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THE DARBOUX-STEFAN PROBLEMS WITH NONLOCAL CONDITIONS FOR ONE-DIMENSIONAL HYPERBOLIC EQUATIONS AND SYSTEMS

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Boundary value problems with nonlocal conditions (undivided and integral) for a strictly hyperbolic equation of arbitrary order and a system of hyperbolic equations of the first order in the case of a degenerate initial condition interval to a point are considered, the case where the boundaries of the domain are unknown ahead is also considered.

Key words: hyperbolic equations, Darboux problem, hyperbolic Stefan problem, nonlocal conditions, characteristics.

1. Introduction

This paper is a transposition and some generalization of research results in [1] to the case when the domain of finding a solution of a mixed problem for a strictly hyperbolic equation of arbitrary order is a curvilinear sector in the plane. Problems in such domains are called Darboux problems [2].

For a strictly hyperbolic equation we consider a problem with nonlocal (undivided and integral) boundary conditions in the curvilinear sector and also the hyperbolic Stefan problem, a problem for which domain boundaries are unknown a priori in the upper half-plane of the plane xOt. Such problems have important practical applications and arise in many applied evolution processes (see, for instance, [1]–[7]).

2. Statement of the problem

Let G be a curvilinear sector of the upper half-plane $t>0$ of the plane xOt, bounded by the curves γ_0 and γ_{m+1} which are given by equations $x=a_0(t)$, $x=a_{m+1}(t)$, $m\geqslant 0$, $a_0(0)=a_{m+1}(0)=0$, $a_{m+1}(t)>a_0(t)$ for all $t>0$ respectively. The curves $\gamma_s : x=a_s(t)$,

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$s=\overline{0,m+1}$, $a_s \in C^1(R_+)$ $(R_+=[0,\infty))$, $a_{s+1}(t) > a_s(t)$ for all $t>0$, $a_s(0)=0$ divide G into $m+1$ connectivity components G^s $(s=\overline{0,m})$, that are numbered from left to right.

For each $s=\overline{0,m}$ in G^s, is given a strictly hyperbolic equation of order $n \geqslant 2$

$$A^s u \equiv \sum_{i=0}^{n} A_i^s(x,t,\partial_x,\partial_t)u^s(x,t) = f^s(x,t), \tag{1}$$

where $A_i^s(x,t,\partial_x,\partial_t)$ is a linear homogeneous differential operator of order i, for each $s=\overline{0,m}$:

$$A_i^s(x,t,\partial_x,\partial_t)u^s(x,t) \equiv \sum_{j=0}^{i} A_{ij}^s(x,t)\frac{\partial^i u^s}{\partial x^j \partial t^{i-j}},$$

which coefficients $A_{ij}^s(x,t)$ are square matrices of order n, with $A_{n0}^s(x,t) \equiv I$, $s=\overline{0,m}$. Suppose $A_{nj}^s \in C^1(\bar{G}^s)$, $j=\overline{1,n}$; A_{ij}^s, $f^s \in C(\bar{G}^s)$, $i=\overline{0,n-1}$, $j=\overline{0,i}$; $s=\overline{0,m}$.

We will understand the hyperbolicity of equation (1) in the sense that in decomposition

$$A_n^s(x,t,\xi,\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i^s(x,t)\xi)$$

the functions $\lambda_i^s(x,t)$ are real and different for all $(x,t) \in \bar{G}^s$, whence $\lambda_i^s(x,t) \in C^1(\bar{G}^s)$. Note that the condition $\lambda_i^s \neq \lambda_k^s$ for $j \neq k$ is weighted here, which is not required for the case of first order hyperbolic systems. Moreover, for all $t \geqslant 0$ and at each $s=\overline{0,m}$ conditions

$$\begin{gathered}
\lambda_i^s(a_s(t),t) - a_s'(t) > 0, \quad i=\overline{1,p_s},\\
\lambda_i^s(a_s(t),t) - a_s'(t) < 0, \quad i=\overline{p_s+1,n},\\
\lambda_i^s(a_{s+1}(t),t) - a_{s+1}'(t) > 0, \quad i=\overline{1,q_s},\\
\lambda_i^s(a_{s+1}(t),t) - a_{s+1}'(t) < 0, \quad i=\overline{q_s+1,n},\\
0 \leqslant p_s, \quad q_s \leqslant n, \quad s=\overline{0,m}
\end{gathered} \tag{2}$$

are fulfilled.

Since $p_s(q_s)$ is the number of indices i for which $\lambda_i^s(0,0) > a_s'(0)$ (respectively a_{s+1}') and $a_s'(0) \leqslant a_{s+1}'(0)$, then $p_s \geqslant q_s$ for all $s=\overline{0,m}$. Let $N=\sum_{s=0}^{m}(p_s-q_s)+(m+1)n$. For the equation (1), set the conditions replacing the boundary conditions to γ_0 and γ_{m+1} and the conjugate conditions to $\gamma_1,\ldots,\gamma_m$ if $m>0$:

$$\sum_{s=0}^{m}\sum_{i=0}^{n-1}\left[\sum_{k=s}^{s+1} B_{is}^{kp}(t,\partial_x,\partial_t)u^s(x,t)\Big|_{x=a_k(t)} + \int_{a_s(t)}^{a_{s+1}(t)} C_{is}^p(y,t,\partial_y,\partial_t)u^s(y,t)dy\right] = h^p(t), \quad p=\overline{1,q}, \tag{3}$$

$$\sum_{s=0}^{m}\sum_{i=0}^{n-1}\int\limits_{a_s(t)}^{a_{s+1}(t)} C_{is}^{p}(y,t,\partial_y,\partial_t)u^s(y,t)dy = h^p(t), \tag{4}$$

$$p=\overline{1+q,n},\quad t\geqslant 0,\quad 0\leqslant q\leqslant N,$$

$$\frac{\partial^{k+l}u^s(0,0)}{\partial x^k\partial t^l} = u_s^{kl},\quad k+l=\overline{0,n-2},\quad s=\overline{0,m}. \tag{5}$$

Here $B_{is}^{kp}(t,\partial_x,\partial_t)$ and $C_{is}^{p}(y,t,\partial_y,\partial_t)$ are given linear homogeneous differential operators of order i with continuous coefficients depending on $t\in[0,+\infty)$ and $(y,t)\in\bar{G}^s$; $h^p(t)$ are given at $t\geqslant 0$ continuous functions, with $h^p\in C^1(\mathbb{R}_+)$ and $h^p(0)=0$ for $p=\overline{q+1,N}$; u_s^{kl} are given numbers.

3. Existence and uniqueness of problem solution

Suppose that the conditions specified in paragraph 2 are fulfilled; $a_s'(t)\neq 0$ for all $s=\overline{0,m+1}$ and $t\in\mathbb{R}_+$; the operator coefficients C_{is}^{p} and the free terms $h^p(t)$ for $p=\overline{q+1,N}$ are functions from classes $C^1(\bar{G}^s)$ and $C^1(\mathbb{R}_+)$ respectively.

Before defining the notion of piecewise continuous generalized solution of the problem (1)-(5), let us first transform it by assuming that the desired solution has piecewise continuous derivatives of all orders $\leqslant n$ and all equalities are satisfied in the ordinary way.

Using the considerations made in [8], consider for each $s=\overline{0,m}$ operators

$$M_i^s(x,t,\partial_x,\partial_t)u^s=\sum_{k=1}^{n} b_{ik}^s(x,t)\partial_t^{k-1}\partial_x^{n-k}u^s,\quad i=\overline{1,n},\quad s=\overline{0,m} \tag{6}$$

with a characteristic form

$$\sum_{k=1}^{n} b_{ik}^s(x,t)\lambda^{k-1}\xi^{n-k}=\prod_{j\neq i}(\lambda-\lambda_j^s(x,t)\xi). \tag{7}$$

The formulas (6) define for each $s=\overline{0,m}$ a set of n linearly independent forms from the derivatives $\partial_t^{i-1}\partial_x^{n-i}u^s$; conversely, these derivatives can be found as linear combinations of $M_i^s(x,t,\partial_x,\partial_t)u^s$, namely

$$\partial_t^{i-1}\partial_x^{n-i}u^s=\sum_{k=1}^{n} c_{ik}^s(x,t)M_k^s(x,t,\partial_x,\partial_t)u^s, \tag{8}$$

where the matrix $c_{ik}^s(x,t)$ is inverse to the matrix $b_{ik}^s(x,t)$. It is not difficult to check that

$$c_{ik}^s(x,t)=\frac{(\lambda_k^s(x,t))^{i-1}}{\prod\limits_{j\neq k}(\lambda_k^s(x,t)-\lambda_j^s(x,t))},\quad s=\overline{0,m}. \tag{9}$$

Indeed, from (8) we get

$$\lambda^{i-1}=\sum_{r=1}^{n} c_{ir}^s(x,t)M_r^s(x,t,1,\lambda),\quad i=\overline{1,n},\quad s=\overline{0,m}.$$

According to (7),

$$\lambda^{i-1}=\sum_{r=1}^{n}c^s_{ir}(x,t)\prod_{j\neq r}(\lambda-\lambda^s_j(x,t)),\quad i=1,n,\quad s=\overline{0,m}.$$

Substitute $\lambda=\lambda^s_k(x,t)$ in the last equality. Then we get

$$(\lambda^s_k(x,t))^{i-1}=\sum_{r=1}^{n}c^s_{ir}(x,t)\prod_{j\neq r}(\lambda^s_k(x,t)-\lambda^s_j(x,t))=$$
$$=c^s_{ik}(x,t)\prod_{j\neq k}(\lambda^s_k(x,t)-\lambda^s_j(x,t)),\quad i=\overline{1,n},\quad s=\overline{0,m}.$$

From this we immediately obtain (9).

Thus, an arbitrary linear homogeneous differential operator of order $n-1$ with continuous coefficients can, moreover, be uniquely represented as a linear combination of operators $M^s_i(x,t,\partial_x,\partial_t)$, with the coefficients in this representation being continuous functions from (x,t). If the coefficients of a given operator are continuously differentiable, then the coefficients in the representation are continuously differentiable [1, 8].

Using this, we express the principal parts of the operator $B^{kp}_{is}(t,\partial_x,\partial_t)$ and $C^p_{is}(y,t,\partial_y,\partial_t)$ in (3)-(4) in terms of $M^s_i(x,t,\partial_x,\partial_t)$, $i=\overline{1,n}$:

$$B^{kp}_{n-1,s}(t,\partial_x,\partial_t)=\sum_{i=1}^{n}\alpha^{kp}_{is}(t)M^s_i(a_k(t),t,\partial_x,\partial_t),\tag{10}$$

$$C^{p}_{n-1,s}(y,t,\partial_y,\partial_t)=\sum_{i=1}^{n}\beta^{p}_{is}(y,t)M^s_i(y,t,\partial_y,\partial_t).\tag{11}$$

According to the above, all coefficients $\alpha^{kp}_{is}(t)$, $\beta^p_{is}(y,t)$ are continuous, and coefficients $\beta^p_{is}(y,t)$ at $p=\overline{q+1,N}$ are continuously differentiable.

Let

$$\alpha^1_s(t)=\left\|\alpha^{sp}_{is}(t)\right\|,\ p=\overline{1,q},\ i=\overline{1,p_s};\ \alpha^2_s(t)=\left\|\alpha^{s+1,p}_{is}(t)\right\|,\ p=\overline{1,q},\ i=\overline{q_s+1,n};$$
$$\alpha^3_s(0)=-\left\|\alpha^{s+1,p}_{is}(0)\right\|,\ p=\overline{1,q},\ i=\overline{1,q_s};$$
$$\alpha^4_s(0)=-\left\|\alpha^{sp}_{is}(0)\right\|,\ p=\overline{1,q},\ i=\overline{p_s+1,n};$$
$$\beta^1_s(t)=\left\|\beta^p_{is}(a_s(t),t)\Big(\lambda^s_i(a_s(t),t)-a'_s(t)\Big)\right\|,\ p=\overline{q+1,N},\ i=\overline{1,p_s};$$
$$\beta^2_s(t)=-\left\|\beta^p_{is}(a_{s+1}(t),t)\Big(\lambda^s_i(a_{s+1}(t),t)-a'_{s+1}(t)\Big)\right\|,\ p=\overline{q+1,N},\ i=\overline{q_s+1,n};$$
$$\beta^3_s(0)=\left\|\beta^p_{is}(0,0)\Big(\lambda^s_i(0,0)-a'_{s+1}(0)\Big)\right\|,\ p=\overline{q+1,N},\ i=\overline{1,q_s};$$
$$\beta^4_s(0)=-\left\|\beta^p_{is}(0,0)\Big(\lambda^s_i(0,0)-a'_s(0)\Big)\right\|,\ p=\overline{q+1,N},\ i=\overline{p_s+1,n};$$
$$s=\overline{0,m},\ t\geqslant 0$$

and besides that, let us introduce square matrices of order N

$$A(t)=\begin{Vmatrix} \alpha_0^1(t) & \dots & \alpha_m^1(t) & \alpha_0^2(t) & \dots & \alpha_m^2(t) \\ \beta_0^1(t) & \dots & \beta_m^1(t) & \beta_0^2(t) & \dots & \beta_m^2(t) \end{Vmatrix},$$

$$B(0)=\begin{Vmatrix} \alpha_0^3(0)0_0^1 & \dots & \alpha_m^3(0)0_m^1 & 0_0^1\alpha_0^4(0) & \dots & 0_m^1\alpha_m^4(0) \\ \beta_0^3(0)0_0^2 & \dots & \beta_m^3(0)0_m^2 & 0_0^2\beta_0^4(0) & \dots & 0_m^2\beta_m^4(0) \end{Vmatrix}.$$

Here 0_s^k are zero matrices of dimension $q\times(p_s-q_s)$ if $k=1$ and of dimension $(N-q)(p_s-q_s)$ if $k=2$ ($s=\overline{0,m}$).

Let us assume that

$$\det A(t)\neq 0,\ \forall\, t\geqslant 0, \tag{12}$$

$$|A(0)^{-1}B(0)|<1 \tag{13}$$

and at the point $(0,0)$ the $\sum\limits_{s=0}^{m}(p_s-q_s)$ agreement conditions

$$\sum_{p=1}^{N}(\delta_{l_i^s,p}-\delta_{k_i^s,p})H^p(0)=0,\ i=\overline{q_s+1,p_s},\ s=\overline{0,m} \tag{14}$$

are fulfilled, where δ_{jp} are matrix elements $[I-A(0)^{-1}B(0)]^{-1}$,

$$l_i^s=i,\ k_i^s=ns+\sum_{r=0}^{m}p_r-\sum_{r=0}^{s}q_r+i,$$

$$H^p(0)=h^p(0)\ (p=\overline{1,q}),\ H^p(0)=h^{p\prime}(0)\ (p=\overline{q+1,N}).$$

The operators $M_i^s(x,t,\partial_x,\partial_t)$ $(i=\overline{1,n})$ defined by formulas (6) and (7) have the property that the principal parts of operators A^s and

$$(\partial_t+\lambda_i^s(x,t)\partial_x)M_i^s(x,t,\partial_x,\partial_t)u^s,\quad s=\overline{0,m}$$

are the same for arbitrary $i=\overline{1,n}$.

By putting $v_i^s(x,t)=M_i^s(x,t,\partial_x,\partial_t)u^s$ $(i=1,n)$, we can write equation (1) for each $s=\overline{0,m}$ in each of following n forms:

$$\frac{\partial v_i^s}{\partial t}+\lambda_i^s(x,t)\frac{\partial v_i^s}{\partial x}=\sum_{k=1}^{n}a_{ik}^s(x,t)v_k^s(x,t)+S_i^s(x,t,\partial_x,\partial_t)u^s+f^s(x,t), \tag{15}$$

$$i=\overline{1,n},\quad s=\overline{0,m},$$

where the coefficients of $a_{ik}^s(x,t)$ and the linear differential operators $S_i^s(x,t,\partial_x,\partial_t)$ of order $n-2$ are obviously determined by the coefficients of equation (1) .

The conditions (3)-(4) taking into account (10)-(11) give equalities

$$\sum_{s=0}^{m}\sum_{i=1}^{n}\left[\sum_{k=s}^{s+1}\alpha_{is}^{kp}(t)v_i^s(a_k(t),t)+\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^p(y,t)dy\right]= \tag{16}$$

$$=\sum_{s=0}^{m}\sum_{i=0}^{n-2}\left[\sum_{k=s}^{s+1}B_{is}^{kp}(t,\partial_x,\partial_t)u^s(x,t)\bigg|_{x=a_k(t)}\right.-$$

$$- \int\limits_{a_s(t)}^{a_{s+1}(t)} C_{is}^{p}(y,t,\partial_y,\partial_t)u^s(y,t)dy\Bigg] + h^p(t) \equiv H_1^p(t,u), \quad p=\overline{1,q};$$

$$(17) \quad \sum_{s=0}^{m}\sum_{i=1}^{n} \int\limits_{a_s(t)}^{a_{s+1}(t)} \beta_{is}^{p}(y,t)v_i^s(y,t)dy =$$

$$= -\sum_{s=0}^{m}\sum_{i=0}^{n-2} \int\limits_{a_s(t)}^{a_{s+1}(t)} C_{is}^{p}(y,t,\partial_y,\partial_t)u^s(y,t)dy + h^p(t) \equiv H_2^p(t,u),$$

$$p=\overline{1+q,N}; \quad t \geqslant 0.$$

Let us now choose for an arbitrary point $(x,t) \in \bar{G}^s$ a line l with equation

$$\psi(\tau,x,t) = a_s(\tau) + \frac{a_{s+1}(\tau)-a_s(\tau)}{a_{s+1}(t)-a_s(t)}(x-a_s(t)), \quad (0 \leqslant \tau \leqslant t).$$

Then for arbitrary $i=\overline{0,n-2}$, $j=\overline{0,i}$ the representation

$$(18) \quad \left.\frac{\partial^i u^s}{\partial x^j \partial t^{i-j}}\right|_{(x,t)} = \sum_{k=i}^{n-2}\sum_{l=j}^{k-i+j} g_{ij}^{kls}(x,t)\left.\frac{\partial^k u^s}{\partial x^l \partial t^{k-l}}\right|_{(0,0)} +$$

$$+ \int\limits_0^t \sum_{k=1}^{n} G_{ij}^{ks}(\tau,x,t)v_k^s(\psi(\tau,x,t),\tau)d\tau$$

holds. In order to obtain this representation, we have to express the integrand function in equality

$$\left.\frac{\partial^i u^s}{\partial x^j \partial t^{i-j}}\right|_{(x,t)} = \left.\frac{\partial^i u^s}{\partial x^j \partial t^{i-j}}\right|_{(0,0)} + \int\limits_0^t \frac{d}{d\tau}\left(\left.\frac{\partial^i u^s}{\partial x^j \partial t^{i-j}}\right|_{(\psi(\tau,x,t),\tau)}\right)d\tau$$

by the formula for the derivative of a complex function; then apply a similar transformation to each of the derivatives of order $i+1$ obtained, and so on, including the derivatives of order $n-1$, which have to be expressed in v_k^s according to the formulas (8); now using the standard permutation of integration bounds we have to convert the multiple integrals into single integrals.

Substituting the expression (18) into the equation (15) subject to the conditions (5) leads to a system of Volterra integro-differential equations of the form

$$(19) \quad \frac{\partial v_i^s}{\partial t} + \lambda_i^s(x,t)\frac{\partial v_i^s}{\partial x} = \sum_{k=1}^{n} a_{ik}^s(x,t)v_k^s(x,t) + \int\limits_0^t \sum_{k=1}^{n} Q_{ik}^s(\tau,x,t)v_k^s(\psi(\tau,x,t),\tau)d\tau +$$

$$+ \sum_{k=0}^{n-2}\sum_{l=0}^{k} u_s^{kl} g_{kl}^{is}(x,t) + f^s(x,t), \quad (x,t) \in \bar{G}^s, \quad s=\overline{0,m}, \quad i=\overline{1,n}.$$

The right-hand sides of the additional conditions (16)-(17) are transformed accordingly, so that these conditions will have the form

$$\sum_{s=0}^{m}\sum_{i=1}^{n}\left[\sum_{k=s}^{s+1}\alpha_{is}^{kp}(t)v_i^s(a_k(t),t)+\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^{p}(y,t)v_i^s(y,t)\right]= \tag{20}$$

$$=\tilde{H}_1^p(t,v)+\sum_{k=0}^{n-2}\sum_{l=0}^{k}\tilde{h}_{kl1}^{p}(t)u_s^{kl},\ \ p=\overline{1,q},$$

$$\sum_{s=0}^{m}\sum_{i=1}^{n}\int\limits_{a_s(t)}^{a_{s+1}(t)}\beta_{is}^{p}(y,t)v_i^s(y,t)dy=\tilde{H}_2^p(t,v)+\sum_{k=0}^{n-2}\sum_{l=0}^{k}\tilde{h}_{kl2}^{p}(t)u_s^{kl}, \tag{21}$$

$$p=\overline{1+q,N};\ \ t\geqslant 0.$$

We can now define a piecewise continuous solution of the problem (1)-(5). It is according to the formula (18) that we will so call the piecewise continuous function u for which

$$u^s(x,t)=\sum_{k=0}^{n-2}\sum_{l=0}^{k}g_{0,0}^{kls}(x,t)u_s^{kl}+\int\limits_{0}^{t}\sum_{k=1}^{n}G_{0,0}^{ks}(\tau,x,t)v_k^s(\psi(\tau,x,t),\tau)d\tau,$$

where the vector-function v is the piecewise continuous generalized solution of the problem (19)-(21), that is, the piecewise continuous function which for all (x,t) satisfies the integro-functional equation obtained by integrating the equation (19) along the corresponding characteristics $\xi=\varphi_i^s(\tau,x,t)$, as solutions of the Cauchy problem

$$\frac{d\xi}{d\tau}=\lambda_i^s(\xi,\tau),\ \ \xi(t)=x,\ (x,t)\in\bar{G}^s,\ i=\overline{1,n},\ s=\overline{0,m},$$

and for all t also satisfies the ratio (20) and (21).

From the considerations made in deriving the relations (19)–(21), it follows that the solution of the problem (1)–(5) as a function having piecewise continuous derivatives of all orders $\leqslant n$ and satisfying equality (1)–(5) for all $(x,t)\in G\setminus\cup\gamma_m$, is also a piecewise continuous generalized solution of this problem. The converse statement, of course, need not necessarily hold. It is easy to check if the piecewise continuous generalized solution u of the problem (1)–(5) has piecewise continuous derivatives of all orders $\leqslant n$.

Although the equations (19) and the additional conditions (20)–(21) have a more general form due to the presence of additional Volterra integral terms, all considerations made by the method of characteristics in [9] apply directly to the obtained problem (19)–(21). Thus, we have the following theorem:

Theorem 1. *Let the given functions in the problem* (1)$-$(5) *satisfy all the assumptions formulated in paragraphs 2–3, and for the system of equations* (19) *with additional conditions* (20)$-$(21), (12) *and* (13) *and the agreement conditions* (14) *are fulfilled. Then this problem has in* $\bar{G}$ *a unique piecewise continuous generalized solution.*

4. The case of the Darboux-Stefan problem

We will consider domains $G^s_\varepsilon = \{(x,t) \in G^s : 0 < t \leqslant \varepsilon\}$ and smooth lines γ_s of the same type as in paragraph 2 but now let us assume that these lines (with equations $x = a_s(t)$, $a_s(0) = 0$, $s = \overline{0, m+1}$) are not predefined but are constructed in the process of solving the problem; only the values of $a'_s(0)$ are given, and

$$a'_0(0) < a'_1(0) < \ldots < a'_{m+1}(0).$$

In the domain G_ε, we will consider the hyperbolic system

$$\frac{\partial u_i}{\partial t} + \lambda_i(x,t)\frac{\partial u_i}{\partial x} = \sum_{i=1}^{n} a_{ij}(x,t)u_j + f_i(x,t), \quad i = \overline{1,n} \tag{22}$$

with piecewise continuous (uniformly continuous in each domain G^s_ε) functions a_{ij} and f_i and piecewise smooth (having uniformly continuous derivatives in each domain G^s_ε) functions λ_i. The meaning of the notations λ^s_i etc. is the same as before. Since the boundaries of the domains G^s_ε are not predefined, we will assume that all functions λ^s_i, a^s_{ij}, f^s_i $(i,j = \overline{1,n},\ s = \overline{0,m})$ are given for $\geqslant 0$ in some neighborhood of point $(0,0)$ and are continuous together with $\dfrac{\partial \lambda^s_i}{\partial x}$ and $\dfrac{\partial \lambda^s_i}{\partial t}$; the functions λ_i, etc. are combined from them after constructing all the functions $a_s(t)$, i.e. domains G^s_ε, according to the rule:

$$\lambda_i(x,t) = \lambda^s_i(x,t), \quad a_s(t) < x < a_{s+1}(t), \quad s = \overline{0,m}.$$

Suppose that for each $s = \overline{0,m}$

$$\lambda^s_i(0,0) - a'_s(0) > 0 \quad (i = \overline{1,p_s}),$$
$$\lambda^s_i(0,0) - a'_{s+1}(0) < 0 \quad (i = \overline{p_s+1,n}),$$

where $0 \leqslant p_s \leqslant n$. Let us assume $N = (m+1)n$.

Consider the following problem: for some $\varepsilon > 0$ we need to find functions $a_i(t)$, $i = \overline{0,m+1}$, and in the corresponding domain G_ε the solution $u(x,t)$ of the system (22), such that conditions

$$\sum_{i=1}^{n}\sum_{s=0}^{m}\left[\sum_{k=s}^{s+1} \alpha^{kp}_{is}(a(t),t)u^s_i(a_k(t),t) + \int_{a_s(t)}^{a_{s+1}(t)} \beta^p_{is}(y,t)u^s_i(y,t)dy\right] = \\ = h^p(a(t),t), \quad p = \overline{1,N}, \quad t \in [0,\varepsilon]; \tag{23}$$

$$a'_r(t) = \sum_{j=1}^{n}\sum_{s=0}^{m}\sum_{k=s}^{s+1} \gamma^{ks}_{rj}(a(t),t)u^s_j(a_k(t),t) + H_r(a(t),t), \quad r = \overline{0,m+1}, \quad t \in [0,\varepsilon], \tag{24}$$

where $a(t) = (a_0(t), \ldots, a_{m+1}(t))$ are fulfilled.

This problem is an example of a multiphase two-phase hyperbolic Stefan problem for (22) system, in the case of degeneracy of the initial condition interval to a single point.

Let us introduce matrices

$$\alpha^1_s(a,t) = \|\alpha^{sp}_{is}(a,t)\|, \quad p = \overline{1,N} \quad i = \overline{1,p_s},$$
$$\alpha^2_s(a,t) = \|\alpha^{s+1,p}_{is}(a,t)\|, \quad p = \overline{1,N} \quad i = \overline{1+p_s,n},$$

and assume

$$A(a,t) = \|\alpha_0^1(a,t)\dots\alpha_m^1(a,t)\alpha_0^2(a,t)\dots\alpha_m^2(a,t)\|,$$
$$B(a,t) = (-1)\|\alpha_0^2(a,t)\dots\alpha_m^2(a,t)\alpha_0^1(a,t)\dots\alpha_m^1(a,t)\|.$$

5. Solvability of the Darboux-Stefan problem

By a piecewise continuous generalized solution of the problem (22)—(24) we will understand the set of functions $a_s(t)$ $(s=\overline{0,m+1},\ 0\leqslant t\leqslant\varepsilon)$ for some $\varepsilon>0$ and a piecewise continuous generalized solution $u(x,t)$ in G_ε of the problem (22), (23) satisfying the condition (24) for all $t\in[0,\varepsilon]$.

Theorem 2. *Let*

1) *coefficients* $\lambda_i^s\in C^2(\bar U_0)$ $\quad(i=\overline{1,n}, s=\overline{0,m})$, *where*
$$\bar U_0=\{(x,t):|x|\leqslant\varepsilon_0, 0\leqslant t\leqslant\varepsilon_0\}$$
for some $\varepsilon_0>0$;
2) *coefficients* a_{ij}^s *and free terms* f_i^s $\quad(j,i=\overline{1,n},\ s=\overline{0,m})$ *belong to class* $C^1(\bar U_0)$;
3) *coefficients* $\alpha_{is}^{kp}, h^p\in C^1([-\varepsilon_0,\varepsilon_0]^{m+1}\times[0,\varepsilon_0])$, $\beta_{is}^p\in C^1(\bar U_0)$, $(i=\overline{1,n},\ s=\overline{0,m},$ $p=\overline{1,N},\ k=\overline{s,s+1})$;
4) *coefficients* $\gamma_{rj}^{sk}, H_r\in C^1([-\varepsilon_0,\varepsilon_0]^{m+1}\times[0,\varepsilon_0])$;
5) $\det A(0,0)\neq 0$;
6) $|A^{-1}(0,0)B(0,0)|<1$ (*one of the standard matrix norms is denoted by the norm* $|\cdot|$);
7) $|H_r(0,0)|<1$ $(r=\overline{0,m+1})$;
8) $\det\|\alpha_{is}^{sp}(0,0)+\alpha_{is}^{s+1,p}(0,0)\|\neq 0$, *where the rows of this matrix are numbered with values* $p=\overline{1,N}$ *and the columns with pairs* (s,i), $s=\overline{0,m}$, $i=\overline{1,n}$;
9) *the agreement conditions are fulfilled*
$$\sum_{i=1}^{n}\sum_{s=0}^{m}[\alpha_{is}^{sp}(0,0)+\alpha_{is}^{s+1,p}(0,0)]u_i^s(0,0)=h^p(0,0),$$
whence, according to 8), we can determine all values $u_i^s(0,0)$ *that must satisfy*
$$a_r'(0)=\sum_{s=0}^{m}\sum_{j=1}^{n}[\gamma_{rj}^{ss}(0,0)+\gamma_{rj}^{s+1,s}(0,0)]u_j^s(0,0)+H_r(0,0),\quad r=\overline{0,m+1}.$$

Then there exists $\varepsilon\in(0,\varepsilon_0]$ *such that the problem* (22)—(24) *has in* $\bar G_\varepsilon$ *a unique piecewise smooth generalized solution defined for all* $t\in[0,\varepsilon]$.

Proof. Denote given values $a_s'(0)$ by $(a_s')_0$, set some values $\varepsilon\in(0,\varepsilon_0]$, $h>0$ and denote by D_ε^h the set of functions $a=(a_0,\dots,a_{m+1})\in[C^1[0,\varepsilon]]^{m+2}$, for which
$$|a_s(t)|<\varepsilon,\ \ |a_s'(t)-(a_s')_0|\leqslant h,\ \ 0\leqslant t\leqslant\varepsilon,\ \ s=\overline{0,m+1}.$$

Let us consider ε and h sufficiently small such that under given assumptions each function $a\in D_\varepsilon^h$ corresponds to a piecewise continuous solution in $\bar G_\varepsilon=\bar G_{\varepsilon,a}$ of the corresponding problem at fixed $a_s(t)$; denote this solution by $U(x,t;a)$ (its value for fixed x, t is a functional with respect to a).

It is easy to prove that for arbitrary $j=\overline{1,n}$, $s=\overline{0,m}$, $k=\overline{s,s+1}$ the dependence $U_j^s(a_k(t),t;a)$ in the uniform deviation metric on a as element $[C^1[0,\varepsilon]]^{m+2}$ satisfies the Lipschitz condition

$$(25)\quad \exists L\geqslant 0:\forall a^1,a^2\in D_\varepsilon^h,$$
$$\max_{0\leqslant t\leqslant\varepsilon}|U^s(a_k^1(t),t;a^1)-U^s(a_k^2(t),t;a^2)|\leqslant L[\max_{0\leqslant t\leqslant\varepsilon}|a^1(t)-a^2(t)|+$$
$$+\max_{0\leqslant t\leqslant\varepsilon}|a^{1'}(t)-a^{2'}(t)|],$$

where the vertical lines denote any of the norms in $\mathbb{R}^{n+2}$ (its choice determines L).

The relation (25) can be obtained by using a priori estimates of [1] for the solution through given functions, from which it follows, in particular, that all values

$$(26)\qquad |U_j^s(x,t;a)|\leqslant U_0=const\ \ (j=\overline{1,n},\ s=\overline{0,m},\ (x,t)\in\bar{G}_\varepsilon^s,\ a\in D_\varepsilon^h).$$

A similar statement would hold for the derivatives $\dfrac{\partial u_j^s}{\partial x}$ and $\dfrac{\partial u_j^s}{\partial t}$:

$$(27)\qquad |{U_{jx}^s}'(x,t;a)|\leqslant U_1=\mathrm{const},\qquad |{U_{jt}^s}'(x,t;a)|\leqslant U_2=\mathrm{const}\,.$$

Since we only need to satisfy the condition (24), let us consider on D_ε^h the operator $A:a\to Aa$, that acts according to rule

$$(Aa)_r(t)=\int\limits_0^t\left[\sum_{j=1}^n\sum_{s=0}^m\sum_{k=s}^{s+1}\gamma_{rj}^{ks}(a(\tau),\tau)U_j^s(a_k(\tau),\tau;a)+H_r(a(\tau),\tau)\right]d\tau,$$
$$r=\overline{0,m+1},\quad t\in[0,\varepsilon].$$

The desired solution is its fixed point. It follows from the agreement condition 9) that if for a fixed h it is sufficient to reduce ε, then the operator A maps D_ε^h into itself and in the metric $[C^1[0,\varepsilon]]^{m+2}$ is contractive. Therefore by Banach theorem the existence and uniqueness of a fixed point of the operator, i.e. the desired solution follows [10].

This completes the proof of Theorem 2. □

6. Remarks

1. The problem (1)—(5) does not exclude the case where some of the curves γ_m are characteristics of the equation (1). The only thing that changes is the number of conditions (3)—(4) [7].

2. If the characteristics of the equation (1) coming from the intersection point of the boundary curves do not fall into the domain G, then the number of boundary conditions (3)—(4) is $(m+1)n$.

3. It is easy to construct examples like those of [1], which point out the importance of the conditions (12)—(13) for the problem (1)—(5).

4. The boundary movement conditions in the problem (22)—(24) can be given in a more general form, e.g.

$$a_r'(t)=g_r(a(t),t,u^s(a(t),t)),\quad r=\overline{0,m+1},\quad s=\overline{0,m},$$
$$a(t)=(a_0(t),\dots,a_{m+1}(t)),$$

$$u^s(a(t),t) = (u_1^s(a(t),t),\dots,u_n^s(a(t),t)).$$

In this case, the functions $g_r(a(t),t,u^s(a(t),t))$, in addition to being continuous in a set of variables, should also be required to satisfy a local Lipschitz condition for the variables $a(t)$ and $u^s(x,t)$ [10].

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ЗАДАЧІ ДАРБУ-СТЕФАНА З НЕЛОКАЛЬНИМИ УМОВАМИ ДЛЯ ОДНОВИМІРНИХ ГІПЕРБОЛІЧНИХ РІВНЯНЬ І СИСТЕМ

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Розглянуто крайові задачі з нелокальними умовами (нерозділеними та інтегральними) для строго гіперболічного рівняння довільного порядку та системи гіперболічних рівнянь першого порядку у випадку виродження інтервалу задання початкових умов в точку, причому розглянуто також випадок, коли межі області наперед невідомі.

Ключові слова: гіперболічні рівняння, задача Дарбу, гіперболічна задача Стефана, нелокальні умови, характеристики.