ON ENDOMORPHISMS OF THE INVERSE SEMIGROUP OF CONVEX ORDER ISOMORPHISMS OF THE SET $\omega$ OF A BOUNDED RANK WHICH ARE GENERATED BY REES CONGRUENCES

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Let $I^n_\omega(\rightarrow \text{conv})$ be the inverse semigroup of convex order isomorphisms of the rank $\leq n$. Let $\operatorname{End}(I^n_\omega(\rightarrow \text{conv}))$ be a subsemigroup of $\operatorname{End}(I^n_\omega(\rightarrow \text{conv}))$ which consists of $a \in \operatorname{End}(I^n_\omega(\rightarrow \text{conv}))$ such that the image $(\alpha)a$ is isomorphic to a subsemigroup of the semigroup of $\omega \times \omega$-matrix units for all $\alpha \in I^n_\omega(\rightarrow \text{conv})$. We describe the semigroup $\operatorname{End}(I^n_\omega(\rightarrow \text{conv}))$ of all endomorphisms of the monoid $I^n_\omega(\rightarrow \text{conv})$ up to its ideal $\operatorname{End}^1(I^n_\omega(\rightarrow \text{conv}))$.

Key words: bicyclic extension, inverse semigroup, endomorphism, automorphism, the semigroup of $\omega \times \omega$-matrix units.

We shall follow the terminology of [1, 2, 9, 10]. By $\mathbb{N}$ and $\omega$ we denote the set of all positive integers and the set of all non-negative integers, respectively.

Let $\mathcal{P}(\omega)$ be the family of all subsets of $\omega$. For any $F \in \mathcal{P}(\omega)$ and $n, m \in \omega$ we put $n - m + F = \{n - m + k : k \in F\}$ if $F \neq \emptyset$ and $n - m + \emptyset = \emptyset$. A subfamily $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called $\omega$-closed if $F_1 \cap (-n + F_2) \subseteq \mathcal{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathcal{F}$.

We denote $[0; 0] = \{0\}$ and $[0; k] = \{0, \ldots, k\}$ for any positive integer $k$. The set $[0; k]$, $k \in \omega$, is called an initial interval of $\omega$.

A nonempty set $S$ with a binary associative operation is called a semigroup. By $(\omega, +)$ we denote the set $\omega$ with the usual addition $(x, y) \mapsto x + y$. We consider the following ideal $I_n = \{x \in \omega : x \geq n\}$ of $(\omega, +)$. Define $(\omega, +) = (\omega, +)/I_n$.

A semigroup $S$ is called inverse if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element $x^{-1}$ is called the inverse of $x \in S$. If $S$ is an inverse semigroup, then the mapping $\operatorname{inv} : S \to S$ which assigns to every element $x$ of $S$ its inverse element $x^{-1}$ is called the inversion.
If $S$ is a semigroup, then we shall denote the subset of all idempotents in $S$ by $E(S)$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a band (or the band of $S$). Then the semigroup operation on $S$ determines the following partial order $\preceq$ on $E(S)$: $e \preceq f$ if and only if $ef = fe = e$. This order is called the natural partial order on $E(S)$. A semilattice is a commutative semigroup of idempotents.

For semigroups $S$ and $T$ a map $h: S \to T$ is called:

- a homomorphism if $h(s_1 \cdot s_2) = h(s_1) \cdot h(s_2)$ for all $s_1, s_2 \in S$;
- an annihilating homomorphism if $h$ is a homomorphism and $h(s_1) = h(s_2)$ for all $s_1, s_2 \in S$;
- an isomorphism if $h: S \to T$ is a bijective homomorphism.

For a semigroup $S$ a homomorphism (an isomorphism) $h: S \to S$ is called an endomorphism (automorphism) of $S$. For simplicity of calculation, the image of $s \in S$ under an endomorphism $\tau$ of a semigroup $S$ we shall denote it by $(s)_\tau$.

A congruence on a semigroup $S$ is an equivalence relation $\mathcal{C}$ on $S$ such that $(s, t) \in \mathcal{C}$ implies $(as, at), (sb, tb) \in \mathcal{C}$ for all $a, b \in S$. Every congruence $\mathcal{C}$ on a semigroup $S$ generates the associated natural homomorphism $E_\mathcal{C}: S \to S/\mathcal{C}$ which assigns to each element $s$ of $S$ its congruence class $[s]_\mathcal{C}$ in the quotient semigroup $S/\mathcal{C}$. Also every homomorphism $h: S \to T$ of semigroups $S$ and $T$ generates the congruence $\mathcal{C}_h$ on $S$: $(s_1, s_2) \in \mathcal{C}_h$ if and only if $(s_1)h = (s_2)h$.

A nonempty subset $I$ of a semigroup $S$ is called an ideal of $S$ if

$$SIS = \{asb: s \in I, a, b \in S\} \subseteq I.$$ 

Every ideal $I$ of a semigroup $S$ generates the congruence $\mathcal{C}_I = (I \times I) \cup \Delta_S$ on $S$, which is called the Rees congruence on $S$. An endomorphism $\tau$ of a semigroup $S$ is said to be Rees if $\tau$ generates a Rees congruence $\mathcal{C}_\tau$ on $S$.

Let $\mathcal{A}_\lambda$ denote the set of all partial one-to-one transformations of $\lambda$ together with the following semigroup operation:

$$x(\alpha)\beta = (xa)\beta \quad \text{if} \quad x \in \text{dom}(\alpha) = \{ y \in \text{dom}\alpha: yo \in \text{dom}\beta \}$$

for $\alpha, \beta \in \mathcal{A}_\lambda$.

The semigroup $\mathcal{I}_\lambda$ is called the symmetric inverse semigroup over the cardinal $\lambda$ (see [1]). For any $\alpha \in \mathcal{A}_\lambda$ the cardinality of $\text{dom}\alpha$ is called the rank of $\alpha$ and it is denoted by $\text{rank}\alpha$. The symmetric inverse semigroup was introduced by V. V. Wagner [11] and it plays a major role in the theory of semigroups.

Put $\mathcal{I}^n_\lambda = \{ \alpha \in \mathcal{A}_\lambda: \text{rank}\alpha \leq n \}$, for $n = 1, 2, 3, \ldots$. Obviously, $\mathcal{I}^n_\lambda (n = 1, 2, 3, \ldots)$ is an inverse semigroup, $\mathcal{I}^n_\lambda$ is an ideal of $\mathcal{A}_\lambda$ for each $n = 1, 2, 3, \ldots$. The semigroup $\mathcal{I}^\lambda_\lambda$ is called the symmetric inverse semigroup of finite transformations of the rank $\leq n$ [7]. By

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix}$$

we denote a partial one-to-one transformation which maps $x_1$ onto $y_1$, $x_2$ onto $y_2$, ..., and $x_n$ onto $y_n$. Obviously, in such a case we have $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$ ($i, j = 1, 2, 3, \ldots, n$). The empty partial map $\mathcal{O}: \lambda \to \lambda$ is denoted by $0$. It is obvious that $0$ is zero of the semigroup $\mathcal{I}^\lambda_\lambda$.

For a partially ordered set $(P, \leq)$, a subset $X$ of $P$ is called order-convex, if $x \leq z \leq y$ and $(x, y) \subseteq X$ implies that $z \in X$, for all $x, y, z \in P$ [8]. It is obvious that the set of all
partial order isomorphisms between convex subsets of \((\omega, \subseteq)\) under the composition of partial self-maps forms an inverse subsemigroup of the symmetric inverse semigroup \(\mathcal{I}_\omega\) over the set \(\omega\). We denote this semigroup by \(\mathcal{I}_\omega(\text{conv})\). We put \(\mathcal{I}_\omega^n(\text{conv}) = \mathcal{I}_\omega(\text{conv}) \cap \mathcal{I}_\omega^n\) and it is obvious that \(\mathcal{I}_\omega^n(\text{conv})\) is closed under the semigroup operation of \(\mathcal{I}_\omega^n\).

The semigroup \(\mathcal{I}_\omega^n(\text{conv})\) is called the inverse semigroup of convex order isomorphisms of \((\omega, \subseteq)\) of the rank \(\leq n\). Obviously that every non-zero element of the semigroup \(\mathcal{I}_\omega^n(\text{conv})\) of the rank \(k \leq n\) has a form

\[
\begin{pmatrix} i & i+1 & \cdots & i+k-1 \\ j & j+1 & \cdots & j+k-1 \end{pmatrix}
\]

for some \(i, j \in \omega\).

The bicyclic monoid \(\mathcal{C}(p, q)\) is the semigroup with the identity \(1\) generated by two elements \(p\) and \(q\) subject to the condition \(pq = 1\). The semigroup operation on \(\mathcal{C}(p, q)\) is determined as follows:

\[
q^k p^l \cdot q^m p^n = q^{k + m - \min\{l, m\}} p^{l + n - \min\{l, m\}}.
\]

It is well known that the bicyclic monoid \(\mathcal{C}(p, q)\) is a bisimple (and hence simple) combinatorial \(E\)-unitary inverse semigroup and every non-trivial congruence on \(\mathcal{C}(p, q)\) is a group congruence [1].

On the set \(B_\omega = \omega \times \omega\) we define the semigroup operation \(\cdot\) in the following way

\[(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2), & \text{if } j_1 \leq i_2; \\ (i_1, j_1 - i_2 + j_2), & \text{if } j_1 \geq i_2. \end{cases}\]

It is well known that the semigroup \(B_\omega\) is isomorphic to the bicyclic monoid by the mapping \(b : \mathcal{C}(p, q) \to B_\omega, q^i p^j \mapsto (k, l)\) (see: [1, Section 1.12] or [10, Exercise IV.1.11(ii)])

Next we shall describe the construction which is introduced in [4].

Let \(B_\omega\) be the bicyclic monoid and \(\mathcal{F}\) be an \(\omega\)-closed subfamily of \(\mathcal{P}(\omega)\). On the set \(B_\omega \times \mathcal{F}\) we define the semigroup operation \(\cdot\) in the following way

\[(i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 \leq i_2; \\ (i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 \geq i_2. \end{cases}\]

In [4] is proved that if the family \(\mathcal{F} \subseteq \mathcal{P}(\omega)\) is \(\omega\)-closed then \((B_\omega \times \mathcal{F}, \cdot)\) is a semigroup. Moreover, if an \(\omega\)-closed family \(\mathcal{F} \subseteq \mathcal{P}(\omega)\) contains the empty set \(\emptyset\) then the set \(I = \{(i, j, \emptyset) : i, j \in \omega\}\) is an ideal of the semigroup \((B_\omega \times \mathcal{F}, \cdot)\). For any \(\omega\)-closed family \(\mathcal{F} \subseteq \mathcal{P}(\omega)\) the following semigroup

\[B_\omega^\mathcal{F} = \{(B_\omega \times \mathcal{F}, \cdot)/I, \text{ if } \emptyset \in \mathcal{F}; \allowbreak \allowbreak (B_\omega \times \mathcal{F}, \cdot), \text{ if } \emptyset \notin \mathcal{F}\}
\]

is defined in [4]. The semigroup \(B_\omega^\mathcal{F}\) generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in [4] that \(B_\omega^\mathcal{F}\) is combinatorial inverse semigroup and Green’s relations, the natural partial order on \(B_\omega^\mathcal{F}\) and its set of idempotents are described. The criteria of simplicity, \(0\)-simplicity, bisimplicity, \(0\)-bisimplicity of the semigroup \(B_\omega^\mathcal{F}\) and when \(B_\omega^\mathcal{F}\) has the identity, is isomorphic to the bicyclic semigroup or the countable semigroup of matrix units are given. In particularly in [4] is proved that the semigroup \(B_\omega^\mathcal{F}\) is isomorphic to the semigroup of \(\omega \times \omega\)-matrix units if and only if \(\mathcal{F}\) consists of a singleton set and the empty set.
The semigroup $B^\mathcal{F}_\omega$ in the case when the family $\mathcal{F}$ consists of the empty set and some singleton subsets of $\omega$ is studied in [3]. It is proved that the semigroup $B^\mathcal{F}_\omega$ is isomorphic to the subsemigroup $\mathcal{R}^\mathcal{F}_\omega(F_{\text{min}})$ of the Brandt $\omega$-extension of the subsemilattice $(F, \text{min})$ of $(\omega, \text{min})$, where $F = \bigcup \mathcal{F}$. Also topologizations of the semigroup $B^\mathcal{F}_\omega$ and its closure in semitopological semigroups are studied.

For any $n \in \omega$ we put $\mathcal{F}_n = \{[0; k] : k = 0, \ldots, n\}$. It is obvious that $\mathcal{F}_n$ is an $\omega$-closed family of $\omega$.

In the paper [5] we study the semigroup $B^\mathcal{F}_n$. It is shown that the Green relations $\mathcal{D}$ and $\mathcal{J}$ coincide in $B^\mathcal{F}_n$, the semigroup $B^\mathcal{F}_n$ is isomorphic to the semigroup $\mathcal{J}^{n+1}(\text{conv})$, and $B^\mathcal{F}_n$ admits only Rees congruences. Also in [5], we study shift-continuous topologies of the semigroup $B^\mathcal{F}_n$. In particular, we prove that for any shift-continuous $T_1$-topology $\tau$ on the semigroup $B^\mathcal{F}_n$, every non-zero element of $B^\mathcal{F}_n$ is an isolated point of $(B^\mathcal{F}_n, \tau)$, $B^\mathcal{F}_n$ admits the unique compact shift-continuous $T_1$-topology, and every $\omega$-compact shift-continuous $T_1$-topology is compact, where $\omega$ is the discrete infinite countable space.

We describe the closure of the semigroup $B^\mathcal{F}_n$ in a Hausdorff semitopological semigroup and prove the criterium when a topological inverse semigroup $B^\mathcal{F}_n$ is $H$-closed in the class of Hausdorff topological semigroups.

In the paper [6] injective endomorphisms of the semigroup $B^\mathcal{F}_n$ for a positive integer $n \geq 2$ are described. In particular, it is proved that for $n \geq 1$, the semigroup of injective endomorphisms of the semigroup $B^\mathcal{F}_n$ is isomorphic to $(\omega, +)$. Also, there the structure of the semigroup $\text{End}(\mathcal{B}_n)$ of all endomorphisms of the semigroup of $\lambda \times \lambda$-matrix units $\mathcal{B}_n$ is described.

This paper is a continuation of the investigation which are presented in [5, 6]. Let $\text{End}^1(\mathcal{J}^n(\text{conv}))$ be a subsemigroup of $\text{End}(\mathcal{J}^n(\text{conv}))$ which consists of $a \in \text{End}(\mathcal{J}^n(\text{conv}))$ such that the image $(a)n$ is isomorphic to a subsemigroup of the semigroup of $\omega \times \omega$-matrix units for all $\alpha \in \mathcal{J}^n(\text{conv})$. We describe the semigroup $\text{End}^1(\mathcal{J}^n(\text{conv}))$ of all endomorphisms of the monoid $\mathcal{J}^n(\text{conv})$ up to its ideal $\text{End}^1(\mathcal{J}^n(\text{conv}))$.

By Theorem 1 of [5], for any $n \in \omega$ the semigroup $B^\mathcal{F}_n$ is isomorphic to the semigroup $\mathcal{J}^{n+1}(\text{conv})$ by the mapping $\mathcal{J} : B^\mathcal{F}_n \rightarrow \mathcal{J}^{n+1}(\text{conv})$, defined by the formulae $(0)\mathcal{J} = 0$ and

$$(i, j, [0; k])\mathcal{J} = \begin{pmatrix} i+k & \cdots & i+1 \\ j+k & \cdots & j+1 \end{pmatrix}.$$ 

Later we study endomorphisms of the semigroup $\mathcal{J}^n(\text{conv})$.

By Theorem 2 of [5] for an arbitrary $n \in \omega$ the semigroup $B^\mathcal{F}_n$ (and hence the semigroup $\mathcal{J}^n(\text{conv})$) admits only Rees congruences. Moreover, by Theorem 3 of [5] for any homomorphism $h$ from $B^\mathcal{F}_n$ into a semigroup $S$ the image $(B^\mathcal{F}_n)h$ is either isomorphic to $B^\mathcal{F}_k$ for some $k = 0, 1, \ldots, n$, or is a singleton. Also, Lemma 1 of [6] states that if $n$ is any positive integer and $a$ is an arbitrary non-annihilating endomorphism of the semigroup $\mathcal{J}^n(\text{conv})$ then $(0)a = 0$.

By Proposition 3 of [5] for any non-negative integer $n$ the map $h_0 : B^\mathcal{F}_n \rightarrow B^\mathcal{F}_n$ defined by the formulae $(0)h_0 = 0$ and

$$(i, j, [0; k])h_0 = \begin{cases} 0, & \text{if } k = 0; \\ (i, j, [0; k-1]), & \text{if } k = 1, \ldots, n. \end{cases}$$
is an endomorphism of $B_n$. Using the isomorphism $J: B_n \to \mathcal{U}^{m+1}_n(\text{conv})$ we get that the endomorphism $h_0$ of $B_n$ generates the following endomorphism $t_1: \mathcal{U}^{m+1}_n(\text{conv}) \to \mathcal{U}^m_n(\text{conv})$ ($m \in \mathbb{N}$) which is defined by the formulae

$$(0)_{t_1} = 0, \quad (i)_{t_1} = 0, \quad (j + 1)_{t_1} = (j), \quad \ldots, \quad (j + k - 1 + i + k)_{t_1} = (j + k - 1 + i)$$

for all $i, j \in \omega$ and $k = 1, \ldots, m$. It is obvious that so defined endomorphism $t_1$ of $\mathcal{U}^m_n(\text{conv})$ generates the Rees congruence $\mathcal{C}_{t_1}$ which is generated by the ideal $\mathcal{U}^1_n(\text{conv})$. Also for $p = 1, \ldots, m$ the mapping $t_p = t_1 \circ \cdots \circ t_1$ is an endomorphism of $\mathcal{U}^m_n(\text{conv})$ and $t_p$ generates the Rees congruence $\mathcal{C}_{t_p}$ which is generated by the ideal $\mathcal{U}^p_n(\text{conv})$ of the semigroup $\mathcal{U}_n^m(\text{conv})$. Later for $p = 1, \ldots, m$ the above determined endomorphism $t_p$ we call the $p$-canonical Rees endomorphism of the semigroup $\mathcal{U}_n^m(\text{conv})$.

Later we study endomorphisms of the semigroup $\mathcal{U}_n^m(\text{conv})$ for any positive integer $n$.

By Corollary 1 of [6] for any positive integer $n$ and arbitrary $i_0 \in \omega$ the map $\epsilon_{i_0}: \mathcal{U}^n_\omega(\text{conv}) \to \mathcal{U}^n_\omega(\text{conv})$ defined by the formulae $(0)_{\epsilon_{i_0}} = 0$ and

$$(i + i + 1 \ldots + i + k)_{\epsilon_{i_0}} = \left(\begin{array}{c} i + i + 1 \ldots + i + k \\ i + i + 1 \ldots + i + k \\ \vdots \\ i + i + 1 \ldots + i + k \end{array}\right), \quad k = 0, \ldots, n - 1,$$

is an endomorphism of the semigroup $\mathcal{U}^n_\omega(\text{conv})$, and moreover it is injective. It is obvious for any $i_0 \in \omega$ the endomorphism $\epsilon_{i_0}$ generates the identity congruence on the semigroup $\mathcal{U}^n_\omega(\text{conv})$. Also, by Theorem 1 of [6] for any positive integer $n \geq 2$ for every injective endomorphism $\alpha: \mathcal{U}^n_\omega(\text{conv}) \to \mathcal{U}^n_\omega(\text{conv})$ there exists $i_0 \in \omega$ such that $\alpha = \epsilon_{i_0}$.

Fix an arbitrary $i_0 \in \omega$. Then we have that

$$(0)_{t_1} = (0)_{t_1} = 0,$$

$$(i)_{t_1} = (i)_{t_1} = 0, \quad (j + 1)_{t_1} = (j)_{i_0} = (j + i_0),$$

$$\ldots,$$

$$(i + i + 1 \ldots + i + k)_{t_1} = (i + i + 1 \ldots + i + k)_{t_1} = (i + i + 1 \ldots + i + k + i + i + 1)_{i_0} = (i + i + 1 \ldots + i + k + i + i + 1)_{j + i + 1 \ldots + i + k + i + i + 1}$$

and

$$(0)_{t_1} = (0)_{t_1} = 0,$$

$$(i)_{t_1} = (i)_{t_1} = 0, \quad (j + 1)_{t_1} = (j + 1)_{i_0} = (j + i_0),$$

$$\ldots,$$

$$(i + i + 1 \ldots + i + k)_{t_1} = (i + i + 1 \ldots + i + k)_{t_1} = (i + i + 1 \ldots + i + k + i + i + 1)_{j + i + 1 \ldots + i + k + i + i + 1}$$

for all $i, j \in \omega$ and $k = 1, \ldots, n$. This implies that $t_1 \circ t_1 = t_1 \circ t_1$. Then the definition of the $p$-canonical Rees endomorphism $t_1$ of the semigroup $\mathcal{U}^{m+1}_n(\text{conv})$ implies the following lemma.
Lemma 1. Let \( n \) be a positive integer \( \geq 2 \). Then for any \( p = 1, \ldots, n - 1 \) and \( i_0 \in \omega \) the \( p \)-canonical Rees endomorphism \( \tau_1 \) and injective endomorphism \( \epsilon_{i_0} \) of the semigroup \( \mathcal{I}_n^\omega(\text{conv}) \) commute, i.e., \( \epsilon_{i_0} \circ \tau_p = \tau_p \circ \epsilon_{i_0} \).

By \( \text{End}(\mathcal{I}_n^\omega(\text{conv})) \) we denote the semigroup of all endomorphisms of the monoid \( \mathcal{I}_n^\omega(\text{conv}) \). We define

\[
\text{End}^1(\mathcal{I}_n^\omega(\text{conv})) = \{ a \in \text{End}(\mathcal{I}_n^\omega(\text{conv})) | (\mathcal{I}_n^\omega(\text{conv}))a \subseteq \mathcal{I}_1^\omega(\text{conv}) \}.
\]

Observe that the set \( \text{End}^1(\mathcal{I}_n^\omega(\text{conv})) \) is an ideal of \( \text{End}(\mathcal{I}_n^\omega(\text{conv})) \). Indeed, let \( b \in \text{End}(\mathcal{I}_n^\omega(\text{conv})) \) and \( a \in \text{End}^1(\mathcal{I}_n^\omega(\text{conv})) \). Then for any \( \alpha \in \mathcal{I}_n^\omega(\text{conv}) \) the definition of the monoid \( \mathcal{I}_n^\omega(\text{conv}) \) implies that

\[
(\alpha)(a \circ b) \in ((\mathcal{I}_n^\omega(\text{conv}))a)b \subseteq (\mathcal{I}_1^\omega(\text{conv}))b \subseteq \mathcal{I}_1^\omega(\text{conv}),
\]

and

\[
(\alpha)(b \circ a) \in ((\mathcal{I}_n^\omega(\text{conv}))b)a \subseteq (\mathcal{I}_1^\omega(\text{conv}))a \subseteq \mathcal{I}_1^\omega(\text{conv}).
\]

Let \( a \in \text{End}^1(\mathcal{I}_n^\omega(\text{conv})) \). By Theorems 1 and 3 of \([5]\) the image \( (\mathcal{I}_n^\omega(\text{conv}))a \) is isomorphic to the semigroup \( \mathcal{I}_1^\omega(\text{conv}) \), which is isomorphic to the semigroup of \( \omega \times \omega \)-matrix units \( B_\omega \). This implies that there exists an isomorphism \( \epsilon : \mathcal{I}_1^\omega(\text{conv}) \to (\mathcal{I}_n^\omega(\text{conv}))a \). Then we have that \( a = \tau_{n-1} \circ \epsilon \), where \( \tau_{n-1} \) is the \((n-1)\)-canonical Rees endomorphism of \( \mathcal{I}_n^\omega(\text{conv}) \).

We denote

\[
\text{End}^*(\mathcal{I}_n^\omega(\text{conv})) = \text{End}(\mathcal{I}_n^\omega(\text{conv})) \setminus \text{End}^1(\mathcal{I}_n^\omega(\text{conv})).
\]

It is obvious that \( a \in \text{End}^*(\mathcal{I}_n^\omega(\text{conv})) \) if and only if

\[
(\mathcal{I}_n^\omega(\text{conv}))a \cap (\mathcal{I}_n^\omega(\text{conv})) \neq \emptyset.
\]

Let \( b \in \text{End}^*(\mathcal{I}_n^\omega(\text{conv})) \). Theorems 1 and 3 of \([5]\), and an equality \( \text{ran} \ b = k \), implies that the image \( (\mathcal{I}_n^\omega(\text{conv}))b \) is isomorphic to the semigroup \( \mathcal{I}_k^\omega(\text{conv}) \) for any \( k \in \{2, 3, \ldots, n\} \). Then there exists an isomorphism \( \epsilon_{i_0} : \mathcal{I}_k^\omega(\text{conv}) \to (\mathcal{I}_n^\omega(\text{conv}))b \) such that \( (\mathcal{I}_n^\omega(\text{conv}))b = (\mathcal{I}_k^\omega(\text{conv})) \epsilon_{i_0} \). Hence, \( b = \epsilon_{i_0} \circ \tau_{n-k} \), where \( \tau_{n-k} \) is the \((n-k)\)-canonical Rees endomorphism of the monoid \( \mathcal{I}_n^\omega(\text{conv}) \).

The above arguments imply the following theorem.

Theorem 1. The semigroup \( \text{End}(\mathcal{I}_n^\omega(\text{conv})) \) of all endomorphisms of the semigroup \( \mathcal{I}_n^\omega(\text{conv}) \) is the disjoint union of the set \( \text{End}^*(\mathcal{I}_n^\omega(\text{conv})) \) and the ideal \( \text{End}^1(\mathcal{I}_n^\omega(\text{conv})) \). Moreover,

- for any \( a \in \text{End}^*(\mathcal{I}_n^\omega(\text{conv})) \) we have that \( a = \tau_{n-1} \circ \epsilon \), and
- for any \( b \in \text{End}^1(\mathcal{I}_n^\omega(\text{conv})) \) we have that \( b = \epsilon_{i_0} \circ \tau_{n-k} \).

Simple verifications show that for any \( p_1 \)- and \( p_2 \)-canonical Rees endomorphisms \( \tau_{p_1} \) and \( \tau_{p_2} \) we have that

\[
\tau_{p_1} \circ \tau_{p_2} = \tau_{p_1+p_2} = \tau_{p_2} \circ \tau_{p_1},
\]

and moreover, in the case when \( p_1 + p_2 \geq n \), \( \tau_{p_1} \circ \tau_{p_2} \) is the annihilating endomorphisms of the monoid \( \mathcal{I}_n^\omega(\text{conv}) \), i.e., \( (\alpha)(\tau_{p_1} \circ \tau_{p_2}) = 0 \), for all \( \alpha \in \mathcal{I}_n^\omega(\text{conv}) \). This implies the following proposition.

Proposition 1. For any positive integer \( n \), the semigroup of \( p \)-canonical Rees endomorphisms of the semigroup \( \mathcal{I}_n^\omega(\text{conv}) \) is isomorphic to the semigroup \( (\omega_n, +) \).
By Theorem 2 of [6] for \( n \geq 2 \) the semigroup of injective endomorphisms of the semigroup \( \mathcal{I}_n^\omega(\text{conv}) \) is isomorphic to the semigroup \( (\omega, +) \).

Let \( I_0^\omega = \{(0, j) \mid j \in \omega\} \) be a subset of the direct product of the semigroups \( (\omega_{n-1}, +) \) and \( (\omega, +) \). It is obvious that \( I_0^\omega \) is an ideal of the semigroup \( (\omega_{n-1}, +) \times (\omega, +) \).

This implies the following theorem.

**Theorem 2.** For any positive integer \( n \) the semigroup \( \text{End}(\mathcal{I}_n^\omega(\text{conv}))/\text{End}_1(\mathcal{I}_n^\omega(\text{conv})) \) is isomorphic to the Rees quotient semigroup \( (\omega_{n-1}, +) \times (\omega, +))/I_0^\omega \).

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ПРО ЕНДОМОРФІЗМИ ІНВЕРСНОЇ НАПІВГРУПИ ПОРЯДКОВО ОПУКЛИХ ІЗОМОРФІЗМІВ МНОЖИННИ \( \omega \) ОБМЕЖЕНОГО РАНГУ, ЯКІ ПОРУДЖЕНИ КОНГРУЕНЦІЯМИ РІСА

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Нехай \( \mathcal{K}^\omega_{(\text{conv})} \) — інверсна напівгрупа порядково опуклих ізоморфізмів лінійно впорядкованої множини \( (\omega, \leq) \) рангу \( \leq n \). Нехай \( \text{End}^1(\mathcal{K}^\omega_{(\text{conv})}) \) — піднапівгрупа напівгрупи \( \text{End}(\mathcal{K}^\omega_{(\text{conv})}) \), яка складається з таких елементів \( a \in \text{End}(\mathcal{K}^\omega_{(\text{conv})}) \), що образ \( (a) \) ізоморфній піднапівгрупні напівгрупи \( \omega \times \omega \)-матричних одиниць для всіх \( \alpha \in \mathcal{K}^\omega_{(\text{conv})} \). Ми описуємо напівгрупу \( \text{End}(\mathcal{K}^\omega_{(\text{conv})}) \) усіх ендоарифемів модеї \( \mathcal{K}^\omega_{(\text{conv})} \) за модулем ідеалу \( \text{End}^1(\mathcal{K}^\omega_{(\text{conv})}) \).

Ключові слова: біциклічне розширення, інверсна напівгрупа, ендоарифем, автоморфізм, напівгрупа \( \lambda \times \lambda \)-матричних одиниць.