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REMARKS TO PROPERTIES OF ANALYTIC IN THE UNIT DISK FUNCTIONS P-VALENT ALONG WITH THEIR DERIVATIVES

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Let $K = (k_j)$ be an increasing sequence of integers, $k_0 = 0$, and $E(K)$ be a class of analytic in the disk $\mathbb{D} = \{z : |z| < 1\}$ functions f such that for some $p \in \mathbb{N}$ all derivatives $f^{(k_j)}$ are in the middle p-valent in $\mathbb{D}$. It is proved that every function $f \in E(K)$ is entire if and only if

$$\lim_{j\to\infty} \left\{ \ln k_j - \frac{1}{k_j} \sum_{s=1}^{j} (k_s - k_{s-1}) \ln (k_s - k_{s-1}) \right\} = +\infty.$$

Key words: entire function, p-valent function.

1. Introduction

Let the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

be an analytic in the disk $\mathbb{D} = \{z : |z| < 1\}$ and $n(w)$ be the number of roots of the equation $f(z) = w$ in $\mathbb{D}$. The function is called p-valent in $\mathbb{D}$ if $n(w) \le p$ for all $w \in \mathbb{C}$ and $n(w) = p$ for some $w \in \mathbb{C}$. As in [1, p. 26, 33] we put

$$p(\varrho) = \frac{1}{2\pi} \int_0^{2\pi} n(\varrho e^{i\theta}) d\theta,$$

$$W(R) = \int_0^R p(\varrho) d(\varrho^2) = \frac{1}{\pi} \int_0^{2\pi} \int_0^R n(\varrho e^{i\theta}) \varrho d\varrho d\theta.$$

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If $W(R)\le pR^2$, $p>0$, for all $R\in(0,+\infty)$ then f is called in the middle p-valent in $\mathbb{D}$. Clearly, if f is p-valent in $\mathbb{D}$ then f is in the middle p-valent in $\mathbb{D}$.

S. M. Shah [2] proved that if the analytic in $\mathbb{D}$ function f and all its derivatives $f^{(n)}$ are univalent in $\mathbb{D}$ then f is an entire function, and conjectured that for an increasing sequence (k_j) such that $\sum\limits_{j=1}^{\infty}\frac{1}{k_j}=+\infty$ a function f and all its derivatives $f^{(k_j)}$ are univalent in $\mathbb{D}$ then f is an entire function. In [3] this conjecture is disproved and it is proved that the condition

$$\lim_{j\to\infty}\left\{\ln k_j-\frac{1}{k_j}\sum_{s=1}^{j}(k_s-k_{s-1})\ln(k_s-k_{s-1})\right\}=+\infty \tag{2}$$

is necessary and sufficient in order that for each analytic in $\mathbb{D}$ function f the univalence of $f^{(n_p)}$ in $\mathbb{D}$ for all $p\in\mathbb{Z}_+$, $p_0=0$, implies that f is an entire function.

In the proposed note will be shown that similar results are correct for functions in the middle p-valent in $\mathbb{D}$.

2. Functions with all derivatives in the middle p-valent

The following theorem is true.

Theorem 1. *If analytic in $\mathbb{D}$ function f and all its derivatives $f^{(n)}$ are in the middle p-valent in $\mathbb{D}$ then f is an entire function of exponential type.*

Proof. It is known [1, p. 83] that if function (1) is in the middle p-valent in $\mathbb{D}$ then

$$|a_n|<\begin{cases} A_p\mu_p n^{2p-1}, & \text{if } p>1/4;\\ A_p|a_0|n^{-1/2}\ln(n+1), & \text{if } p=1/4;\\ A_p|a_0|n^{-1/2}\ln^{-1/2}(n+1), & \text{if } p<1/4,\end{cases}$$

where $\mu_p=\max\{|a_\nu|:\,\nu\le p\}$ and $A_p=(p+2)2^{3p-1}\exp\{p\pi^2+1/2\}$. It is clear that the function f is in the middle p_1-valent in $\mathbb{D}$ and $p_2\ge p_1$ then f is in the middle p_2-valent. Therefore, we can assume that $p\in\mathbb{N}$ and use only the estimate $|a_n|<A_p\mu_p n^{2p-1}$.

Since the derivative $f^{(k)}(z)=\sum\limits_{n=0}^{\infty}\frac{(n+k)!}{n!}a_{n+k}z^n$ is in the middle p_1-valent in $\mathbb{D}$, we have

$$\frac{(n+k)!}{n!}|a_{n+k}|\le A_p n^{2p-1}\max\left\{\frac{(\nu+k)!}{\nu!}|a_{\nu+k}|:\,\nu\le p\right\},$$

whence for $n=p+1$ we get

$$|a_{k+p+1}|\le A_p(p+1)^{2p-1}(p+1)!\times$$
$$\times\max\left\{\frac{|a_{k+p}|}{k+p+1},\frac{|a_{k+p-1}|}{(k+p+1)(k+p)},\dots,\frac{|a_{k+1}|}{(k+p+1)\dots(k+2)},\frac{|a_k|}{(k+p+1)\dots(k+1)}\right\}.$$

Therefore, for all $n\ge p$

$$|a_{n+1}|\le\frac{B_p}{n+1}\max\left\{|a_n|,\frac{|a_{n-1}|}{n},\dots,\frac{|a_{n-p+1}|}{n\dots(n-p+2)},\frac{|a_{n-p}|}{n\dots(n-p+1)}\right\}.$$

where $B_p = A_p(p+1)^{2p-1}(p+1)!$. Using this inequality, we get by induction

$$|a_{n+1}| \le$$
$$\le \frac{B_p}{n+1}\max\left\{\frac{B_p}{n}\max\left\{|a_{n-1}|, \frac{|a_{n-2}|}{n-1},\dots,\frac{|a_{n-p}|}{(n-1)\dots(n-p+1)}, \frac{|a_{n-1-p}|}{(n-1)\dots(n-p)}\right\},\right.$$
$$\left.\frac{|a_{n-1}|}{n},\dots,\frac{|a_{n-p+1}|}{n\dots(n-p+2)}, \frac{|a_{n-p}|}{n\dots(n-p+1)}\right\} =$$
$$= \frac{B_p^2}{(n+1)n}\max\left\{|a_{n-1}|, \frac{|a_{n-2}|}{n-1},\dots,\frac{|a_{n-p}|}{(n-1)\dots(n-p+1)}, \frac{|a_{n-1-p}|}{(n-1)\dots(n-p)}\right\} \le \dots \le$$
$$\le \frac{B_p^{n-p-1}}{(n+1)n\dots(p+1)}\max\left\{|a_p|, \frac{|a_{p-1}|}{p},\dots,\frac{|a_1|}{p!}, \frac{|a_0|}{p!}\right\} = C_p\frac{B_p^n}{(n+1)!}, \quad C_p = \mathrm{const} > 0.$$

Hence it follows that f is an entire function of exponential type $\sigma \le B_p$. □

3. Functions with a sequence of derivatives in the middle p-valent

Now suppose that there exists an increasing sequence (k_j) such that all derivatives $f^{(k_j)}$ are in the middle p-valent in $\mathbb{D}$. We can consider that $k_0 = 0$. Then as above for $n \ge 1$ and $j \ge 0$ we have

$$|a_{k_j+n}| \le A_p\frac{n^{2p-1}n!}{(k_j+n)!}\max\left\{\frac{(k_j+\nu)!}{\nu!}|a_{k_j+\nu}| : \nu \le p\right\}.$$

In particular,

$$|a_{k_j+\nu}| = |a_{k_{j-1}+k_j-k_{j-1}+\nu}| \le$$
$$\le A_p\frac{(k_j-k_{j-1}+\nu)^{2p-1}(k_j-k_{j-1}+\nu)!}{(k_j+\nu)!}\max_{0\le\mu\le p}\left\{\frac{(k_{j-1}+\mu)!}{\mu!}|a_{k_{j-1}+\mu}|\right\}. \tag{3}$$

Let $1 \le m \le k_{j+1} - k_j$. Then from (2) and (3) we get

$$|a_{k_j+m}| \le A_p\frac{m^{2p-1}m!}{(k_j+m)!}\max_{0\le\nu\le p}\left\{\frac{(k_j+\nu)!}{\nu!}A_p\frac{(k_j-k_{j-1}+\nu)^{2p-1}(k_j-k_{j-1}+\nu)!}{(k_j+\nu)!}\right\}\times$$
$$\times\max_{0\le\mu\le p}\left\{\frac{(k_{j-1}+\mu)!}{\mu!}|a_{k_{j-1}+\mu}|\right\}$$

and, since $\frac{(m+\nu)!}{\nu!} \le \frac{(m+p)!}{p!}$ for all $m > 1$ and $0 \le \mu \le p$, we have

$$|a_{k_j+m}| \le \frac{A_p^2}{p!}\frac{m^{2p-1}m!(k_j-k_{j-1}+p)^{2p-1}(k_j-k_{j-1}+p)!}{(k_j+m)!}\max_{0\le\nu\le p}\left\{\frac{(k_{j-1}+\nu)!}{\nu!}|a_{k_{j-1}+\nu}|\right\}.$$

Applying to $|a_{k_{j-1}+\nu}|$ inequality (3), from here as above we obtain

$$|a_{k_j+m}| \le \frac{A_p^3}{(p!)^2}\times$$
$$\times\frac{m^{2p-1}m!(k_j-k_{j-1}+p)^{2p-1}(k_j-k_{j-1}+p)!(k_{j-1}-k_{j-2}+p)^{2p-1}(k_{j-1}-k_{j-2}+p)!}{(k_j+m)!}\times$$
$$\times\max_{0\le\nu\le p}\left\{\frac{(k_{j-2}+\nu)!}{\nu!}|a_{k_{j-2}+\nu}|\right\}.$$

Continuing this process we come to the inequality

$$(4)\qquad |a_{k_j+m}| \le B_p\left(\frac{A_p}{p!}\right)^j\frac{m^{2p-1}m!}{(k_j+m)!}\prod_{s=1}^{j}(k_s-k_{s-1}+p)^{2p-1}\prod_{s=1}^{j}(k_s-k_{s-1}+p)!.$$

which is correct for all $j\ge 0$, $1\le m\le k_{j+1}-k_j$ and some constant $B_p>0$. Using (4) we prove the following theorem.

Theorem 2. *If an increasing sequence (k_j) satisfies condition (2), $k_0=0$, and all $f^{(k_j)}$ are in the middle p-valent in $\mathbb{D}$ then f is an entire function.*

Proof. From (4) for $j\ge 0$ and $1\le m\le k_{j+1}-k_j$ we obtain

$$\frac{1}{k_j+m}\ln\frac{1}{|a_{k_j+m}|}\ge -\frac{\ln B_p+j(\ln A_p-\ln p!)+(2p-1)\ln m}{k_j+m}+\frac{\ln(k_j+m)!}{k_j+m}-$$
$$(5)\qquad -\frac{m!}{k_j+m}-\frac{2p-1}{k_j+m}\sum_{s=1}^{j}\ln(k_s-k_{s-1}+p)-\frac{1}{k_j+m}\sum_{s=1}^{j}\ln(k_s-k_{s-1}+p)!.$$

Clearly, the first term in the left hand side of (5) is a bounded value. If we use Stirling formula

$$n!=n^ne^{-n}\sqrt{2\pi n}\exp\{-\theta_n/(12n)\},\quad 0<\theta_n<1,$$

we get

$$\frac{\ln(k_j+m)!}{k_j+m}=\ln(k_j+m)+O(1),$$
$$\frac{\ln m!}{k_j+m}=\frac{m\ln m}{k_j+m}+O(1)$$

and

$$\ln(k_s-k_{s-1}+p)!\ge(k_s-k_{s-1}+p)\ln(k_s-k_{s-1}+p)+\frac{1}{2}\ln(k_s-k_{s-1}+p)+O(1)=$$
$$=(k_s-k_{s-1})\ln(k_s-k_{s-1})+\frac{2p+1}{2}\ln(k_s-k_{s-1}+p)+O(1)$$

as $j\to\infty$ and, since $\sum\limits_{s=1}^{j}\ln(k_s-k_{s-1}+p)\le k_j+jp$, from (5) we obtain

$$\frac{1}{k_j+m}\ln\frac{1}{|a_{k_j+m}|}\ge\frac{1}{k_j+m}\{(k_j+m)\ln(k_j+m)-m\ln m-Q_j\}+O(1),\quad j\to\infty,$$

for all $1 \le m \le k_{j+1} - k_j$, where

$$Q_j = \sum_{s=1}^{j} (k_s - k_{s-1}) \ln(k_s - k_{s-1}).$$

Now we consider the function

$$\Phi(x) = \frac{1}{k_j + x} \left\{ (k_j + x) \ln(k_j + x) - x \ln x - Q_j \right\}, \quad 1 \le x \le k_{j+1} - k_j.$$

Since $\Phi'(x) = (Q_j - k_j \ln x)/(k_j + x)^2$, this function has on $[1, +\infty)$ the only point of the extremum $x = \exp\{Q_j/k_j\}$, which is a maximum point. Therefore,

$$\min\{\Phi(x) : 1 \le x \le k_{j+1} - k_j\} = \min\{\Phi(1), \Phi(k_{j+1} - k_j)\}$$

and for $1 \le m \le k_{j+1} - k_j$ (5) implies

$$\frac{1}{k_j + m} \ln \frac{1}{|a_{k_j+m}|} \ge \min \left\{ \frac{1}{k_j + 1} \left((k_j + 1) \ln(k_j + 1) - Q_j \right), \right.$$

$$\left. \frac{1}{k_{j+1}} \left(k_{j+1} \ln(k_{j+1}) - (k_{j+1} - k_j) \ln(k_{j+1} - k_j) - Q_j \right) \right\} + O(1) \ge$$

$$\ge \min \left\{ \ln k_j - \frac{Q_j}{k_j}, \ln k_{j+1} - \frac{Q_{j+1}}{k_{j+1}} \right\} + O(1), \quad j \to \infty.$$

In view of (2) from the last equation it follows that $\frac{1}{k} \ln \frac{1}{|a_k|} \to +\infty$ as $k \to \infty$, i.e., f is an entire function. Theorem 2 is proved. □

4. Conclusion

Let $K = (k_j)$ be an increasing sequence of integers, $k_0 = 0$, and $E(K)$ be a class of analytic in $\mathbb{D}$ functions f such that for some $p \in \mathbb{N}$ all $f^{(k_j)}$ are in the middle p-valent in $\mathbb{D}$. From Theorem 2 it follows that if the sequence K satisfies (2) then f is an entire function. On the other hand, if (2) not hods then [3] there exists an analytic in $\mathbb{D}$ function f such that all $f^{(k_j)}$ are univalent in $\mathbb{D}$ $f^{(k_j)}$, but f is not an entire function. Therefore, the following statement is true: *every function* $f \in E(K)$ *is entire if and only if the sequence* K *satisfies* (2).

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ЗАУВАЖЕННЯ ДО ВЛАСТИВОСТЕЙ АНАЛІТИЧНИХ В ОДИНИЧНОМУ КРУЗІ ФУНКЦІЙ, P-ЛИСТИХ РАЗОМ ЗІ СВОЇМИ ПОХІДНИМИ

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Нехай $K = (k_j)$ — зростаюча послідовність цілих чисел, $k_0 = 0$, а $E(K)$ — клас таких аналітичних в крузі $\mathbb{D} = \{z : |z| < 1\}$ функцій f, що для деякого $p \in \mathbb{N}$ всі похідні $f^{(k_j)}$ є в середньому p-листими в $\mathbb{D}$. Доведено, що кожна функція $f \in E(K)$ є цілою тоді і тільки тоді, коли

$$\lim_{j\to\infty}\left\{\ln k_j - \frac{1}{k_j}\sum_{s=1}^{j}(k_s - k_{s-1})\ln(k_s - k_{s-1})\right\} = +\infty.$$

Ключові слова: ціла функція, p-листа функція.