so defined endomorphism $\mathfrak{r}_1$ of $\mathscr{I}_{\omega}^{m}(\overrightarrow{\mathrm{conv}})$ generates the Rees congruence $\mathfrak{C}_{\mathfrak{r}_1}$ which is generated by the ideal $\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})$. Also for $p = 1, \ldots, m$ the mapping $\mathfrak{r}_p = \underbrace{\mathfrak{r}_1 \circ \cdots \circ \mathfrak{r}_1}_{p-\text{times}}$ is an endomorphism of $\mathscr{I}_{\omega}^{m}(\overrightarrow{\mathrm{conv}})$ and $\mathfrak{r}_p$ generates the Rees congruence $\mathfrak{C}_{\mathfrak{r}_p}$ which is generated by the ideal $\mathscr{I}_{\omega}^{p}(\overrightarrow{\mathrm{conv}})$ of the semigroup $\mathscr{I}_{\omega}^{m}(\overrightarrow{\mathrm{conv}})$. Later for $p = 1, \ldots, m$ the above determined endomorphism $\mathfrak{r}_p$ we call the *p-canonical Rees endomorphism* of the semigroup $\mathscr{I}_{\omega}^{m}(\overrightarrow{\mathrm{conv}})$.

Later we study endomorphisms of the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ for any positive integer n.

By Corollary 1 of [6] for any positive integer n and arbitrary $i_0 \in \omega$ the map $\mathfrak{e}_{i_0}\colon \mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}) \to \mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ defined by the formulae $(\mathbf{0})\mathfrak{e}_{i_0} = \mathbf{0}$ and

$$\left(\begin{smallmatrix} i & i+1 & \cdots & i+k \\ j & j+1 & \cdots & j+k \end{smallmatrix}\right)\mathfrak{e}_{i_0} = \left(\begin{smallmatrix} i_0+i & i_0+i+1 & \cdots & i_0+i+k \\ i_0+j & i_0+j+1 & \cdots & i_0+j+k \end{smallmatrix}\right), \qquad k = 0, \ldots, n-1,$$

is an endomorphism of the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$, and moreover it is injective. It is obvious for any $i_0 \in \omega$ the endomorphism $\mathfrak{e}_{i_0}$ generates the identity congruence on the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$. Also, by Theorem 1 of [6] for any positive integer $n \geqslant 2$ for every injective endomorphism $\mathfrak{a}\colon \mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}) \to \mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ there exists $i_0 \in \omega$ such that $\mathfrak{a} = \mathfrak{e}_{i_0}$.

Fix an arbitrary $i_0 \in \omega$. Then we have that

$$\begin{aligned}
((\mathbf{0})\mathfrak{r}_1)\mathfrak{e}_{i_0} &= (\mathbf{0})\mathfrak{e}_{i_0} = \mathbf{0},\\
\left(\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)\mathfrak{r}_1\right)\mathfrak{e}_{i_0} &= (\mathbf{0})\mathfrak{e}_{i_0} = \mathbf{0},\\
\left(\left(\begin{smallmatrix} i & i+1 \\ j & j+1 \end{smallmatrix}\right)\mathfrak{r}_1\right)\mathfrak{e}_{i_0} &= \left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)\mathfrak{e}_{i_0} = \left(\begin{smallmatrix} i+i_0 \\ j+i_0 \end{smallmatrix}\right),\\
\cdots & \qquad \cdots,\\
\left(\left(\begin{smallmatrix} i & \cdots & i+k-1 & i+k \\ j & \cdots & j+k-1 & k+k \end{smallmatrix}\right)\mathfrak{r}_1\right)\mathfrak{e}_{i_0} &= \left(\begin{smallmatrix} i & \cdots & i+k-1 \\ j & \cdots & j+k-1 \end{smallmatrix}\right)\mathfrak{e}_{i_0} = \left(\begin{smallmatrix} i+i_0 & \cdots & i+k-1+i_0 \\ j+i_0 & \cdots & j+k-1+i_0 \end{smallmatrix}\right)
\end{aligned}$$

and

$$\begin{aligned}
((\mathbf{0})\mathfrak{e}_{i_0})\mathfrak{r}_1 &= (\mathbf{0})\mathfrak{r}_1 = \mathbf{0},\\
\left(\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)\mathfrak{e}_{i_0}\right)\mathfrak{r}_1 &= \left(\begin{smallmatrix} i+i_0 \\ j+i_0 \end{smallmatrix}\right)\mathfrak{r}_1 = \mathbf{0},\\
\left(\left(\begin{smallmatrix} i & i+1 \\ j & j+1 \end{smallmatrix}\right)\mathfrak{e}_{i_0}\right)\mathfrak{r}_1 &= \left(\begin{smallmatrix} i+i_0 & i+1+i_0 \\ j+i_0 & j+1+i_0 \end{smallmatrix}\right)\mathfrak{r}_1 = \left(\begin{smallmatrix} i+i_0 \\ j+i_0 \end{smallmatrix}\right),\\
\cdots & \qquad \cdots,\\
\left(\left(\begin{smallmatrix} i & \cdots & i+k-1 & i+k \\ j & \cdots & j+k-1 & k+k \end{smallmatrix}\right)\mathfrak{e}_{i_0}\right)\mathfrak{r}_1 &= \left(\begin{smallmatrix} i+i_0 & \cdots & i+k-1+i_0 & i+k+i_0 \\ j+i_0 & \cdots & j+k-1+i_0 & k+k+i_0 \end{smallmatrix}\right)\mathfrak{r}_1 = \left(\begin{smallmatrix} i+i_0 & \cdots & i+k-1+i_0 \\ j+i_0 & \cdots & j+k-1+i_0 \end{smallmatrix}\right)
\end{aligned}$$

for all $i, j \in \omega$ and $k = 1, \ldots, n$. This implies that $\mathfrak{e}_{i_0} \circ \mathfrak{r}_1 = \mathfrak{r}_1 \circ \mathfrak{e}_{i_0}$. Then the definition of the p-canonical Rees endomorphism $\mathfrak{r}_1$ of the semigroup $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$ implies the following lemma.

Lemma 1. *Let n be a positive integer $\geqslant 2$. Then for any $p = 1, \ldots, n-1$ and $i_0 \in \omega$ the p-canonical Rees endomorphism $\mathfrak{r}_1$ and injective endomorphism $\mathfrak{e}_{i_0}$ of the semigroup $\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$ commute, i.e., $\mathfrak{e}_{i_0} \circ \mathfrak{r}_p = \mathfrak{r}_p \circ \mathfrak{e}_{i_0}$.*

By $\mathfrak{End}(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$ we denote the semigroup of all endomorphisms of the monoid $\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$. We define

$$\mathfrak{End}^1(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})) = \left\{\mathfrak{a} \in \mathfrak{End}(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})) | (\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))\mathfrak{a} \subseteq \mathscr{I}_\omega^1(\overrightarrow{\mathrm{conv}})\right\}.$$

Observe that the set $\mathfrak{End}^1(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$ is an ideal of $\mathfrak{End}(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$. Indeed, let $\mathfrak{b} \in \mathfrak{End}(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$ and $\mathfrak{a} \in \mathfrak{End}^1(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$. Then for any $\alpha \in \mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$ the definition of the monoid $\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$ implies that

$$(\alpha)(\mathfrak{a} \circ \mathfrak{b}) \in ((\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))\mathfrak{a})\mathfrak{b} \subseteq (\mathscr{I}_\omega^1(\overrightarrow{\mathrm{conv}}))\mathfrak{b} \subseteq \mathscr{I}_\omega^1(\overrightarrow{\mathrm{conv}}),$$

and

$$(\alpha)(\mathfrak{b} \circ \mathfrak{a}) \in ((\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))\mathfrak{b})\mathfrak{a} \subseteq (\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))\mathfrak{a} \subseteq \mathscr{I}_\omega^1(\overrightarrow{\mathrm{conv}}).$$

Let $\mathfrak{a} \in \mathfrak{End}^1(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$. By Theorems 1 and 3 of [5] the image $(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})\mathfrak{a}$ is isomorphic to the semigroup $\mathscr{I}_\omega^1(\overrightarrow{\mathrm{conv}})$, which is isomorphic to the semigroup of $\omega \times \omega$-matrix units $\boldsymbol{B}_\omega$. This implies that there exists an isomorphism $\mathfrak{e} : \mathscr{I}_\omega^1(\overrightarrow{\mathrm{conv}}) \to (\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))\mathfrak{a}$. Then we have that $\mathfrak{a} = \mathfrak{r}_{n-1} \circ \mathfrak{e}$, where $\mathfrak{r}_{n-1}$ is the $(n-1)$-canonical Rees endomorphism of $\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$.

We denote

$$\mathfrak{End}^*(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})) = \mathfrak{End}(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})) \setminus \mathfrak{End}^1(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})).$$

It is obvious that $\mathfrak{a} \in \mathfrak{End}^*(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$ if and only if

$$(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))\mathfrak{a} \cap (\mathscr{I}_\omega^2(\overrightarrow{\mathrm{conv}}) \setminus \mathscr{I}_\omega^1(\overrightarrow{\mathrm{conv}})) \neq \varnothing.$$

Let $\mathfrak{b} \in \mathfrak{End}^*(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$. Theorems 1 and 3 of [5], and an equality $|\operatorname{ran}\mathfrak{b}| = k$, implies that the image $(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))\mathfrak{b}$ is isomorphic to the semigroup $\mathscr{I}_\omega^k(\overrightarrow{\mathrm{conv}})$ for any $k \in \{2, 3, \ldots, n\}$. Then there exists an isomorphism $\mathfrak{e}_{i_0} : \mathscr{I}_\omega^k(\overrightarrow{\mathrm{conv}}) \to (\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))\mathfrak{b}$ such that $(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))\mathfrak{b} = (\mathscr{I}_\omega^k(\overrightarrow{\mathrm{conv}}))\mathfrak{e}_{i_0}$. Hence, $\mathfrak{b} = \mathfrak{e}_{i_0} \circ \mathfrak{r}_{n-k}$, where $\mathfrak{r}_{n-k}$ is $(n-k)$-canonical Rees endomorphism of the monoid $\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$.

The above arguments imply the following theorem.

Theorem 1. *The semigroup $\mathfrak{End}(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$ of all endomorphisms of the semigroup $\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$ is the disjoint union of the set $\mathfrak{End}^*(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$ and the ideal $\mathfrak{End}^1(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$. Moreover,*

- *for any $\mathfrak{a} \in \mathfrak{End}^*(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$ we have that $\mathfrak{a} = \mathfrak{r}_{n-1} \circ \mathfrak{e}$, and*
- *for any $\mathfrak{b} \in \mathfrak{End}^1(\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}}))$ we have that $\mathfrak{b} = \mathfrak{e}_{i_0} \circ \mathfrak{r}_{n-k}$.*

Simple verifications show that for any p_1- and p_2-canonical Rees endomorphisms $\mathfrak{r}_{p_1}$ and $\mathfrak{r}_{p_2}$ we have that

$$\mathfrak{r}_{p_1} \circ \mathfrak{r}_{p_2} = \mathfrak{r}_{p_1 \dotplus p_2} = \mathfrak{r}_{p_2} \circ \mathfrak{r}_{p_1},$$

and moreover, in the case when $p_1 + p_2 \geqslant n$, $\mathfrak{r}_{p_1} \circ \mathfrak{r}_{p_2}$ is the annihilating endomorphisms of the monoid $\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$, i.e., $(\alpha)(\mathfrak{r}_{p_1} \circ \mathfrak{r}_{p_2}) = \mathbf{0}$, for all $\alpha \in \mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$. This implies the following proposition.

Proposition 1. *For any positive integer n, the semigroup of p-canonical Rees endomorphisms of the semigroup $\mathscr{I}_\omega^n(\overrightarrow{\mathrm{conv}})$ is isomorphic to the semigroup $(\omega_n, \dotplus)$.*

By Theorem 2 of [6] for $n \geqslant 2$ the semigroup of injective endomorphisms of the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ is isomorphic to the semigroup $(\omega, +)$.

Let $I_{\omega}^{0} = \{(0, j) \mid j \in \omega\}$ be a subset of the direct product of the semigroups $(\omega_{n-1}, \dot{+})$ and $(\omega, +)$. It is obvious that I_{ω}^{0} is an ideal of the semigroup $(\omega_{n-1}, \dot{+}) \times (\omega, +)$.

This implies the following theorem.

Theorem 2. *For any positive integer n the semigroup* $\mathfrak{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))/\mathfrak{End}^{1}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ *is isomorphic to the Rees quotient semigroup* $((\omega_{n-1}, \dot{+}) \times (\omega, +))/I_{\omega}^{0}$.

Acknowledgements

The author acknowledges Oleg Gutik and the referee for their useful comments and suggestions.

References

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. I, Amer. Math. Soc. Surveys 7, Providence, R.I., 1961.
2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. II, Amer. Math. Soc. Surveys 7, Providence, R.I., 1967.
3. O. Gutik and O. Lysetska, *On the semigroup* $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ *which is generated by the family* $\mathscr{F}$ *of atomic subsets of* ω, Visn. L'viv. Univ., Ser. Mekh.-Mat. **92** (2021), 34–50. DOI: 10.30970/vmm.2021.92.034-050
4. O. Gutik and M. Mykhalenych, *On some generalization of the bicyclic monoid*, Visnyk Lviv. Univ. Ser. Mech.-Mat. **90** (2020), 5–19 (in Ukrainian). DOI: 10.30970/vmm.2020.90.005-019
5. O. Gutik and O. Popadiuk, *On the semigroup* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ *which is generated by the family* $\mathscr{F}_n$ *of finite bounded intervals of* ω, arXiv:2208.09155, 2022, preprint.
6. O. Gutik and O. Popadiuk, *On the semigroup of injective endomorphisms of the semigroup* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ *which is generated by the family* $\mathscr{F}_n$ *of finite bounded intervals of* ω, Mat. Metody Fiz.-Mekh. Polya **65** (2022), no. 1–2, 42–57.
7. O. V. Gutik and A. R. Reiter, *Symmetric inverse topological semigroups of finite rank* $\leqslant n$, Mat. Metody Fiz.-Mekh. Polya **52** (2009), no. 3, 7–14; **reprinted version:** J. Math. Sc. **171** (2010), no. 4, 425–432. DOI: 10.1007/s10958-010-0147-z
8. E. Harzheim, *Ordered sets*, Springer, New-York, Advances in Math. **7**, 2005.
9. M. Lawson, *Inverse semigroups. The theory of partial symmetries*, Singapore, World Scientific, 1998.
10. M. Petrich, *Inverse semigroups*, John Wiley & Sons, New York, 1984.
11. V. V. Wagner, *Generalized groups*, Dokl. Akad. Nauk SSSR **84** (1952), 1119–1122 (in Russian).

Стаття: надійшла до редколегії 24.04.2022
доопрацьована 07.08.2022
прийнята до друку 27.08.2022

ПРО ЕНДОМОРФІЗМИ ІНВЕРСНОЇ НАПІВГРУПИ ПОРЯДКОВО ОПУКЛИХ ІЗОМОРФІЗМІВ МНОЖИНИ ω ОБМЕЖЕНОГО РАНГУ, ЯКІ ПОРОДЖЕНІ КОНГРУЕНЦІЯМИ РІСА

Ольга ПОПАДЮК

*Львівський національний університет імені Івана Франка,
вул. Університетська, 1, 79000, Львів
e-mail: olha.popadiuk@lnu.edu.ua*

Нехай $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ — інверсна напівгрупа порядково опуклих ізоморфізмів лінійно впорядкованої множини $(\omega, \leqslant)$ рангу $\leqslant n$. Нехай $\mathfrak{End}^1(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ – піднапівгрупа напівгрупи $\mathfrak{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$, яка складається з таких елементів $\mathfrak{a} \in \mathfrak{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$, що образ $(\alpha)\mathfrak{a}$ ізоморфний піднапівгрупі напівгрупи $\omega \times \omega$-матричних одиниць для всіх $\alpha \in \mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$. Ми описуємо напівгрупу $\mathfrak{End}(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$ усіх ендоморфізмів моноїда $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ за модулем ідеала $\mathfrak{End}^1(\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}))$.

Ключові слова: біциклічне розширення, інверсна напівгрупа, ендоморфізм, автоморфізм, напівгрупа $\lambda \times \lambda$-матричних одиниць.