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THE MONOID OF ORDER ISOMORPHISMS BETWEEN PRINCIPAL FILTERS OF $\sigma\mathbb{N}^{\kappa}$

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Consider the following generalization of the bicyclic monoid. Let κ be any infinite cardinal and let $\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$ be the semigroup of all order isomorphisms between principal filters of the set $\sigma\mathbb{N}^{\kappa}$ with the product order. We shall study algebraic properties of the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$, show that it is bisimple, E-unitary, F-inverse semigroup, describe Green's relations on $\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$, describe the group of units $H(\mathbb{I})$ of the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$ and describe its maximal subgroups. We prove that the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$ is isomorphic to the semidirect product $\mathcal{S}_{\kappa} \ltimes \sigma\mathbb{B}^{\kappa}$ of the semigroup $\sigma\mathbb{B}^{\kappa}$ by the group $\mathcal{S}_{\kappa}$, show that every non-identity congruence $\mathfrak{C}$ on the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$ is a group congruence and describe the least group congruence on $\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$.

Key words: Semigroup, inverse semigroup, partial map, permutation group, least group congruence, bicyclic monoid, semidirect product

1. Introduction and preliminaries

In this paper, we shall denote the set of integers by $\mathbb{Z}$, the set of positive integers by $\mathbb{N}$, the set of all maps from cardinal κ to the set X by X^{κ} and the symmetric group of degree κ by $\mathcal{S}_{\kappa}$, i.e., $\mathcal{S}_{\kappa}$ is the group of all bijections of the set κ. For set X, by id_X we denote the identity map $id_X \colon X \to X$, $id_X \colon x \mapsto x$ for any $x \in X$. For map $f \colon X \to Y$ and for subset $A \subset X$ we denote $(A)f = \{(x)f \mid x \in X\}$.

Let $(X, \leqslant)$ be a partially ordered set (a poset). For an arbitrary $x \in X$ we denote

$$\uparrow x = \{y \in X : x \leqslant y\} \qquad \text{and} \qquad \downarrow x = \{y \in X : y \leqslant x\}.$$

The sets $\uparrow x$ and $\downarrow x$ are called the *principal filter* and the *principal ideal*, respectively, generated by the element $x \in X$. A map $\alpha \colon (X, \leqslant) \to (Y, \leqslant)$ from poset $(X, \leqslant)$ into a poset $(Y, \leqslant)$ is called *monotone* or *order preserving* if $x \leqslant y$ in $(X, \leqslant)$ implies that

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$x\alpha \leqslant y\alpha$ in $(Y, \leqslant)$. A monotone map $\alpha\colon (X, \leqslant) \to (Y, \leqslant)$ is said to be *order isomorphism* if it is bijective and its converse $\alpha^{-1}\colon (Y, \leqslant) \to (X, \leqslant)$ is monotone.

An semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication. The semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents.

If S is a semigroup, then we shall denote the Green relations on S by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ (see [9]). A semigroup S is called *simple* if S does not contain proper two-sided ideals and *bisimple* if S has only one $\mathscr{D}$-class.

Hereafter we shall assume that λ is an infinite cardinal. If $\alpha\colon \lambda \rightharpoonup \lambda$ is a partial map, then we shall denote the domain and the range of α by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively.

Let $\mathscr{I}_\lambda$ be the set of all partial one-to-one transformations of a cardinal λ together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \quad \text{if} \quad x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \mid y\alpha \in \operatorname{dom}\beta\}, \quad \text{for } \alpha, \beta \in \mathscr{I}_\lambda.$$

The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the cardinal λ (see [9, Section 1.9]). The symmetric inverse semigroup was introduced by Wagner [29] and it plays a major role in the theory of semigroups.

The *bicyclic semigroup* (or the *bicyclic monoid*) $\mathscr{C}(p, q)$ is the semigroup with the identity 1 generated by elements p and q subject only to the condition $pq = 1$.

The bicyclic semigroup plays an important role in the algebraic theory of semigroups and the theory of topological semigroups. For instance, a well-known Andersen's result [1] states that a (0-)simple semigroup with an idempotent is completely (0-)simple if and only if it does not contain an isomorphic copy of the bicyclic semigroup.

The bicyclic monoid admits only the discrete semigroup topology. Bertman and West in [7] extended this result for the case of semitopological semigroups. Stable and Γ-compact topological semigroups do not contain the bicyclic monoid [2, 22]. The problem of an embedding of the bicyclic monoid into compact-like topological semigroups was studied in [3, 4, 19]. The study of various generalizations of the bicyclic monoid, their algebraic and topological properties, like topologizations, shift-continuous topologizations and embedding into compact-like topological semigroups was conducted in several publications, including [5, 6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, 25, 18].

Remark 1. We observe that the bicyclic semigroup is isomorphic to the semigroup $\mathscr{C}_{\mathbb{N}}(\alpha, \beta)$ which is generated by partial transformations α and β of the set of positive integers $\mathbb{N}$, defined as follows: $(n)\alpha = n + 1$ if $n \geqslant 1$ and $(n)\beta = n - 1$ if $n > 1$ (see Exercise IV.1.11(ii) in [27]).

Taking into account this remark, we shall consider the following generalization of the bicyclic semigroup. For an arbitrary positive integer $n \geqslant 2$ by $(\mathbb{N}^n, \leqslant)$ we denote the n-th power of the set of positive integers $\mathbb{N}$ with the product order:

$$(x_1, \ldots, x_n) \leqslant (y_1, \ldots, y_n) \qquad \text{if and only if} \qquad x_i \leqslant y_i \quad \text{for all} \quad i = 1, \ldots, n.$$

It is obvious that the set of all order isomorphisms between principal filters of the poset $(\mathbb{N}^n, \leqslant)$ with the operation of the composition of partial maps forms a semigroup. Denote this semigroup by $\mathcal{IPF}(\mathbb{N}^n)$. The structure of the semigroup $\mathcal{IPF}(\mathbb{N}^n)$ was introduced and studied in [15]. There was shown that $\mathcal{IPF}(\mathbb{N}^n)$ is a bisimple, E-unitary, F-inverse monoid, described Green's relations on $\mathcal{IPF}(\mathbb{N}^n)$ and its maximal subgroups. It was proved that $\mathcal{IPF}(\mathbb{N}^n)$ is isomorphic to the semidirect product of the direct n-th power of the bicyclic monoid $\mathscr{C}^n(p,q)$ by the group of permutation $\mathcal{S}_n$, every non-identity congruence on $\mathcal{IPF}(\mathbb{N}^n)$ is group and was described the least group congruence on $\mathcal{IPF}(\mathbb{N}^n)$. It was shown that every shift-continuous topology on $\mathcal{IPF}(\mathbb{N}^n)$ is discrete and discussed embedding of the semigroup $\mathcal{IPF}(\mathbb{N}^n)$ into compact-like topological semigroups. In [25] it was proved that a Hausdorff locally compact semitopological semigroup $\mathcal{IPF}(\mathbb{N}^n)$ with an adjoined zero is either compact or discrete. In this paper we shall extend this generalization from $\mathbb{N}^n$ to $\sigma\mathbb{N}^\kappa$ for any infinite cardinal κ.

For any infinite cardinal κ consider the subset $\sigma\mathbb{N}^\kappa$ of $\mathbb{N}^\kappa$ which contains all maps a such that the set $\{x \in \kappa \mid (x)\,a \neq 1\}$ is finite, i.e.,

$$\sigma\mathbb{N}^\kappa = \{a \in \mathbb{N}^\kappa \mid \{x \in \kappa \mid (x)\,a \neq 1\} \text{ is finite }\}.$$

Similarly define $\sigma\mathbb{Z}^\kappa$ as the subset of $\mathbb{Z}^\kappa$ which contains all maps a such that the set $\{x \in \kappa \mid (x)\,a \neq 0\}$ is finite.

By $\mathbf{1}$ we shall denote the element of the $\mathbb{N}^\kappa$ such that $(x)\,\mathbf{1} = 1$ for any $x \in \kappa$.

On the set $\mathbb{Z}^\kappa$ consider the product order $\leqslant$:

$$a \leqslant b \qquad \text{if and only if} \qquad (x)\,a \leqslant (x)\,b \quad \text{for all} \quad x \in \kappa.$$

Also, consider the pointwise operations $+$, $-$, max and min on the set $\mathbb{Z}^\kappa$. For any $a, b \in \mathbb{Z}^\kappa$ define

$$\begin{aligned}
&(x)\,(a+b) = (x)\,a + (x)\,b,\\
&(x)\,(a-b) = (x)\,a - (x)\,b,\\
&(x)\,(\max\{a,b\}) = \max\{(x)\,a, (x)\,b\},\\
&(x)\,(\min\{a,b\}) = \min\{(x)\,a, (x)\,b\}
\end{aligned}$$

for any $x \in \kappa$. It is obvious that the set $\sigma\mathbb{Z}^\kappa$ is closed under these operations. The set $\sigma\mathbb{N}^\kappa$ is also closed under the operation max and min but not for $+$ and $-$. Moreover

$$a+b, a-b \notin \sigma\mathbb{N}^\kappa \qquad \text{for any} \qquad a, b \in \sigma\mathbb{N}^\kappa.$$

But

$$a+b-\mathbf{1} \in \sigma\mathbb{N}^\kappa \qquad \text{for any} \qquad a, b \in \sigma\mathbb{N}^\kappa,$$

and

$$a-b+\mathbf{1} \in \sigma\mathbb{N}^\kappa \qquad \text{for any} \qquad a \in \sigma\mathbb{N}^\kappa \qquad \text{and} \qquad b \in {\downarrow}a.$$

Let κ by any infinite cardinal. Define the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ as the set of all order isomorphisms between principal filters of the poset $(\sigma\mathbb{N}^\kappa, \leqslant)$ with the operation of the composition of partial maps, i.e.,

$$\mathcal{IPF}(\sigma\mathbb{N}^\kappa) = (\{\alpha\colon {\uparrow}a \to {\uparrow}b \mid a, b \in \sigma\mathbb{N}^\kappa \text{ and } \alpha \text{ is an order isomorphism}\}, \circ).$$

Consider the following notation. For any $\alpha \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ by d_α and r_α we denote the elements of $\sigma\mathbb{N}^\kappa$ such that $\operatorname{dom}\alpha = {\uparrow}d_\alpha$ and $\operatorname{ran}\alpha = {\uparrow}r_\alpha$

Also we define the maps $\lambda_\alpha, \rho_\alpha \in \mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ in the following way:

$$\operatorname{dom}\rho_\alpha = \operatorname{dom}\alpha = {\uparrow}d_\alpha, \quad \operatorname{ran}\rho_\alpha = \sigma\mathbb{N}^\kappa, \quad (a)\rho_\alpha = a - d_\alpha + \mathbf{1} \quad \text{for } a \in \operatorname{dom}\rho_\alpha;$$

$$\operatorname{ran}\lambda_\alpha = \operatorname{ran}\alpha = {\uparrow}r_\alpha, \quad \operatorname{dom}\lambda_\alpha = \sigma\mathbb{N}^\kappa, \quad (a)\lambda_\alpha = a + r_\alpha - \mathbf{1} \quad \text{for } a \in \operatorname{dom}\lambda_\alpha.$$

Since $a + r_\alpha - \mathbf{1} \in \sigma\mathbb{N}^\kappa$ for any $a \in \operatorname{dom}\lambda_\alpha$ we have that λ_α is well-defined. Similarly, $a - d_\alpha + \mathbf{1} \in \sigma\mathbb{N}^\kappa$ for any $a \in \operatorname{dom}\rho_\alpha$, so ρ_α is well-defined too. We note that the definition of $\lambda_\alpha, \rho_\alpha$ implies that $\lambda_{\lambda_\alpha} = \lambda_\alpha$ and $\rho_{\rho_\alpha} = \rho_\alpha$.

For any infinite cardinal κ and for any bijection $g \in \mathcal{S}_\kappa$ define the selfmap $\mathcal{F}_g \colon \mathbb{Z}^\kappa \to \mathbb{Z}^\kappa$ by formula:

$$(x)(a)\mathcal{F}_g = \big((x)g^{-1}\big)a, \ a \in \mathbb{Z}^\kappa, \ x \in \kappa.$$

2. Algebraic properties of the semigroup $\mathscr{IPF}(\sigma\mathbb{N}^\kappa)$

Proposition 1. *For any infinite cardinal κ the following statements hold:*

(i) *$\mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ is an inverse semigroup;*

(ii) *the semilattice $E(\mathscr{IPF}(\sigma\mathbb{N}^\kappa))$ is isomorphic to the semilattice $(\sigma\mathbb{N}^\kappa, \max)$ by the mapping $\varepsilon \mapsto d_\varepsilon$;*

(iii) *$\alpha\mathscr{L}\beta$ in $\mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$;*

(iv) *$\alpha\mathscr{R}\beta$ in $\mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ if and only if $\operatorname{ran}\alpha = \operatorname{ran}\beta$;*

(v) *$\alpha\mathscr{H}\beta$ in $\mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$ and $\operatorname{ran}\alpha = \operatorname{ran}\beta$;*

(vi) *for any idempotents $\varepsilon, \iota \in \mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ there exist elements $\alpha, \beta \in \mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ such that $\alpha\beta = \varepsilon$ and $\beta\alpha = \iota$, hence $\mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ is bisimple which implies that it is simple.*

Proof. (i) The definition of the semigroup $\mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ implies that $\mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ is an inverse subsemigroup of the symmetric inverse monoid $\mathscr{I}_{\sigma\mathbb{N}^\kappa}$ over the set $\sigma\mathbb{N}^\kappa$.

(ii) implies from statement (i).

(iii)–(v) follow from statement (i) and Proposition 3.2.11(1)–(3) of [23].

(vi) Fix arbitrary idempotents $\varepsilon, \iota \in \mathscr{IPF}(\sigma\mathbb{N}^\kappa)$. Define a partial map $\alpha \colon \sigma\mathbb{N}^\kappa \rightharpoonup \sigma\mathbb{N}^\kappa$ in the following way:

$$\operatorname{dom}\alpha = \operatorname{dom}\varepsilon, \qquad \operatorname{ran}\alpha = \operatorname{dom}\iota \qquad \text{and} \qquad (z)\alpha = z - d_\varepsilon + d_\iota, \quad \text{for any} \quad z \in \operatorname{dom}\alpha.$$

Since $\varepsilon, \iota \in \mathscr{IPF}(\sigma\mathbb{N}^\kappa)$, the partial map α is well-defined and $\alpha \in \mathscr{IPF}(\sigma\mathbb{N}^\kappa)$. Then $\alpha\alpha^{-1} = \varepsilon$ and $\alpha^{-1}\alpha = \iota$ and hence we put $\beta = \alpha^{-1}$. Lemma 1.1 from [26] implies that $\mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ is bisimple and hence simple. □

For any positive integer $k \geqslant 2$ and for any $x \in \kappa$, consider the map $k_x \colon \kappa \to \mathbb{N}$ defined by

$$(t)k_x = \begin{cases} k, & \text{if } \ t = x, \\ 1, & \text{otherwise.} \end{cases}$$

Lemma 1. *For any infinite cardinal κ and for any bijection $g \in \mathcal{S}_\kappa$, the following statements hold:*

(i) *The selfmap $\mathcal{F}_g$ is an order automorphism of the poset $(\mathbb{Z}^\kappa, \leqslant)$, and $(\mathcal{F}_g)^{-1} = \mathcal{F}_{g^{-1}}$.*

(ii) *$(\sigma\mathbb{N}^\kappa)\mathcal{F}_g = \sigma\mathbb{N}^\kappa$.*

(iii) *$(\sigma\mathbb{Z}^\kappa)\mathcal{F}_g = \sigma\mathbb{Z}^\kappa$.*

(iv) $\mathcal{F}_{gh} = \mathcal{F}_g\mathcal{F}_h$ *for any* $h \in \mathcal{S}_\kappa$.
(v) *For any* $k \in \mathbb{N}$ *and for any* $x \in \kappa$: $(k_x)\,\mathcal{F}_g = k_{(x)g}$.
(vi) $(\mathbf{1})\,\mathcal{F}_g = \mathbf{1}$.
(vii) *For any* $h \in \mathcal{S}_\kappa$: $g \neq h \implies \mathcal{F}_g \neq \mathcal{F}_h$.
(viii) *For any* $a, b \in \mathbb{Z}^\kappa$: $(a + b)\,\mathcal{F}_g = (a)\,\mathcal{F}_g + (b)\,\mathcal{F}_g$.
(ix) *For any* $a, b \in \mathbb{Z}^\kappa$: $(a - b)\,\mathcal{F}_g = (a)\,\mathcal{F}_g - (b)\,\mathcal{F}_g$.
(x) *For any* $a, b \in \mathbb{Z}^\kappa$: $(\max\{a, b\})\,\mathcal{F}_g = \max\{(a)\,\mathcal{F}_g, (b)\,\mathcal{F}_g\}$.
(xi) *For any* $a, b \in \mathbb{Z}^\kappa$: $(\min\{a, b\})\,\mathcal{F}_g = \min\{(a)\,\mathcal{F}_g, (b)\,\mathcal{F}_g\}$.

Proof. (i) Show that $\mathcal{F}_g$ is an order isomorphism. Fix distinct $a, b \in \mathbb{Z}^\kappa$. Then there exists $x \in \kappa$ such that $(x)\,a \neq (x)\,b$. For $y = (x)\,g$, we have that $x = (y)\,g^{-1}$, then $\big((y)\,g^{-1}\big)\,a \neq \big((y)\,g^{-1}\big)\,b$ implies that $(a)\,\mathcal{F}_g \neq (b)\,\mathcal{F}_g$, so $\mathcal{F}_g$ is injective.

For any $a \in \mathbb{Z}^\kappa$, consider the map b: $(x)\,b = ((x)\,g)\,a$ for any $x \in \kappa$, then

$$(x)\,(b)\,\mathcal{F}_g = \big((x)\,g^{-1}\big)\,b = \big(\big((x)\,g^{-1}\big)\,g\big)\,a = (x)\,a$$

for any $x \in \kappa$, so $\mathcal{F}_g$ is surjective and moreover its converse $(\mathcal{F}_g)^{-1}$ is equals to the $\mathcal{F}_{g^{-1}}$.

Let $a, b \in \mathbb{Z}^\kappa$ and $a \leqslant b$. For any $x \in \kappa$ we have that $\big((x)\,g^{-1}\big)\,a \leqslant \big((x)\,g^{-1}\big)\,b$ which implies that $(x)\,(a)\,\mathcal{F}_g \leqslant (x)\,(b)\,\mathcal{F}_g$, i.e., $(a)\,\mathcal{F}_g \leqslant (b)\,\mathcal{F}_g$, so $\mathcal{F}_g$ is monotone and such is $\mathcal{F}_g^{-1}$, therefore $\mathcal{F}_g$ is an order isomorphism.

(ii) Fix an element $a \in \sigma\mathbb{N}^\kappa$. Since $(x)\,(a)\,\mathcal{F}_g = \big((x)\,g^{-1}\big)\,a \in \mathbb{N}$ for any $x \in \kappa$ we have that $(a)\,\mathcal{F}_g \in \mathbb{N}^\kappa$. Consider the set $A = \{x \in \kappa \mid (x)\,a \neq 1\}$ and suppose that $(x)\,(a)\,\mathcal{F}_g \neq 1$ for some $x \in \kappa$, then $\big((x)\,g^{-1}\big)\,a \neq 1$ and therefore $(x)\,g^{-1} \in A$, so $x \in (A)\,g$. Since the set A is finite and g is a bijection, we have that the set $(A)\,g$ is finite as well. So $(a)\,\mathcal{F}_g \in \sigma\mathbb{N}^\kappa$, therefore $(\sigma\mathbb{N}^\kappa)\,\mathcal{F}_g \subset \sigma\mathbb{N}^\kappa$. By proved above, we have that $(a)\,\mathcal{F}_{g^{-1}} \in \sigma\mathbb{N}^\kappa$, then $\big((a)\,\mathcal{F}_{g^{-1}}\big)\,\mathcal{F}_g = a$ implies that $\sigma\mathbb{N}^\kappa \subset (\sigma\mathbb{N}^\kappa)\,\mathcal{F}_g$.

(iii) The proof is similar to the proof of (ii).

(iv) For any $h \in \mathcal{S}_\kappa$, $a \in \mathbb{Z}^\kappa$ and $x \in \kappa$ we have that

$$\begin{aligned}(x)\,(a)\,\mathcal{F}_{gh} &= \Big((x)\,(gh)^{-1}\Big)\,a = \\ &= \big((x)\,\big(h^{-1}g^{-1}\big)\big)\,a = \\ &= \big(\big((x)\,h^{-1}\big)\,g^{-1}\big)\,a = \\ &= \big((x)\,h^{-1}\big)\,(a)\,\mathcal{F}_g = \\ &= (x)\,((a)\,\mathcal{F}_g)\,\mathcal{F}_h = \\ &= (x)\,(a)\,(\mathcal{F}_g\mathcal{F}_h)\,.\end{aligned}$$

(v) Let $k \in \mathbb{N}$ and $x \in \kappa$. Then for any $t \in \kappa$ we have that

$$\begin{aligned}(t)\,(k_x)\,\mathcal{F}_g &= \big((t)\,g^{-1}\big)\,k_x = \\ &= \begin{cases} k, & \text{if } (t)\,g^{-1} = x \\ 1, & \text{otherwise} \end{cases} = \\ &= \begin{cases} k, & \text{if } t = (x)\,g \\ 1, & \text{otherwise} \end{cases} = \\ &= (t)\,k_{(x)g}.\end{aligned}$$

(vi) For any $t \in \kappa$ we have that $(t)\,(\mathbf{1})\,\mathcal{F}_g = \big((t)\,g^{-1}\big)\,\mathbf{1} = 1$.

(vii) Let $h \in \mathcal{S}_\kappa$ and $g \neq h$. Then there exists $x \in \kappa$ such that $(x)\,g^{-1} \neq (x)\,h^{-1}$. Consider the image of $2_{(x)g^{-1}}$ under the maps $\mathcal{F}_g$ and $\mathcal{F}_h$. Statement (v) and the inequality $(x)\,g^{-1} \neq (x)\,h^{-1}$ imply that:

$$\big(2_{(x)g^{-1}}\big)\,\mathcal{F}_g = 2_x \neq 2_{((x)g^{-1})h} = \big(2_{(x)g^{-1}}\big)\,\mathcal{F}_h.$$

$(viii)$ For any $a, b \in \mathbb{Z}^\kappa$ and for any $x \in \kappa$ we have that

$$\begin{aligned}(x)\,(a+b)\,\mathcal{F}_g &= \big((x)\,g^{-1}\big)\,(a+b) = \\ &= \big((x)\,g^{-1}\big)\,a + \big((x)\,g^{-1}\big)\,b = \\ &= (x)\,(a)\,\mathcal{F}_g + (x)\,(b)\,\mathcal{F}_g.\end{aligned}$$

Proof of statements (ix) and (xi) are similar to the proof of $(viii)$. □

For any infinite cardinal κ and for any bijection $g \in \mathcal{S}_\kappa$ define the map $\mathcal{F}_g^\circ\colon \sigma\mathbb{N}^\kappa \to \sigma\mathbb{N}^\kappa$ as the restriction of the map $\mathcal{F}_g$ to the set $\sigma\mathbb{N}^\kappa$. By statement (ii) of Lemma 1, the map $\mathcal{F}_g^\circ$ is well-defined and $\mathcal{F}_g^\circ$ is a bijection. This and statement (i) of Lemma 1 imply that the map $\mathcal{F}_g^\circ$ is an order isomorphism of the poset $(\sigma\mathbb{N}^\kappa, \leqslant)$. Similarly, define the map $\mathcal{F}_g^\diamond\colon \sigma\mathbb{Z}^\kappa \to \sigma\mathbb{Z}^\kappa$ as the restriction of the map $\mathcal{F}_g$ to the set $\sigma\mathbb{Z}^\kappa$. And similarly, statement (iii) of Lemma 1 implies that the map $\mathcal{F}_g^\diamond$ is well-defined and $\mathcal{F}_g^\diamond$ is a bijection.

The proof to the next lemma is similar to the proof of Lemma 1.

Lemma 2. *For any infinite cardinal κ and for any bijection $g \in \mathcal{S}_\kappa$ statements $(iv)-(xi)$ of Lemma 1 also hold for $\mathcal{F}_g^\circ$ and $\mathcal{F}_g^\diamond$.*

We shall denote by $\mathbb{I}$ the identity map of $\sigma\mathbb{N}^\kappa$. It is obvious that $\mathbb{I}$ is the unit element of the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$. Also by $H(\mathbb{I})$ we shall denote the group of units of $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$. It is clear that $\alpha \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ is an element of $H(\mathbb{I})$ if and only if it is an order isomorphism of the poset $(\sigma\mathbb{N}^\kappa, \leqslant)$.

Lemma 3. *Let κ be any infinite cardinal and $\alpha \in H(\mathbb{I})$. Then $(\mathbf{1})\,\alpha = \mathbf{1}$ and for any $x \in \kappa$ there exists $y \in \kappa$ such that $(k_x)\,\alpha = k_y$ for any positive integer $k \geqslant 2$.*

Proof. Consider $(\mathbf{1})\,\alpha$. Statement $\mathbf{1} \leqslant (\mathbf{1})\,\alpha$ implies that $(\mathbf{1})\,\alpha^{-1} \leqslant ((\mathbf{1})\,\alpha)\,\alpha^{-1} = \mathbf{1}$, so $(\mathbf{1})\,\alpha = \mathbf{1}$.

Now, consider any $x \in \kappa$ and consider $(2_x)\,\alpha$. Since $\mathbf{1} = (\mathbf{1})\,\alpha \neq (2_x)\,\alpha$, there exists $y \in \kappa$ such that $2_y \leqslant (2_x)\,\alpha$, and the inequality $(2_y)\,\alpha^{-1} \leqslant 2_x$ implies that $(2_x)\,\alpha = 2_y$.

Let $k \geqslant 2$ be a positive integer, suppose that for any positive integer $n \leqslant k$ the statement of the lemma holds.

For any $x \in \kappa$ consider the image $((k+1)_x)\,\alpha$. There exists $z \in \kappa$ such that $(k+1)_z \leqslant ((k+1)_x)\,\alpha$. Suppose the contrary that $(k+1)_z \nleqslant ((k+1)_x)\,\alpha$ for any $z \in \kappa$. Since

$$((k+1)_x)\,\alpha \notin \{\mathbf{1}, 2_z, 3_z, \ldots, k_z \mid z \in \kappa\},$$

there exist two distinct elements $z_1, z_2 \in \kappa$ such that

$$1 < (z_1)\,((k+1)_x)\,\alpha < k+1 \quad \text{and} \quad 1 < (z_2)\,((k+1)_x)\,\alpha < k+1.$$

Hence we have that

$$2_{z_1} \leqslant ((k+1)_x)\,\alpha \quad \text{and} \quad 2_{z_2} \leqslant ((k+1)_x)\,\alpha,$$

and then

$$(2_{z_1})\,\alpha^{-1} \leqslant (k+1)_x \quad \text{and} \quad (2_{z_2})\,\alpha^{-1} \leqslant (k+1)_x\,.$$

Since $(2_{z_1})\,\alpha^{-1} = 2_{z'_1}$ and $(2_{z_2})\,\alpha^{-1} = 2_{z'_2}$ for some z'_1, z'_2 we have that $z'_1 = z'_2$. Then $2_{z_1} = 2_{z_2}$ and hence $z_1 = z_2$, which contradicts $z_1 \neq z_2$. Thus, $((k+1)_z)\,\alpha^{-1} \leqslant (k+1)_x$. Since $((k+1)_z)\,\alpha^{-1} \notin \{\mathbf{1}, 2_x, 3_x, \ldots, k_x\}$, we have that $((k+1)_z)\,\alpha^{-1} = (k+1)_x$, and hence $((k+1)_x)\,\alpha = (k+1)_z$. We shall prove that $x = y$. The relation $2_x < (k+1)_x$ implies that $(2_x)\,\alpha < ((k+1)_x)\,\alpha$. Since $(2_x)\,\alpha = 2_y$ and $((k+1)_x)\,\alpha = (k+1)_z$ we have that $2_y < (k+1)_z$, so $z = y$. □

For any $x \in \kappa$, consider the map $\pi_x \colon \sigma\mathbb{N}^\kappa \to \sigma\mathbb{N}^\kappa$ defined by the formula:

$$(t)\,(a)\,\pi_x = \begin{cases} (t)\,a, & \text{if } t = x; \\ 1, & \text{otherwise,} \end{cases}$$

for any $a \in \sigma\mathbb{N}^\kappa$ and $t \in \kappa$.

Lemma 4. *Let κ be any infinite cardinal and $\alpha \in H(\mathbb{I})$ such that the equality $(2_x)\,\alpha = 2_x$ holds for any $x \in \kappa$. Then α is the identity map.*

Proof. Let $a \in \sigma\mathbb{N}^\kappa$. Since the inequality $(a)\,\pi_x \leqslant a$ holds for any $x \in \kappa$ and α is an order isomorphism, it follows that $((a)\,\pi_x)\,\alpha \leqslant (a)\,\alpha$. By Lemma 3 and by the lemma assumption we have that $((a)\,\pi_x)\,\alpha = (a)\,\pi_x$, so $(a)\,\pi_x \leqslant (a)\,\alpha$ for any $x \in \kappa$ and therefore $a \leqslant (a)\,\alpha$.

So, we have that $a \leqslant (a)\,\alpha$ for any $a \in \sigma\mathbb{N}^\kappa$ and for any α that satisfies the lemma assumption. Applying this result to the element $(a)\,\alpha$ and the map α^{-1} we have that $(a)\,\alpha \leqslant ((a)\,\alpha)\,\alpha^{-1} = a$.

The inequalities $a \leqslant (a)\,\alpha$ and $(a)\,\alpha \leqslant a$ imply that $(a)\,\alpha = a$. □

Theorem 1. *For any infinite cardinal κ, the group of units $H(\mathbb{I})$ of the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ is isomorphic to the group $\mathcal{S}_\kappa$ of all bijections of the cardinal κ. Moreover $\alpha \in H(\mathbb{I})$ if and only if $\alpha = \mathcal{F}^\circ_g$ for some $g \in \mathcal{S}_\kappa$.*

Proof. Define the map $\mathcal{F} \colon \mathcal{S}_\kappa \to H(\mathbb{I})$ in the following way:

$$\forall g \in \mathcal{S}_\kappa \quad (g)\,\mathcal{F} = \mathcal{F}^\circ_g,$$

Since $\mathcal{F}^\circ_g$ is an order automorphism of the poset $(\sigma\mathbb{N}^\kappa, \leqslant)$ we have that the map $\mathcal{F}^\circ_g$ is an element of the group of units $H(\mathbb{I})$, so $\mathcal{F}$ is well-defined. Next, we shall show that the map $\mathcal{F}$ is an isomorphism.

Statement (iv) of Lemma 1 implies that the map $\mathcal{F}$ is a homomorphism and statement (vii) of Lemma 1 implies that $\mathcal{F}$ is injective.

We shall show that $\mathcal{F}$ is surjective. Let $\alpha \in H(\mathbb{I})$. Lemma 3 implies that for any $x \in \kappa$ there exists $y \in \kappa$ such that $(2_x)\,\alpha = 2_y$. We define the map $g \colon \kappa \to \kappa$ in the following way: $(x)\,g = y$. Since α is a bijection so is g.

Now consider the composition $\alpha \circ \mathcal{F}^\circ_{g^{-1}}$. Let $x \in \kappa$. The definition of the map g implies that

$$(2_x)\left(\alpha \circ \mathcal{F}^\circ_{g^{-1}}\right) = \left(2_{(x)g}\right)\mathcal{F}^\circ_{g^{-1}}$$

and statement (v) of Lemma 1 implies that $\left(2_{(x)g}\right)\mathcal{F}^{\circ}_{g^{-1}} = 2_x$, so $(2_x)\left(\alpha\circ\mathcal{F}^{\circ}_{g^{-1}}\right) = 2_x$. By Lemma 4, $\alpha\circ\mathcal{F}^{\circ}_{g^{-1}}$ is identity map, therefore $\alpha = \left(\mathcal{F}^{\circ}_{g^{-1}}\right)^{-1} = \mathcal{F}^{\circ}_{g}$. □

Theorems 2.3 and 2.20 from [9] and Theorem 1 imply the following corollary.

Corollary 1. *For any infinite cardinal κ every maximal subgroup of the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$ is isomorphic to the group $\mathcal{S}_\kappa$ of all bijections of the cardinal κ.*

Proposition 2. *For any infinite cardinal κ and for any $\alpha\in\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$ there exists a unique bijection $g_\alpha\in\mathcal{S}_\kappa$ such that $\alpha=\rho_\alpha\mathcal{F}^{\circ}_{g_\alpha}\lambda_\alpha$.*

Proof. Let $\alpha\in\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$. For the element $\rho_\alpha^{-1}\alpha\lambda_\alpha^{-1}$ we have that

$$\rho_\alpha\rho_\alpha^{-1}\alpha\lambda_\alpha^{-1}\lambda_\alpha=\varepsilon\alpha\iota,$$

where ε and ι are idempotents with $\operatorname{dom}\varepsilon=\operatorname{dom}\alpha$ and $\operatorname{dom}\iota=\operatorname{ran}\alpha$, so $\varepsilon\alpha\iota=\alpha$. Since

$$\operatorname{dom}\left(\rho_\alpha^{-1}\alpha\lambda_\alpha^{-1}\right)=\operatorname{ran}\left(\rho_\alpha^{-1}\alpha\lambda_\alpha^{-1}\right)=\sigma\mathbb{N}^{\kappa},$$

we have that $\rho_\alpha^{-1}\alpha\lambda_\alpha^{-1}\in H(\mathbb{I})$. By Theorem 1, for $\rho_\alpha^{-1}\alpha\lambda_\alpha^{-1}$ there exists a bijection $g_\alpha\in\mathcal{S}_\kappa$ such that $\rho_\alpha^{-1}\alpha\lambda_\alpha^{-1}=\mathcal{F}^{\circ}_{g_\alpha}$.

Suppose that there exists $h\in\mathcal{S}_\kappa$ such that $\alpha=\rho_\alpha\mathcal{F}^{\circ}_{h}\lambda_\alpha$. Then the equality

$$\rho_\alpha\mathcal{F}^{\circ}_{h}\lambda_\alpha=\rho_\alpha\mathcal{F}^{\circ}_{g_\alpha}\lambda_\alpha$$

implies that

$$\left(\rho_\alpha^{-1}\rho_\alpha\right)\mathcal{F}^{\circ}_{h}\left(\lambda_\alpha\lambda_\alpha^{-1}\right)=\left(\rho_\alpha^{-1}\rho_\alpha\right)\mathcal{F}^{\circ}_{g_\alpha}\left(\lambda_\alpha\lambda_\alpha^{-1}\right).$$

The definition of $\lambda_\alpha,\rho_\alpha$ implies that

$$\rho_\alpha^{-1}\rho_\alpha=\lambda_\alpha\lambda_\alpha^{-1}=\mathbb{I},$$

so $\mathcal{F}^{\circ}_{h}=\mathcal{F}^{\circ}_{g_\alpha}$. Statement (v) of Lemma 1 implies that $h=g_\alpha$. □

The following corollary states that every order isomorphism α in the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$ can be uniquely represented as a composition of three basic transformations: shifting to the origin of coordinates, an order isomorphism of entire $\sigma\mathbb{N}^{\kappa}$, and then shifting to the range of α.

Corollary 2. *For any infinite cardinal κ and for any element $\alpha\in\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$ the representation $\alpha=\rho_\alpha\mathcal{F}^{\circ}_{g_\alpha}\lambda_\alpha$ is unique.*

For any $\alpha\in\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$ we shall use this notation g_α to denote the element of S_κ that implements this representation $\alpha=\rho_\alpha\mathcal{F}^{\circ}_{g_\alpha}\lambda_\alpha$.

Lemma 5. *Let κ be any infinite cardinal and $\alpha,\beta\in\mathcal{IPF}(\sigma\mathbb{N}^{\kappa})$, then*

$$\begin{aligned}
&d_{\alpha\beta}=\left(\max\{r_\alpha,d_\beta\}-r_\alpha\right)\mathcal{F}^{-1}_{g_\alpha}+d_\alpha;\\
&r_{\alpha\beta}=\left(\max\{r_\alpha,d_\beta\}-d_\beta\right)\mathcal{F}_{g_\beta}+r_\beta;\\
&\mathcal{F}^{\circ}_{g_{\alpha\beta}}=\mathcal{F}^{\circ}_{g_\alpha}\mathcal{F}^{\circ}_{g_\beta}.
\end{aligned}$$

Proof. By the definition of the composition of the partial maps:

$$\begin{aligned}\operatorname{dom}(\alpha\beta) &= (\operatorname{ran}\alpha\cap\operatorname{dom}\beta)\,\alpha^{-1} =\\ &= ({\uparrow}r_\alpha\cap{\uparrow}d_\beta)\alpha^{-1} =\\ &= ({\uparrow}\max\{r_\alpha,d_\beta\})\,\alpha^{-1}.\end{aligned}$$

Since α is an order isomorphism we get that

$$({\uparrow}\max\{r_\alpha,d_\beta\})\alpha^{-1} = {\uparrow}\left[(\max\{r_\alpha,d_\beta\})\alpha^{-1}\right],$$

and then, by Corollary 2 and by Lemma 1[(vi), $(viii)$],

$$\begin{aligned}\operatorname{dom}(\alpha\beta) &= {\uparrow}\left[(\max\{r_\alpha,d_\beta\})\alpha^{-1}\right] =\\ &= {\uparrow}\left([\max\{r_\alpha,d_\beta\}]\,\lambda_\alpha^{-1}\left(\mathcal{F}^\circ_{g_\alpha}\right)^{-1}\rho_\alpha^{-1}\right) =\\ &= {\uparrow}\left([\max\{r_\alpha,d_\beta\}-r_\alpha+\mathbf{1}]\left(\mathcal{F}^\circ_{g_\alpha}\right)^{-1}\rho_\alpha^{-1}\right) =\\ &= {\uparrow}\left(\left[(\max\{r_\alpha,d_\beta\}-r_\alpha)\,\mathcal{F}^{-1}_{g_\alpha}+\mathbf{1}\right]\rho_\alpha^{-1}\right) =\\ &= {\uparrow}\left[(\max\{r_\alpha,d_\beta\}-r_\alpha)\,\mathcal{F}^{-1}_{g_\alpha}+d_\alpha\right].\end{aligned}$$

Similarly, by the definition of the range of the composition of the partial maps:

$$\begin{aligned}\operatorname{ran}(\alpha\beta) &= (\operatorname{ran}\alpha\cap\operatorname{dom}\beta)\,\beta =\\ &= ({\uparrow}r_\alpha\cap{\uparrow}d_\beta)\beta =\\ &= ({\uparrow}\max\{r_\alpha,d_\beta\})\,\beta.\end{aligned}$$

Since β is an order isomorphism we get that

$$({\uparrow}\max\{r_\alpha,d_\beta\})\beta = {\uparrow}\left[(\max\{r_\alpha,d_\beta\})\beta\right],$$

and then, by Corollary 2 and by Lemma 1[(vi), $(viii)$],

$$\begin{aligned}\operatorname{ran}(\alpha\beta) &= {\uparrow}\left[(\max\{r_\alpha,d_\beta\})\beta\right] =\\ &= {\uparrow}\left([\max\{r_\alpha,d_\beta\}]\,\lambda_\beta\mathcal{F}^\circ_{g_\beta}\rho_\beta\right) =\\ &= {\uparrow}\left([\max\{r_\alpha,d_\beta\}-d_\beta+\mathbf{1}]\,\mathcal{F}^\circ_{g_\beta}\rho_\beta\right) =\\ &= {\uparrow}\left(\left[(\max\{r_\alpha,d_\beta\}-d_\beta)\,\mathcal{F}_{g_\beta}+\mathbf{1}\right]\rho_\beta\right) =\\ &= {\uparrow}\left[(\max\{r_\alpha,d_\beta\}-d_\beta)\,\mathcal{F}_{g_\beta}+r_\beta\right].\end{aligned}$$

We shall prove that

$$\alpha\beta = \rho_{\alpha\beta}\mathcal{F}^\circ_{g_\alpha}\mathcal{F}^\circ_{g_\beta}\lambda_{\alpha\beta}.$$

The definition of the maps $\rho_{\alpha\beta}$, $\mathcal{F}^\circ_{g_\alpha}$, $\mathcal{F}^\circ_{g_\beta}$, $\lambda_{\alpha\beta}$ and the definition of the composition of the partial maps imply that

$$\operatorname{dom}\left(\rho_{\alpha\beta}\mathcal{F}^\circ_{g_\alpha}\mathcal{F}^\circ_{g_\beta}\lambda_{\alpha\beta}\right) = \operatorname{dom}(\alpha\beta)$$

and

$$\operatorname{ran}\left(\rho_{\alpha\beta}\mathcal{F}^\circ_{g_\alpha}\mathcal{F}^\circ_{g_\beta}\lambda_{\alpha\beta}\right) = \operatorname{ran}(\alpha\beta).$$

Now consider any $a \in \operatorname{dom}(\alpha\beta)$ and the representation $a = d_{\alpha\beta} + a - d_{\alpha\beta}$. Denote $a - d_{\alpha\beta}$ by b, then a has the representation $a = d_{\alpha\beta} + b$. And consider the images of a under the maps $\alpha\beta$ and $\rho_{\alpha\beta}\mathcal{F}^{\circ}_{g_\alpha}\mathcal{F}^{\circ}_{g_\beta}\lambda_{\alpha\beta}$:

$$\begin{aligned}
(a)\,\alpha\beta &= (d_{\alpha\beta} + b)\,\alpha\beta = \\
&= \big([\max\{r_\alpha, d_\beta\} - r_\alpha]\,\mathcal{F}^{-1}_{g_\alpha} + d_\alpha + b\big)\,\alpha\beta = \\
&= \big([\max\{r_\alpha, d_\beta\} - r_\alpha]\,\mathcal{F}^{-1}_{g_\alpha} + d_\alpha + b\big)\,\rho_\alpha\mathcal{F}^{\circ}_{g_\alpha}\lambda_\alpha\rho_\beta\mathcal{F}^{\circ}_{g_\beta}\lambda_\beta = \\
&= \big([\max\{r_\alpha, d_\beta\} - r_\alpha]\,\mathcal{F}^{-1}_{g_\alpha} + \mathbf{1} + b\big)\,\mathcal{F}^{\circ}_{g_\alpha}\lambda_\alpha\rho_\beta\mathcal{F}^{\circ}_{g_\beta}\lambda_\beta = \\
&= (\max\{r_\alpha, d_\beta\} - r_\alpha + \mathbf{1} + (b)\,\mathcal{F}_{g_\alpha})\,\lambda_\alpha\rho_\beta\mathcal{F}^{\circ}_{g_\beta}\lambda_\beta = \\
&= (\max\{r_\alpha, d_\beta\} + (b)\,\mathcal{F}_{g_\alpha})\,\rho_\beta\mathcal{F}^{\circ}_{g_\beta}\lambda_\beta = \\
&= (\max\{r_\alpha, d_\beta\} - d_\beta + \mathbf{1} + (b)\,\mathcal{F}_{g_\alpha})\,\mathcal{F}^{\circ}_{g_\beta}\lambda_\beta = \\
&= \big([\max\{r_\alpha, d_\beta\} - d_\beta]\,\mathcal{F}_{g_\beta} + \mathbf{1} + (b)\,\mathcal{F}_{g_\alpha}\mathcal{F}_{g_\beta}\big)\,\lambda_\beta = \\
&= [\max\{r_\alpha, d_\beta\} - d_\beta]\,\mathcal{F}_{g_\beta} + (b)\,\mathcal{F}_{g_\alpha}\mathcal{F}_{g_\beta} + r_\beta = \\
&= r_{\alpha\beta} + (b)\,\mathcal{F}_{g_\alpha}\mathcal{F}_{g_\beta};
\end{aligned}$$

$$\begin{aligned}
(a)\,\rho_{\alpha\beta}\mathcal{F}^{\circ}_{g_\alpha}\mathcal{F}^{\circ}_{g_\beta}\lambda_{\alpha\beta} &= (d_{\alpha\beta} + b)\,\rho_{\alpha\beta}\mathcal{F}^{\circ}_{g_\alpha}\mathcal{F}^{\circ}_{g_\beta}\lambda_{\alpha\beta} = \\
&= (b + \mathbf{1})\,\mathcal{F}^{\circ}_{g_\alpha}\mathcal{F}^{\circ}_{g_\beta}\lambda_{\alpha\beta} = \\
&= \big((b)\,\mathcal{F}_{g_\alpha}\mathcal{F}_{g_\beta} + \mathbf{1}\big)\,\lambda_{\alpha\beta} = \\
&= (b)\,\mathcal{F}_{g_\alpha}\mathcal{F}_{g_\beta} + r_{\alpha\beta}.
\end{aligned}$$

We have that $\alpha\beta = \rho_{\alpha\beta}\mathcal{F}^{\circ}_{g_\alpha}\mathcal{F}^{\circ}_{g_\beta}\lambda_{\alpha\beta}$, so by Corollary 2 $\mathcal{F}^{\circ}_{g_{\alpha\beta}} = \mathcal{F}^{\circ}_{g_\alpha}\mathcal{F}^{\circ}_{g_\beta}$. □

Corollary 3. *For any infinite cardinal κ and for any elements $\alpha, \beta \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ the bijection $g_{\alpha\beta}$ is equals to $g_\alpha g_\beta$.*

Corollary 4. *Let κ be any infinite cardinal and ε be the idempotent of the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$, then $g_\varepsilon = id_\kappa$, $\mathcal{F}^{\circ}_{g_\varepsilon} = \mathbb{I}$.*

Remark 2. In the bicyclic semigroup $\mathscr{C}(p,q)$ the semigroup operation is determined in the following way:

$$p^i q^j \cdot p^k q^l = \begin{cases} p^i q^{j-k+l}, & \text{if } j > k; \\ p^i q^l, & \text{if } j = k; \\ p^{i-j+k} q^l, & \text{if } j < k, \end{cases}$$

which is equivalent to the following formula:

$$p^i q^j \cdot p^k q^l = p^{i+\max\{j,k\}-j} q^{l+\max\{j,k\}-k}.$$

We note that the bicyclic semigroup $\mathscr{C}(p,q)$ is isomorphic to the semigroup $(\mathbb{N}\times\mathbb{N}, *)$ which is defined on the square $\mathbb{N}\times\mathbb{N}$ of the set of all positive integers with the following multiplication:

$$(i,j) * (k,l) = (i + \max\{j,k\} - j, l + \max\{j,k\} - k)\,. \tag{1}$$

To see this, it is sufficiently to check that the map

$$f\colon \mathscr{C}(p,q) \to \mathbb{N}\times\mathbb{N} : p^i q^j \overset{f}{\mapsto} (i+1, j+1)$$

is an isomorphism between semigroups $\mathscr{C}(p,q)$ and $(\mathbb{N}\times\mathbb{N}, *)$.

In this paper we shall use the semigroup $(\mathbb{N}\times\mathbb{N},*)$ as a representation of the bicyclic semigroup $\mathscr{C}(p,q)$ and we shall denote the semigroup $(\mathbb{N}\times\mathbb{N},*)$ by $\mathbb{B}$.

For any infinite cardinal κ, define the semigroup $\sigma\mathbb{B}^\kappa$ as the set $\sigma\mathbb{N}^\kappa\times\sigma\mathbb{N}^\kappa$ with the multiplications $*_\kappa$ which is similar to (1):

$$(a,b)*_\kappa(c,d)=(a+\max\{b,c\}-b,d+\max\{b,c\}-c)\,,\ \text{where } a,b,c,d\in\sigma\mathbb{N}^\kappa. \tag{2}$$

We can observe that the semigroup $\sigma\mathbb{B}^\kappa$, as defined by the multiplication operation $*_\kappa$ in (2), is indeed isomorphic to the σ-product of κ many copies of the bicyclic monoid.

For any $g\in\mathcal{S}_\kappa$ consider a map $\Phi_g\colon\sigma\mathbb{B}^\kappa\to\sigma\mathbb{B}^\kappa$ defined in the following way: for any $(a,b)\in\sigma\mathbb{B}^\kappa$ define

$$((a,b))\,\Phi_g=\left((a)\,\mathcal{F}^\circ_g,(b)\,\mathcal{F}^\circ_g\right).$$

Statements (i) and (ii) of Lemma 1 imply that the map Φ_g is well-defined and Φ_g is a bijection.

Check that the map Φ_g is an automorphism of $\sigma\mathbb{B}^\kappa$. For any $(a,b),(c,d)\in\sigma\mathbb{B}^\kappa$, by statements $(xiii)-(x)$ of Lemma 1:

$$\begin{aligned}
&\big((a,b)*_\kappa(c,d)\big)\Phi_g=((a+\max\{b,c\}-b,d+\max\{b,c\}-c))\,\Phi_g=\\
&=\left((a+\max\{b,c\}-b)\,\mathcal{F}^\circ_g,(d+\max\{b,c\}-c)\,\mathcal{F}^\circ_g\right)=\\
&=((a)\,\mathcal{F}_g+\max\{(b)\,\mathcal{F}_g,(c)\,\mathcal{F}_g\}-(b)\,\mathcal{F}_g,(d)\,\mathcal{F}_g+\max\{(b)\,\mathcal{F}_g,(c)\,\mathcal{F}_g\}-(c)\,\mathcal{F}_g)=\\
&=((a)\,\mathcal{F}_g,(b)\,\mathcal{F}_g)*_\kappa((c)\,\mathcal{F}_g,(d)\,\mathcal{F}_g)=\left((a)\,\mathcal{F}^\circ_g,(b)\,\mathcal{F}^\circ_g\right)*_\kappa\left((c)\,\mathcal{F}^\circ_g,(d)\,\mathcal{F}^\circ_g\right)=\\
&=(a,b)\,\Phi_g*_\kappa(c,d)\,\Phi_g.
\end{aligned}$$

Let κ be any infinite cardinal and $\mathbf{Aut}(\sigma\mathbb{B}^\kappa)$ be the group of automorphisms of the semigroup $\sigma\mathbb{B}^\kappa$. Consider the map $\Phi\colon\mathcal{S}_\kappa\to\mathbf{Aut}\,(\sigma\mathbb{B}^\kappa)$ for any $g\in\mathcal{S}_\kappa$ define $(g)\,\Phi=\Phi_g$. Statement (vii) of Lemma 1 implies that Φ is injective. Next, we show that the map Φ is a homomorphism. For any $g,h\in\mathcal{S}_\kappa$ consider the image of their composition: for any $[a,b]\in\sigma\mathbb{B}^\kappa$

$$\begin{aligned}
([a,b])\,(gh)\,\Phi&=([a,b])\,\Phi_{gh}=\\
&=\left[(a)\,\mathcal{F}^\circ_{gh},(b)\,\mathcal{F}^\circ_{gh}\right].
\end{aligned}$$

Statement (iv) of Lemma 1 implies that

$$\left[(a)\,\mathcal{F}^\circ_{gh},(b)\,\mathcal{F}^\circ_{gh}\right]=\left[(a)\,\mathcal{F}^\circ_g\mathcal{F}^\circ_h,(b)\,\mathcal{F}^\circ_g\mathcal{F}^\circ_h\right],$$

and since

$$\begin{aligned}
\left[(a)\,\mathcal{F}^\circ_g\mathcal{F}^\circ_h,(b)\,\mathcal{F}^\circ_g\mathcal{F}^\circ_h\right]&=\left(\left[(a)\,\mathcal{F}^\circ_g,(b)\,\mathcal{F}^\circ_g\right]\right)\Phi_h=\\
&=([a,b])\,\Phi_g\Phi_h=\\
&=([a,b])\,(g)\,\Phi\,(h)\,\Phi,
\end{aligned}$$

we have that

$$([a,b])\,(gh)\,\Phi=([a,b])\,(g)\,\Phi\,(h)\,\Phi,$$

i.e., Φ is a homomorphism.

For any infinite cardinal κ consider the semidirect product $\mathcal{S}_\kappa\ltimes_\Phi\sigma\mathbb{B}^\kappa$ of the semigroup $\sigma\mathbb{B}^\kappa$ by the group $\mathcal{S}_\kappa$ as the set $\mathcal{S}_\kappa\times\sigma\mathbb{B}^\kappa$ with the operation:

$$(g,[a,b])\,(h,[c,d])=(gh,([a,b])\,\Phi_h*_\kappa[c,d])\qquad\text{for }\ (g,[a,b])\,,(h,[c,d])\in\mathcal{S}_\kappa\times\sigma\mathbb{B}^\kappa.$$

Define the map $\Psi\colon \mathscr{IPF}(\sigma\mathbb{N}^\kappa) \to \mathcal{S}_\kappa \ltimes_\Phi \sigma\mathbb{B}^\kappa$ by the formula:

$$(\alpha)\Psi = \big(g_\alpha, \big[(d_\alpha)\mathcal{F}^\circ_{g_\alpha}, r_\alpha\big]\big).$$

The definition of $d_\alpha, r_\alpha, g_\alpha$ and $\mathcal{F}^\circ_{g_\alpha}$ implies that the map Ψ is well-defined.

Theorem 2. *For any infinite cardinal κ the semigroup $\mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ is isomorphic to the semidirect product $\mathcal{S}_\kappa \ltimes_\Phi \sigma\mathbb{B}^\kappa$ of the semigroup $\sigma\mathbb{B}^\kappa$ by the group $\mathcal{S}_\kappa$.*

Proof. Consider the map Ψ. Corollary 2 implies that Ψ is a bijection. We shall prove that Ψ is also a homomorphism.

For any $\alpha, \beta \in \mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ we have that $(\alpha\beta)\Psi = \Big(g_{\alpha\beta}, \Big[(d_{\alpha\beta})\mathcal{F}^\circ_{g_{\alpha\beta}}, r_{\alpha\beta}\Big]\Big)$. Corollary 3 and Lemma 5 imply that

$$\begin{aligned}
&\Big(g_{\alpha\beta}, \Big[(d_{\alpha\beta})\mathcal{F}^\circ_{g_{\alpha\beta}}, r_{\alpha\beta}\Big]\Big) = \\
&= \Big(g_\alpha g_\beta, \Big[\big((\max\{r_\alpha, d_\beta\} - r_\alpha)\mathcal{F}^{-1}_{g_\alpha} + d_\alpha\big)\mathcal{F}^\circ_{g_\alpha}\mathcal{F}^\circ_{g_\beta}, (\max\{r_\alpha, d_\beta\} - d_\beta)\mathcal{F}_{g_\beta} + r_\beta\Big]\Big).
\end{aligned}$$

Lemma 1, the definition of the operation $*_\kappa$, and the definition of the map Φ imply that

$$\begin{aligned}
&\Big(g_\alpha g_\beta, \Big[\big((\max\{r_\alpha, d_\beta\} - r_\alpha)\mathcal{F}^{-1}_{g_\alpha} + d_\alpha\big)\mathcal{F}^\circ_{g_\alpha}\mathcal{F}^\circ_{g_\beta}, (\max\{r_\alpha, d_\beta\} - d_\beta)\mathcal{F}_{g_\beta} + r_\beta\Big]\Big) = \\
&= \big(g_\alpha g_\beta, \big[\max\{(r_\alpha)\mathcal{F}_{g_\beta}, (d_\beta)\mathcal{F}_{g_\beta}\} - (r_\alpha)\mathcal{F}_{g_\beta} + (d_\alpha)\mathcal{F}_{g_\alpha}\mathcal{F}_{g_\beta}, \max\{(r_\alpha)\mathcal{F}_{g_\beta}, \\
&\qquad (d_\beta)\mathcal{F}_{g_\beta}\} - (d_\beta)\mathcal{F}_{g_\beta} + r_\beta\big]\big) = \\
&= \big(g_\alpha g_\beta, \big[(d_\alpha)\mathcal{F}_{g_\alpha}\mathcal{F}_{g_\beta}, (r_\alpha)\mathcal{F}_{g_\beta}\big] *_\kappa \big[(d_\beta)\mathcal{F}_{g_\beta}, r_\beta\big]\big) = \\
&= \Big(g_\alpha g_\beta, \Big[(d_\alpha)\mathcal{F}^\circ_{g_\alpha}\mathcal{F}^\circ_{g_\beta}, (r_\alpha)\mathcal{F}^\circ_{g_\beta}\Big] *_\kappa \Big[(d_\beta)\mathcal{F}^\circ_{g_\beta}, r_\beta\Big]\Big) = \\
&= \Big(g_\alpha g_\beta, \big(\big[(d_\alpha)\mathcal{F}^\circ_{g_\alpha}, r_\alpha\big]\big)\Phi_{g_\beta} *_\kappa \Big[(d_\beta)\mathcal{F}^\circ_{g_\beta}, r_\beta\Big]\Big) = \\
&= \big(g_\alpha, \big[(d_\alpha)\mathcal{F}^\circ_{g_\alpha}, r_\alpha\big]\big)\Big(g_\beta, \Big[(d_\beta)\mathcal{F}^\circ_{g_\beta}, r_\beta\Big]\Big) \\
&= (\alpha)\Psi\,(\beta)\Psi.
\end{aligned}$$

□

For any $\alpha \in \mathscr{IPF}(\sigma\mathbb{N}^\kappa)$, let $\big(g_\alpha, \big[(d_\alpha)\mathcal{F}^\circ_{g_\alpha}, r_\alpha\big]\big) = (\alpha)\Psi$ be the image of the element α by the isomorphism $\Psi\colon \mathscr{IPF}(\sigma\mathbb{N}^\kappa) \to \mathcal{S}_\kappa \ltimes_\Phi \sigma\mathbb{B}^\kappa$ which is defined above the proof of Theorem 2.

Every inverse semigroup S admits the *least group* congruence $\mathfrak{C}_{\mathbf{mg}}$ (see [27, Section III]):

$$s\mathfrak{C}_{\mathbf{mg}}t \quad \text{if and only if} \quad \text{there exists an idempotent} \quad e \in S \quad \text{such that} \quad se = te.$$

Proposition 3. *For any infinite cardinal κ, any element $\alpha \in \mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ and for any idempotent $\varepsilon \in \mathscr{IPF}(\sigma\mathbb{N}^\kappa)$ we have:*

$$\begin{aligned}
(\alpha\varepsilon)\Psi &= \big(g_\alpha, \big[(d_\alpha)\mathcal{F}^\circ_{g_\alpha}, r_\alpha\big]\big)(id_\kappa, [d_\varepsilon, d_\varepsilon]) = \\
&= \big(g_\alpha, \big[\max\{r_\alpha, d_\varepsilon\} - r_\alpha + (d_\alpha)\mathcal{F}^\circ_{g_\alpha}, \max\{r_\alpha, d_\varepsilon\}\big]\big); \\
(\varepsilon\alpha)\Psi &= (id_\kappa, [d_\varepsilon, d_\varepsilon])\big(g_\alpha, \big[(d_\alpha)\mathcal{F}^\circ_{g_\alpha}, r_\alpha\big]\big) = \\
&= \big(g_\alpha, \big[(\max\{d_\varepsilon, d_\alpha\})\mathcal{F}^\circ_{g_\alpha}, (\max\{d_\varepsilon, d_\alpha\})\mathcal{F}^\circ_{g_\alpha} - (d_\alpha)\mathcal{F}^\circ_{g_\alpha} + r_\alpha\big]\big).
\end{aligned}$$

Proof. By Corollary 4, g_ε is the identity permutation, i.e., $g_\varepsilon = id_\kappa$ and $\mathcal{F}^\circ_{g_\varepsilon} = \mathbb{I}$. Since $\operatorname{dom}\varepsilon = \operatorname{ran}\varepsilon$ we have that $d_\varepsilon = r_\varepsilon$ and then $(d_\varepsilon)\,\mathcal{F}^\circ_{g_\varepsilon} = d_\varepsilon = r_\varepsilon$, so

$$\left(g_\varepsilon, \left[(d_\varepsilon)\,\mathcal{F}^\circ_{g_\varepsilon}, r_\varepsilon\right]\right) = \left(id_\kappa, [d_\varepsilon, d_\varepsilon]\right).$$

Then the definition of the multiplication in $\mathcal{S}_\kappa \ltimes_\Phi \sigma\mathbb{B}^\kappa$ completes the proof of the proposition. □

The following theorem describes the least group congruence on the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$.

Theorem 3. *Let κ be any infinite cardinal. Then $\alpha\mathfrak{C}_{\mathbf{mg}}\beta$ in the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ if and only if*

$$g_\alpha = g_\beta \qquad \text{and} \qquad (d_\alpha)\,\mathcal{F}^\circ_{g_\alpha} - r_\alpha = (d_\beta)\,\mathcal{F}^\circ_{g_\beta} - r_\beta.$$

Proof. Fix an idempotent ε in $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$. By Proposition 3,

$$\left(g_\alpha, \left[(d_\alpha)\,\mathcal{F}^\circ_{g_\alpha}, r_\alpha\right]\right)\left(id_\kappa, [d_\varepsilon, d_\varepsilon]\right) = \left(g_\alpha, \left[\max\{r_\alpha, d_\varepsilon\} - r_\alpha + (d_\alpha)\,\mathcal{F}^\circ_{g_\alpha}, \max\{r_\alpha, d_\varepsilon\}\right]\right),$$

$$\left(g_\beta, \left[(d_\beta)\,\mathcal{F}^\circ_{g_\beta}, r_\beta\right]\right)\left(id_\kappa, [d_\varepsilon, d_\varepsilon]\right) = \left(g_\beta, \left[\max\{r_\beta, d_\varepsilon\} - r_\beta + (d_\beta)\,\mathcal{F}^\circ_{g_\beta}, \max\{r_\beta, d_\varepsilon\}\right]\right),$$

so the equality $\alpha\varepsilon = \beta\varepsilon$ holds if and only if

$$g_\alpha = g_\beta \qquad \text{and} \qquad (d_\alpha)\,\mathcal{F}^\circ_{g_\alpha} - r_\alpha = (d_\beta)\,\mathcal{F}^\circ_{g_\beta} - r_\beta.$$

□

For any infinite cardinal κ, by $\sigma\mathbb{Z}^\kappa_+$ we shall denote the group $(\sigma\mathbb{Z}^\kappa, +)$. Let $\mathbf{Aut}(\sigma\mathbb{Z}^\kappa_+)$ be the group of automorphisms of the group $\sigma\mathbb{Z}^\kappa_+$. Consider the map $\Theta\colon \mathcal{S}_\kappa \to \mathbf{Aut}\left(\sigma\mathbb{Z}^\kappa_+\right)$: for any $g \in \mathcal{S}_\kappa$ define $(g)\,\Theta = \mathcal{F}^\diamond_g$.

Statements (i), (iii) and $(viii)$ of Lemma 1 imply that for any $g \in \mathcal{S}$ the map $\mathcal{F}^\diamond_g$ is an isomorphism of the group $\sigma\mathbb{Z}^\kappa_+$, so the map Θ is well-defined. Next, statements (iv) and (vii) of Lemma 1 imply that the map Θ is an injective homomorphism.

Consider the semidirect product $\mathcal{S}_\kappa \ltimes_\Theta (\sigma\mathbb{Z}^\kappa, +)$ as the set $\mathcal{S}_\kappa \times \sigma\mathbb{Z}^\kappa$ with the operation

$$(g, m)\,(h, n) = \left(gh, (m)\,\mathcal{F}^\diamond_h + n\right).$$

Theorem 4. *For any infinite cardinal κ the quotient semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)/\mathfrak{C}_{\mathbf{mg}}$ is isomorphic to the semidirect product $\mathcal{S}_\kappa \ltimes_\Theta (\sigma\mathbb{Z}^\kappa, +)$ of the group $(\sigma\mathbb{Z}^\kappa, +)$ by the group $\mathcal{S}_\kappa$.*

Proof. Define the map $\Upsilon\colon \mathcal{IPF}(\sigma\mathbb{N}^\kappa) \to \mathscr{S}_\kappa \ltimes_\Theta (\sigma\mathbb{Z}^\kappa, +)$ in the following way: for any $\alpha \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ we put

$$(\alpha)\,\Upsilon = \left(g_\alpha, (d_\alpha)\,\mathcal{F}^\circ_{g_\alpha} - r_\alpha\right).$$

Since $a - b \in \sigma\mathbb{Z}^\kappa$ for any $a, b \in \sigma\mathbb{N}^\kappa$ we have that Υ is well-defined.

For any $\alpha, \beta \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ by the definition of Υ we have that

$$(\alpha\beta)\,\Upsilon = \left(g_{\alpha\beta}, (d_{\alpha\beta})\,\mathcal{F}^\circ_{g_{\alpha\beta}} - r_{\alpha\beta}\right),$$

and by Lemma 5

$$(\alpha\beta)\,\Upsilon = \left(g_\alpha g_\beta, \left((\max\{r_\alpha, d_\beta\} - r_\alpha)\,\mathcal{F}^{-1}_{g_\alpha} + d_\alpha\right)\mathcal{F}^\circ_{g_\alpha}\mathcal{F}^\circ_{g_\beta} - (\max\{r_\alpha, d_\beta\} - d_\beta)\,\mathcal{F}_{g_\beta} - r_\beta\right),$$

then, by statements $(viii)$ and (ix) of Lemma 1

$$\begin{aligned}(\alpha\beta)\Upsilon =&\big(g_\alpha g_\beta, (\max\{r_\alpha,d_\beta\})\mathcal{F}_{g_\beta} - (r_\alpha)\mathcal{F}_{g_\beta} + (d_\alpha)\mathcal{F}_{g_\alpha}\mathcal{F}_{g_\beta} - (\max\{r_\alpha,d_\beta\})\mathcal{F}_{g_\beta}+\\ &+(d_\beta)\mathcal{F}_{g_\beta} - r_\beta\big) =\\ =&\big(g_\alpha g_\beta, (d_\alpha)\mathcal{F}_{g_\alpha}\mathcal{F}_{g_\beta} - (r_\alpha)\mathcal{F}_{g_\beta} + (d_\beta)\mathcal{F}_{g_\beta} - r_\beta\big) =\\ =&\big(g_\alpha, (d_\alpha)\mathcal{F}^\circ_{g_\alpha} - r_\alpha\big)\Big(g_\beta, (d_\beta)\mathcal{F}^\circ_{g_\beta} - r_\beta\Big) =\\ =&(\alpha)\Upsilon(\beta)\Upsilon,\end{aligned}$$

and hence Υ is a homomorphism.

Show that the map Υ is surjective. For any $(g,z)\in\mathcal{S}_\kappa\times\sigma\mathbb{Z}^\kappa$, consider the maps $a,b\colon\kappa\to\mathbb{N}$. For any $x\in\kappa$:

$$(x)a=\begin{cases}(x)z, & \text{if } (x)z>0\\ 1, & \text{if } (x)z=0\\ 0, & \text{if } (x)z<0\end{cases} \qquad \text{and} \qquad (x)b=\begin{cases}0, & \text{if } (x)z>0\\ 1, & \text{if } (x)z=0\\ -(x)z, & \text{if } (x)z<0.\end{cases}$$

We have that $a,b\in\sigma\mathbb{N}^\kappa$ and $z=a-b$. Now we consider $\alpha\in\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ such that

$$\begin{aligned}&g_\alpha=g,\\ &d_\alpha=(a)\left(\mathcal{F}^\circ_g\right)^{-1},\\ &r_\alpha=b.\end{aligned}$$

Then

$$\begin{aligned}(\alpha)\Upsilon&=\big(g_\alpha,(d_\alpha)\mathcal{F}^\circ_{g_\alpha}-r_\alpha\big)=\\ &=\Big(g,\Big((a)\left(\mathcal{F}^\circ_g\right)^{-1}\Big)\mathcal{F}^\circ_g-b\Big)=\\ &=(g,a-b)=\\ &=(g,z),\end{aligned}$$

so Υ is surjective.

Also, Theorem 3 implies that $\alpha\mathfrak{C}_{\mathbf{mg}}\beta$ in $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ if and only if $(\alpha)\Upsilon=(\beta)\Upsilon$. This implies that the homomorphism Υ generates the congruences $\mathfrak{C}_{\mathbf{mg}}$ on $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$. □

Every inverse semigroup S admits a partial order:

$$a\preccurlyeq b \qquad \text{if and only if there exists} \qquad e\in E(S) \quad \text{such that} \quad a=be.$$

So defined order is called *the natural partial order* on S. We observe that $a\preccurlyeq b$ in an inverse semigroup S if and only if $a=fb$ for some $f\in E(S)$ (see [23, Lemma 1.4.6]).

This and Proposition 3 imply the following proposition, which describes the natural partial order on the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$.

Proposition 4. *Let κ be any infinite cardinal and let $\alpha,\beta\in\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$. Then the following conditions are equivalent:*

(i) $\alpha\preccurlyeq\beta$;
(ii) $g_\alpha=g_\beta$, $(d_\alpha)\mathcal{F}^\circ_{g_\alpha}-r_\alpha=(d_\beta)\mathcal{F}^\circ_{g_\beta}-r_\beta$ *and* $d_\beta\leqslant d_\alpha$ *in the poset* $(\sigma\mathbb{N}^\kappa,\leqslant)$;
(iii) $g_\alpha=g_\beta$, $(d_\alpha)\mathcal{F}^\circ_{g_\alpha}-r_\alpha=(d_\beta)\mathcal{F}^\circ_{g_\beta}-r_\beta$ *and* $r_\beta\leqslant r_\alpha$ *in the poset* $(\sigma\mathbb{N}^\kappa,\leqslant)$.

An inverse semigroup S is said to be *E-unitary* if $ae \in E(S)$ for some $e \in E(S)$ implies that $a \in E(S)$ [23]. E-unitary inverse semigroups were introduced by Siatô in [28], where they were called "*proper ordered inverse semigroups*".

Proposition 5. *For any infinite cardinal κ, the inverse semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ is E-unitary.*

Proof. Let $\alpha \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$. Suppose that $\alpha\varepsilon$ is an idempotent for some idempotent $\varepsilon \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$. Then Proposition 3 and the definition of idempotents imply that $g_\alpha = id_\kappa$ and $d_\alpha = (d_\alpha)\mathcal{F}_{g_\alpha} = r_\alpha$, so α is an idempotent. □

An inverse semigroup S is called *F-inverse*, if the $\mathfrak{C}_{\mathbf{mg}}$-class $s_{\mathfrak{C}_{\mathbf{mg}}}$ of each element s has the top (biggest) element with the respect to the natural partial order on S [24].

Proposition 6. *For any infinite cardinal κ, the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ is an F-inverse semigroup.*

Proof. Let $\alpha \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$. Consider an element $\beta \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ such that

$$\begin{aligned} g_\beta &= g_\alpha, \\ d_\beta &= d_\alpha - \min\{d_\alpha, (r_\alpha)\left(\mathcal{F}^\circ_{g_\alpha}\right)^{-1}\} + \mathbf{1}, \\ r_\beta &= r_\alpha - \min\{(d_\alpha)\mathcal{F}^\circ_{g_\alpha}, r_\alpha\} + \mathbf{1}. \end{aligned}$$

We have that $\min\{d_\alpha, (r_\alpha)\left(\mathcal{F}^\circ_{g_\alpha}\right)^{-1}\} \in \sigma\mathbb{N}^\kappa$ and $\min\{d_\alpha, (r_\alpha)\left(\mathcal{F}^\circ_{g_\alpha}\right)^{-1}\} \leqslant d_\alpha$, so $d_\beta \in \sigma\mathbb{N}^\kappa$. Similar $r_\beta \in \sigma\mathbb{N}^\kappa$, so β is well-defined. Also, we have that $g_\beta = g_\alpha$ and

$$\begin{aligned} (d_\beta)\mathcal{F}^\circ_{g_\beta} - r_\beta &= \left(d_\alpha - \min\{d_\alpha, (r_\alpha)\left(\mathcal{F}^\circ_{g_\alpha}\right)^{-1}\} + \mathbf{1}\right)\mathcal{F}^\circ_{g_\alpha} - \left(r_\alpha - \min\{(d_\alpha)\mathcal{F}^\circ_{g_\alpha}, r_\alpha\} + \mathbf{1}\right) = \\ &= (d_\alpha)\mathcal{F}^\circ_{g_\alpha} - \min\{(d_\alpha)\mathcal{F}^\circ_{g_\alpha}, r_\alpha\} + \mathbf{1} - r_\alpha + \min\{(d_\alpha)\mathcal{F}^\circ_{g_\alpha}, r_\alpha\} - \mathbf{1} = \\ &= (d_\alpha)\mathcal{F}^\circ_{g_\alpha} - r_\alpha, \end{aligned}$$

then Theorem 3 implies that $\beta\mathfrak{C}_{\mathbf{mg}}\alpha$.

Now, for any $\gamma \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$, such that $\gamma\mathfrak{C}_{\mathbf{mg}}\alpha$, we consider the idempotent ε with $d_\varepsilon = r_\gamma$ and consider the product $(\beta)\Psi(\varepsilon)\Psi$. By Proposition 3

$$\begin{aligned} (\beta)\Psi(\varepsilon)\Psi &= \left(g_\beta, \left[(d_\beta)\mathcal{F}^\circ_{g_\beta}, r_\beta\right]\right)(id_\kappa, [d_\varepsilon, d_\varepsilon]) = \\ &= \left(g_\beta, \left[\max\{r_\beta, d_\varepsilon\} - r_\beta + (d_\beta)\mathcal{F}^\circ_{g_\beta}, \max\{r_\beta, d_\varepsilon\}\right]\right) = \\ &= \left(g_\beta, \left[\max\{r_\beta, r_\gamma\} - r_\beta + (d_\beta)\mathcal{F}^\circ_{g_\beta}, \max\{r_\beta, r_\gamma\}\right]\right). \end{aligned}$$

Since $\gamma\mathfrak{C}_{\mathbf{mg}}\alpha$, by Theorem 3 we have that $g_\gamma = g_\alpha$ and $r_\gamma - (d_\gamma)\mathcal{F}^\circ_{g_\gamma} = r_\alpha - (d_\alpha)\mathcal{F}^\circ_{g_\alpha}$, then for any $x \in \kappa$

$$\begin{aligned} (x)(\max\{r_\beta, r_\gamma\}) &= (x)\left(\max\{r_\alpha - \min\{(d_\alpha)\mathcal{F}^\circ_{g_\alpha}, r_\alpha\} + \mathbf{1}, r_\gamma\}\right) = \\ &= \begin{cases} \max\{(x)r_\alpha - (x)r_\alpha + (x)\mathbf{1}, (x)r_\gamma\}, & \text{if } (x)(d_\alpha)\mathcal{F}^\circ_{g_\alpha} > (x)r_\alpha \\ \max\{(x)r_\alpha - (x)(d_\alpha)\mathcal{F}^\circ_{g_\alpha} + (x)\mathbf{1}, (x)r_\gamma\}, & \text{otherwise} \end{cases} = \\ &= \begin{cases} \max\{1, (x)r_\gamma\}, & \text{if } (x)(d_\alpha)\mathcal{F}^\circ_{g_\alpha} > (x)r_\alpha \\ \max\{(x)r_\gamma - (x)(d_\gamma)\mathcal{F}^\circ_{g_\gamma} + 1, (x)r_\gamma\}, & \text{otherwise} \end{cases} = \\ &= (x)r_\gamma, \end{aligned}$$

so $\max\{r_\beta, r_\gamma\} = r_\gamma$. Also

$$\begin{aligned}\max\{r_\beta, r_\gamma\} - r_\beta + (d_\beta)\,\mathcal{F}^\circ_{g_\beta} &= r_\gamma - r_\beta + (d_\beta)\,\mathcal{F}^\circ_{g_\beta} = \\ &= r_\gamma - r_\alpha + (d_\alpha)\,\mathcal{F}^\circ_{g_\alpha} = \\ &= (d_\gamma)\,\mathcal{F}^\circ_{g_\gamma},\end{aligned}$$

so

$$\Big(g_\beta, \Big[\max\{r_\beta, r_\gamma\} - r_\beta + (d_\beta)\,\mathcal{F}^\circ_{g_\beta}, \max\{r_\beta, r_\gamma\}\Big]\Big) = \Big(g_\gamma, \Big[(d_\gamma)\,\mathcal{F}^\circ_{g_\gamma}, r_\gamma\Big]\Big) = (\gamma)\,\Psi.$$

The equality $(\beta)\,\Psi\,(\varepsilon)\,\Psi = (\gamma)\,\Psi$ implies that $\gamma = \beta\varepsilon$, so $\gamma \preccurlyeq \beta$. This means that the element β is the biggest element in the $\mathfrak{C}_{\mathbf{mg}}$-class of the element α in $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$. $\square$

Lemma 6. *Let κ be any infinite cardinal and let $\mathfrak{C}$ be a congruence on the semigroup $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$ such that $\varepsilon\mathfrak{C}\iota$ for some two distinct idempotents $\varepsilon, \iota \in \mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$. Then $\varsigma\mathfrak{C}\upsilon$ for all idempotents ς, υ of $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$.*

Proof. We observe that without loss of generality we may assume that $\varepsilon \preccurlyeq \iota$ where $\preccurlyeq$ is the natural partial order on the semilattice $E(\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa))$. Indeed, if $\varepsilon, \iota \in E(\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa))$ then $\varepsilon\mathfrak{C}\iota$ implies that $\varepsilon = \varepsilon\varepsilon\mathfrak{C}\iota\varepsilon$, and since the idempotents ε and ι are distinct in $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$ we have that $\iota\varepsilon \preccurlyeq \varepsilon$.

Now, the inequality $\varepsilon \preccurlyeq \iota$ implies that $\operatorname{dom}\varepsilon \subseteq \operatorname{dom}\iota$. Next, we define partial map $\alpha\colon \sigma\mathbb{N}^\kappa \rightharpoonup \sigma\mathbb{N}^\kappa$ in the following way:

$$\operatorname{dom}\alpha = \sigma\mathbb{N}^\kappa, \qquad \operatorname{ran}\alpha = \operatorname{dom}\iota \qquad \text{and} \qquad (z)\,\alpha = z + d_\iota - \mathbf{1}, \quad \text{for any} \quad z \in \operatorname{dom}\alpha.$$

The definition of α implies that $\alpha\iota\alpha^{-1} = \alpha\alpha^{-1} = \mathbb{I}$ and $\alpha^{-1}\alpha = \iota$, and moreover, we have that

$$\begin{aligned}\left(\alpha\varepsilon\alpha^{-1}\right)\left(\alpha\varepsilon\alpha^{-1}\right) &= \alpha\varepsilon\left(\alpha^{-1}\alpha\right)\varepsilon\alpha^{-1} = \\ &= \alpha\varepsilon\iota\varepsilon\alpha^{-1} = \\ &= \alpha\varepsilon\varepsilon\alpha^{-1} = \\ &= \alpha\varepsilon\alpha^{-1},\end{aligned}$$

which implies that $\alpha\varepsilon\alpha^{-1}$ is an idempotent of $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$ such that $\alpha\varepsilon\alpha^{-1} \neq \mathbb{I}$.

Thus, it was shown that there exists a non-unit idempotent ε^* in $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$ such that $\varepsilon^*\mathfrak{C}\mathbb{I}$. This implies that $\varepsilon_0\mathfrak{C}\mathbb{I}$ for any idempotent ε_0 of $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$ such that $\varepsilon^* \preccurlyeq \varepsilon_0 \preccurlyeq \mathbb{I}$. Since $\varepsilon^* \neq \mathbb{I}$ we have that $d_{\varepsilon^*} \neq \mathbf{1}$, so there exists $x \in \kappa$ such that $(x)\,d_{\varepsilon^*} \neq 1$, thus $2_x \leqslant d_{\varepsilon^*}$. Consider an idempotent ε_x in $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$ such that $d_{\varepsilon_x} = 2_x$. Then $d_{\varepsilon_x} = 2_x \leqslant d_{\varepsilon^*}$ implies that $\varepsilon^* \preccurlyeq \varepsilon_x$, so $\varepsilon_x\mathfrak{C}\mathbb{I}$.

Fix an arbitrary $y \in \kappa \setminus \{x\}$. Define a bijection on the set κ in the following way:

$$(x)\,g = y, \qquad (y)\,g = x \qquad \text{and} \qquad (t)\,g = t, \quad \text{for} \quad t \in \kappa \setminus \{x, y\}.$$

Next, consider the map $\mathcal{F}^\circ_g$ as an element of $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$. The definition of g implies that $g^{-1} = g$, then, by Lemma 1(i) we have that $\left(\mathcal{F}^\circ_g\right)^{-1} = \mathcal{F}^\circ_{g^{-1}} = \mathcal{F}^\circ_g$ and then

$$\mathcal{F}^\circ_g\mathbb{I}\mathcal{F}^\circ_g = \mathcal{F}^\circ_g\mathcal{F}^\circ_g = \mathcal{F}^\circ_g\left(\mathcal{F}^\circ_g\right)^{-1} = \mathbb{I}.$$

The calculations

$$\begin{aligned}\left(\mathcal{F}_g^\circ\varepsilon_x\mathcal{F}_g^\circ\right)\Psi &= \left(\mathcal{F}_g^\circ\right)\Psi\left(\varepsilon_{d_x}\right)\Psi\left(\mathcal{F}_g^\circ\right)\Psi = \\ &= (g,[\mathbf{1},\mathbf{1}])\,(id_\kappa,[2_x,2_x])\,(g,[\mathbf{1},\mathbf{1}]) = \\ &= (g,[2_x,2_x])\,(g,[\mathbf{1},\mathbf{1}]) = \\ &= \left(gg,\left[(2_x)\,\mathcal{F}_g^\circ,(2_x)\,\mathcal{F}_g^\circ\right]\right) = \\ &= \left(id_\kappa,\left[2_{(x)g},2_{(x)g}\right]\right) = \\ &= (id_\kappa,[2_y,2_y]) = \\ &= (\varepsilon_y)\,\Psi\end{aligned}$$

shows that $\mathcal{F}_g^\circ\varepsilon_x\mathcal{F}_g^\circ = \varepsilon_y$, where ε_y is an idempotent in $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$ such that $d_{\varepsilon_y} = 2_y$. Then

$$\varepsilon_y = \left(\mathcal{F}_g^\circ\varepsilon_x\mathcal{F}_g^\circ\right)\mathfrak{C}\left(\mathcal{F}_g^\circ\mathbb{I}\mathcal{F}_g^\circ\right) = \mathbb{I}$$

implies that $\varepsilon_y\mathfrak{C}\mathbb{I}$.

The above arguments imply that $\varepsilon_x\mathfrak{C}\mathbb{I}$ for every idempotent $\varepsilon_x \in \mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$ such that ε_x is the identity map of the principal filter ${\uparrow}2_x$ of the poset $(\sigma\mathbb{N}^\kappa,\leqslant)$, $x\in\kappa$. Now, fix an idempotent ζ in $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$ and consider the set $A = \{x\in\kappa \mid (x)\,d_\zeta \neq 1\}$. Since $d_\zeta \in \sigma\mathbb{N}^\kappa$ the set A is finite, so there exists $k\in\mathbb{N}$ such that $A = \{x_1,x_2,\ldots,x_k\}$ for some $x_1,x_2,\ldots,x_k\in\kappa$. Consider the idempotent $\varepsilon_A = \varepsilon_{x_1}\ldots\varepsilon_{x_k}$. Since $\mathfrak{C}$ is congruence, $\varepsilon_{x_i}\mathfrak{C}\mathbb{I}$ for any $x_i\in A$ and A is finite we have that $(\varepsilon_{x_1}\ldots\varepsilon_{x_k})\,\mathfrak{C}\mathbb{I}$. The definition of ε_A and the semigroup operation of $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$ imply that $d_{\varepsilon_A} = 2_A$, where

$$(t)\,2_A = \begin{cases} 2 & \text{if } t\in A\\ 1 & \text{otherwise.}\end{cases}$$

We define the partial map $\gamma\colon \sigma\mathbb{N}^\kappa \rightharpoonup \sigma\mathbb{N}^\kappa$ in the following way:

$$\operatorname{dom}\gamma = \sigma\mathbb{N}^\kappa, \quad \operatorname{ran}\gamma = {\uparrow}2_A \quad \text{and} \quad (z)\,\gamma = z + 2_A - \mathbf{1}, \quad \text{for any} \quad z\in\operatorname{dom}\gamma.$$

The definition of γ implies that that $\gamma\gamma^{-1} = \mathbb{I}$ and $\gamma^{-1}\gamma = \varepsilon_A$. For any positive integer $n\in\mathbb{N}$ consider the idempotent

$$\left(\gamma^{-1}\right)^n\gamma^n = \underbrace{\gamma^{-1}\ldots\gamma^{-1}}_{n\text{-times}}\underbrace{\gamma\ldots\gamma}_{n\text{-times}}.$$

Since $\varepsilon_A = \gamma^{-1}\gamma\mathfrak{C}\mathbb{I}$ we have that $\gamma^{-1}\gamma^{-1}\gamma\gamma\mathfrak{C}\gamma^{-1}\gamma = \varepsilon_A$ and $\gamma^{-1}\gamma^{-1}\gamma\gamma\mathfrak{C}\mathbb{I}$, so by induction $\left(\gamma^{-1}\right)^n\gamma^n\mathfrak{C}\mathbb{I}$, for any $n\in\mathbb{N}$. Also, by induction, we have that $d_{(\gamma^{-1})^n\gamma^n} = (n+1)_A$, where

$$(t)\,(n+1)_A = \begin{cases} n+1 & \text{if } t\in A\\ 1 & \text{otherwise,}\end{cases}$$

for any $n\in\mathbb{N}$. Thus, we have that

$$d_\zeta \leqslant d_{(\gamma^{-1})^m\gamma^m} = (m+1)_A,$$

where $m = \max\{(x)\,d_\zeta \mid x\in\kappa\}$, implies that $\left(\gamma^{-1}\right)^m\gamma^m \preccurlyeq \zeta$, so $\zeta\mathfrak{C}\mathbb{I}$. □

Lemma 7. *Let κ be any infinite cardinal and let $\mathfrak{C}$ be a congruence on the semigroup $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$ such that $\alpha\mathfrak{C}\beta$ for some non-$\mathscr{H}$-equivalent elements $\alpha,\beta\in\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$. Then $\varepsilon\mathfrak{C}\iota$ for all idempotents ε,ι of $\mathcal{IPF}\,(\sigma\mathbb{N}^\kappa)$.*

Proof. Since α and β are not-$\mathscr{H}$-equivalent in $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ we have that either $\alpha\alpha^{-1} \neq \beta\beta^{-1}$ or $\alpha^{-1}\alpha \neq \beta^{-1}\beta$ (see [23, p. 82]). Then Proposition 4 from [23, Section 2.3] implies that $\alpha\alpha^{-1}\mathfrak{C}\beta\beta^{-1}$ and $\alpha^{-1}\alpha\mathfrak{C}\beta^{-1}\beta$ and hence the assumption of Lemma 6 holds. □

Lemma 8. *Let κ be any infinite cardinal and let $\mathfrak{C}$ be a congruence on the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ such that $\alpha\mathfrak{C}\beta$ for some two distinct $\mathscr{H}$-equivalent elements $\alpha, \beta \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$. Then $\varepsilon\mathfrak{C}\iota$ for all idempotents ε, ι of $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$.*

Proof. By Proposition 1(vi) the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ is simple and then Theorem 2.3 from [9] implies that there exist $\mu, \xi \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ such that $f\colon H_\alpha \to H_{\mathbb{I}}\colon \chi \mapsto \mu\chi\xi$ maps α to $\mathbb{I}$ and β to $\gamma \neq \mathbb{I}$, respectively, which implies that $\mathbb{I}\mathfrak{C}\gamma$. Since γ is an element of the group of units of the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$, by Theorem 1, $\gamma = \mathcal{F}^\circ_{g_\gamma}$ and since $\gamma \neq \mathbb{I}$ we have that $g_\gamma \neq id_\kappa$, so there exists $x \in \kappa$ such that $(x)g_\gamma \neq x$. Put ε as the identity map with $d_\varepsilon = 2_x$. Since $\mathfrak{C}$ is a congruence on the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ and $\gamma \in H_{\mathbb{I}}$ we have that

$$\varepsilon = \varepsilon\varepsilon = \varepsilon\mathbb{I}\varepsilon\mathfrak{C}\varepsilon\gamma\varepsilon.$$

Proposition 3 implies that

$$(\varepsilon\gamma\varepsilon)\Psi = \left(g_\gamma, \left[\max\{(2_x)\mathcal{F}^\circ_{g_\gamma}, 2_x\}, \max\{(2_x)\mathcal{F}^\circ_{g_\gamma}, 2_x\}\right]\right).$$

By Lemma 1(v) we have that $(2_x)\mathcal{F}^\circ_{g_\gamma} = 2_{(x)g_\gamma} \neq 2_x$, this and the definition of elements 2_x and $2_{(x)g_\gamma}$ imply that $\max\{(2_x)\mathcal{F}^\circ_{g_\gamma}, 2_x\} \neq 2_x$, so

$$r_{\varepsilon\gamma\varepsilon} = \max\{(2_x)\mathcal{F}^\circ_{g_\gamma}, 2_x\} \neq 2_x = r_\varepsilon,$$

then by Proposition 1(v), $\varepsilon\gamma\varepsilon$ and ε are non-$\mathscr{H}$-equivalent elements in $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$. Next, we apply Lemma 7. □

Theorem 5. *For any infinite cardinal κ every non-identity congruence $\mathfrak{C}$ on the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ is group.*

Proof. For every non-identity congruence $\mathfrak{C}$ on $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ there exist two distinct elements $\alpha, \beta \in \mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ such that $\alpha\mathfrak{C}\beta$. If $\alpha\mathscr{H}\beta$ in $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ then by Lemma 7 all idempotents of the semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ are $\mathfrak{C}$-equivalent, otherwise by Lemma 8 we get the same. Thus, by Lemma II.1.10 of [27] the quotient semigroup $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)/\mathfrak{C}$ has a unique idempotent and hence it is a group. □

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МОНОЇД ПОРЯДКОВИХ ІЗОМОРФІЗМІВ ГОЛОВНИХ ФІЛЬТРІВ МНОЖИНИ $\sigma\mathbb{N}^\kappa$

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Розглянемо таке узагальнення біциклічного моноїда. Для довільного нескінченного кардинала κ розглянемо напівгрупу $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ всіх порядкових ізоморфізмів головних фільтрів множини $\sigma\mathbb{N}^\kappa$ з порядком добутку. Ми дослідимо алгебричні властивості напівгрупи $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$, доведемо, що вона є біпростою, E-унітарною, F-інверсною напівгрупою, опишемо відношення Гріна на напівгрупі $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$, опишемо групу одиниць $H(\mathbb{I})$ цієї напівгрупи і її максимальні підгрупи. Доведемо, що напівгрупа $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ ізоморфна напівпрямому добутку $\mathcal{S}_\kappa \ltimes \sigma\mathbb{B}^\kappa$ напівгрупи $\sigma\mathbb{B}^\kappa$ і групи $\mathcal{S}_\kappa$, доведемо що кожна не тотожна конгруенція $\mathfrak{C}$ на напігрупі $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$ є груповою; опишемо найменшу групову конгруенцію на $\mathcal{IPF}(\sigma\mathbb{N}^\kappa)$.

Ключові слова: напівгрупа, інверсна напівгрупа, часткове відображення, група перестановок, найменша групова конгруенція, біциклічна напівгрупа, напівпрямий добуток.