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SPACES OF IDEMPOTENT MEASURES AND ABSOLUTE RETRACTS

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The aim of this note is to establish the AR-property for some non-compact spaces of idempotent measures (i.e., analogs of probability measures in the idempotent mathematics).

Key words: idempotent measure, absolute retract, convexity.

1. Introduction

In this note we deal with the spaces of idempotent measures. These measures are analogs of probability measures in the idempotent mathematics, i.e. a part of mathematics in which operations of addition and multiplication in $\mathbb{R}$ are replaced by idempotent ones (e.g., max or min). The spaces of idempotent measures (also called Maslov measures) are considered in numerous publications; see, e.g., the survey paper [4] and references therein.

In particular, in [2, 3] it is proved that some spaces of idempotent measures are absolute retracts.

Most of publications concern the compact case. In the present note we consider some non-compact spaces of idempotent measures.

Our main result is Corollary 5 which states that some non-compact spaces of idempotent measures are absolute retracts. To this end, we define a c-structure (in the sense of [8]) on the spaces of idempotent measures making these spaces $l.c$-spaces and then apply results of [7].

As the property of being an absolute retract is a necessary ingredient in characterization theorems for some model spaces in infinite-dimensional topology [1], we expect some applications of our results in this area.

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2. Preliminaries

Let (X, d) be a metric space. We consider the standard metric on $[0,1]$ and the max metric on the product $X \times [0,1]$. By Cl A (resp. Int A) we denote the closure (resp. interior) of a set A in a topological space.

The Hausdorff distance between two non-empty sets A, B in a metric space (X, d) is denoted by $\mathrm{d}_H(A,B)$,

$$\mathrm{d}_H(A,B) = \max\left\{\mathrm{d}(A,B), \mathrm{d}(B,A)\right\} = \max\left\{\sup_{a\in A}\inf_{b\in B}\mathrm{d}(a,b),\ \sup_{b\in B}\inf_{a\in A}\mathrm{d}(b,a)\right\}.$$

Note that the Hausdorff distance can be infinite.

By $\bigoplus$ we denote operation max on $\mathbb{R}$.

The definition of idempotent measure in terms of functionals is given in [9]. In [5] it is formulated in terms of special subsets of $X \times [0,1]$. Let X be a topological space.

Definition 1. The family of subsets $A \subset X \times [0,1]$ satisfying

(1) $\mathrm{Cl}\, A = A$;
(2) A is saturated, i.e. $\forall (x,t) \in A\ \forall t',\ 0 \leqslant t' \leqslant t \Rightarrow (x,t') \in A$;
(3) $X \times \{0\} \subset A$;
(4) $A \cap (X \times \{1\}) \neq \emptyset$

is denoted by $\bar{I}(X)$.

Given $A \in \bar{I}(X)$, we call the set

$$\mathrm{supp}(A) = \mathrm{Cl}\left(\{x \in X \mid \exists t > 0, (x,t) \in A\}\right)$$

the support of A.

The open ball of radius ε and center at A_1 in $\bar{I}(X)$ is the set

$$O_\varepsilon(A_1) = \{A \mid \mathrm{d}_H(A_1, A) < \varepsilon\}.$$

The distance d_2 in $\bar{I}(X) \times \bar{I}(X)$ between $(A_1,A_2),(B_1,B_2)$ is defined as $\max\{\mathrm{d}_H(A_1,B_1), d_H(A_2,B_2)\}$. Then the open ball of radius ε and center at (A_1,B_1) is the set

$$O_\varepsilon(A_1,B_1) = \{(A,B) \mid \mathrm{d}_2((A,B),(A_1,B_1)) < \varepsilon\}.$$

The distance $\mathrm{d}_{2,t}$ in $\bar{I}(X)\times\bar{I}(X)\times[0,1]$ between $(A_1,B_1,t_1),(A_2,B_2,t_2)$ is defined as

$$\max\{\mathrm{d}_2((A_1,A_2),(B_1,B_2)), |t_1 - t_2|\}.$$

The open ball of radius ε and center at (A_1,B_1,t_1) is the set

$$O_\varepsilon(A_1,B_1,t_1) = \{(A,B,t) \mid \mathrm{d}_{2,t}((A,B,t),(A_1,B_1,t_1)) < \varepsilon\}.$$

Lemma 1. *For any $A, B \in \bar{I}(X)$, the Hausdorff distance $d_H(A,B) \leqslant 1$.*

Proof. Let $A, B \in \bar{I}(X)$. If $(x,t) \in A$, then $(x,0) \in B$ and $\mathrm{d}((x,t),(x,0)) \leqslant 1$. We conclude that $B \subset O_\varepsilon(A)$, for any $\varepsilon > 1$. Similarly, $A \subset O_\varepsilon(B)$, for any $\varepsilon > 1$. Therefore, $\mathrm{d}_H(A,B) \leqslant 1$. □

3. Results

Let X be a metric space. Define a map $\bigoplus : \bar{I}(X) \times \bar{I}(X) \to \bar{I}(X)$ as follows:

$$\bigoplus(A, B) = A \bigoplus B = A \cup B,$$

where $A, B \in \bar{I}(X)$. It is obvious that this map is non-expanding and continuous. Using this operation define a map $\lambda : \bar{I}(X) \times \bar{I}(X) \times [0,1] \to \bar{I}(X)$ as follows $\lambda(A, B, t) = A \bigoplus tB$, where $tB = \{(x, ts) \mid (x, s) \in B\}$. This is map has the following property $\lambda(A, B, 0) = A$, $\lambda(A, B, 1) = A \bigoplus B$.

Proposition 1. *λ is a non-expanding map.*

Proof. We show that for arbitrary two distinct points

$$(A_1, B_1, t_1), (A_2, B_2, t_2) \in \bar{I}(X) \times \bar{I}(X) \times [0, 1]$$

we obtain

$$\mathrm{d}_{2,t}((A_1, B_1, t_1), (A_2, B_2, t_2)) \geqslant \mathrm{d}_H(\lambda(A_1, B_1, t_1), \lambda(A_2, B_2, t_2)).$$

Therefore

$$\mathrm{d}_{2,t}((A_1, B_1, t_1), (A_2, B_2, t_2)) = \max\{\mathrm{d}_H(A_1, A_2), \mathrm{d}_H(B_1, B_2), |t_1 - t_2|\}.$$

Let $\mathrm{d}_{2,t}((A_1, B_1, t_1), (A_2, B_2, t_2)) = c$, then

$$\mathrm{d}_H(A_1, A_2) \leqslant c,\ \mathrm{d}_H(B_1, B_2) \leqslant c,\ |t_1 - t_2| \leqslant c.$$

Assume that $\mathrm{d}_H(A_1 \bigoplus t_1B_1, A_2 \bigoplus t_2B_2) > c$. In other words, there exists $(x_1, s_1) \in A_1 \cup t_1B_1$ such that the distance to any point from $A_2 \cup t_2B_2$ is greater than c.

If $(x_1, s_1) \in A_1$ this is a contradiction as $d_H(A_1, A_2) \leqslant c$ and this is meaning that in set A_2 there exists a point with distance less or equal c, thus exiting point for set $A_2 \bigoplus t_2B_2$. If $(x_1, s_1) \in t_1B_1$ then exists $s_1' \in [0, 1]$ and $(x_1, s_1') \in B_1$ where $s_1 = t_1s_1'$, as $d_H(B_1, B_2) \leqslant c$, then exists $(x_2, s_2') \in B_2$ such that $\mathrm{d}((x_1, s_1'), (x_2, s_2')) \leqslant c$. Consider the point $(x_2, s_2) \in t_2B_2$, where $s_2 = t_2s_2'$. The distance $\mathrm{d}((x_1, s_1), (x_2, s_2))$ is equal to $\max\{\mathrm{d}(x_1, x_2), |s_1 - s_2|\}$. From assumption we have $\mathrm{d}(x_1, x_2) \leqslant c$, let us estimate the value $|s_1 - s_2| = |t_1s_1' - t_2s_2'|$. We consider two cases:

a) $t_2 \geqslant t_1$, then

$$|t_1s_1' - t_2s_2'| \leqslant |t_2s_1' - t_2s_2'| = |t_2||s_1' - s_2'| \leqslant |t_2|c \leqslant c,$$

as $t_2 \in [0, 1]$.

b) $t_1 \geqslant t_2$, then

$$|t_1s_1' - t_2s_2'| = |t_2s_2' - t_1s_1'| \leqslant |t_1s_2' - t_1s_1'| = |t_1||s_2' - s_1'| \leqslant |t_1|c \leqslant c,$$

as $t_1 \in [0, 1]$.

Thus, $|s_1 - s_2| \leqslant c$ and this is a contradiction. □

Corollary 1. *λ is a continuous map.*

Definition 2. The set

$$\Delta_{\rm d}^n = \left\{(t_0, \ldots, t_n) | \bigoplus_{i=0}^{n} t_i = 0,\ t_i \in [0,1],\ i = 0, \ldots, n\right\}$$

is called the n-dimensional id-*simplex*.

In [8] the following notion is introduced. Given a topological space Y, a c-structure on Y is an assignment for each non empty finite subset $A \subset Y$ of a non empty contractible subspace $F(A) \subset Y$ such that $F(A) \subset F(B)$ if $A \subset B$. A nonempty subset E of Y is an F-set if $F(A) \subset E$ for any non empty finite subset A of E.

If (Y, d) is a metric space, then we say that $(Y, d; F)$ is a metric l.c-space if open balls are F-sets and if any neighborhood $\{y \in Y \mid \mathrm{d}(y, E) < r\}$ of every F-set $E \subset Y$ is also an F-set.

The following characterization theorem for metric l.c-spaces and its proof were suggested by the referee.

Theorem 1. *Let F be a c-structure on a metric space (X, d). The triple $(X, d; F)$ is a metric l.c-space if and only if $F(\{x\}) = \{x\}$ for all $x \in X$, and every r-neighborhood, $r > 0$, of every F-set is an F-set.*

Proof. To prove the "if", part, observe that every singleton $\{x\} = F(\{x\})$ in X is an F-set and hence every open r-ball is an F-set.

To prove the "only if" part, assume that $(X, d; F)$ is a metric l.c-space. We claim that for every $x \in X$ we have $F(\{x\}) \subset \{x\}$. In the opposite case, there exits $y \in F(\{x\}) \setminus \{x\}$. Choose any positive $r < d(x, y)$. Since the open ball $B_r(x)$ is an F-set, $y \in F(\{x\}) \subset B_r(x)$ and hence $d(y, x) < r < d(y, x)$. This contradiction shows that $F(\{x\}) \subset \{x\}$, therefore $F(\{x\}) = \{x\}$ as $F(\{x\})$ is nonempty.

This theorem easily implies that, for any metric l.c-space $(X, d; F)$, we have $A \subset F(A)$ for any finite nonempty set $A \subset X$.

Definition 3. A subset $\mathcal{B}$ in $\bar{I}(X)$ is called id-*convex* if for any collection $A_1, \ldots, A_n \in \mathcal{B}$ and a vector $(t_1, \ldots, t_n) \in \Delta_{\rm d}^{n-1}$ the set $\bigoplus_{i=1}^n t_i A_i$ is an element of $\mathcal{B}$.

Proposition 2. *Every ε-neighborhood of any id-convex set in $\bar{I}(X)$ is an id-convex set.*

Proof. In other words, it should be shown that for every $\varepsilon > 0$ and for every collection $A_1, \ldots, A_n \in \mathcal{B} \subset \bar{I}(X)$ where $A_i \in \bar{I}(X)$ and a certain set of $C_1, \ldots, C_n \in \bar{I}(X)$, where $\mathrm{d}_H(A_i, C_i) < \varepsilon$ for $i \in 1, \ldots, n$ then $\mathrm{d}_H(\bigcup_{i=1}^n t_i A_i, \bigcup_{i=1}^n t_i C_i) < \varepsilon$ for every $(t_1, \ldots, t_n) \in \Delta_{\rm d}^{n-1}$. Since for every $(x, s) \in \bigcup_{i=1}^n t_i A_i$, there exists $j \in 1, \ldots, n$ such that $(x, s) \in t_j A_j$, then there exists $(x, s') \in A_j$ such that $s = t_j s'$. Then there exists $(y, p') \in C_j$ such that $\mathrm{d}((x, s'), (y, p')) < \varepsilon$, as $\mathrm{d}_H(A_j, C_j) < \varepsilon$ then $(y, p) \in t_j C_j$, where $p = t_j p'$. The distance between (x, s) and (y, p) is equal to

$$\max\{\mathrm{d}(x, y), |s - p|\} = \max\{\mathrm{d}(x, y), |t_j(s' - p')|\} < \varepsilon.$$

□

Note that the one-point sets are clearly id-convex.

If $\mathcal{A} = \{A_1, \dots, A_n\} \subseteq \bar{I}(X)$ is a finite set, we define

$$F(\mathcal{A}) = \left\{ \bigcup_{i=1}^{n} t_i A_i \mid (t_1, \dots, t_n) \in \Delta_{\mathrm{d}}^n \right\}.$$

If $\mathcal{A} \subseteq \mathcal{B}$, then clearly $F(\mathcal{A}) \subseteq F(\mathcal{B})$.

Lemma 2. *$F(\mathcal{A})$ is contractible for any finite $\mathcal{A} = \{A_1, \dots, A_n\} \subseteq \bar{I}(X)$.*

Proof. Define $H\colon F(\mathcal{A}) \times [0,1] \to F(\mathcal{A})$ as follows

$$H\left(\bigcup_{i=1}^{n} t_i A_i, t\right) = \bigcup_{i=1}^{n} \max\{t, t_i\} A_i.$$

It is easy to check that H is well-defined. Clearly, H is a homotopy connecting the identity map with the constant map into $A_1 \cup \dots \cup A_n$. □

Thus we obtain that $(\bar{I}(X), F))$ is a c-space.

Corollary 2. *Every id-convex set in $\bar{I}(X)$ is an F-set.*

The characterization theorem for metric l.c-spaces implies the following statement.

Corollary 3. *$(\bar{I}(X), d_H; F)$ is an l.c-space for every metric space X.*

Recall that a space X is an absolute retract in the class of all metrizable spaces (briefly, $X \in AR(Metric)$), if X is a retract of every metrizable space Y in which X is a closed subset.

The following is a consequence of results of [7, 8].

Corollary 4. *$\bar{I}(X) \in AR(Metric)$ for every metric space X.*

Let X be a metric space. Consider the set $\bar{I}_\beta(X)$ of all subsets A in $\bar{I}(X)$ such that the support of A, i.e., the set $\mathrm{supp}(A)$ is compact. Also by $\bar{I}_\omega(X)$ we denote the family of all sets $A \in \bar{I}_\beta(X)$ such that $\mathrm{supp}(A)$ is a finite set. Note that this definition agrees with the Chigogidze extension [6] of the normal functor (in [9] it is established that the functor of idempotent measures is normal).

We also consider the space $\bar{I}_a(X)$, which consists of all $A \in \bar{I}(X)$ such that $\{x \mid \exists t > 0, (x,t) \in A\}$ is at most countable set.

Now, let $A \in \bar{I}(X)$, $x_0 \in X$ be a base point. Define the number

$$\Phi_{x_0,r}(A) = \max\{t \in [0,1] \mid \exists x \in X,\ \mathrm{d}(x, x_0) \geqslant r, (x,t) \in A\}.$$

Finally, let

$$\bar{I}_s(X) = \{A \in \bar{I}(X), \lim_{r \to \infty} \Phi_{x_0,r}(A) = 0\}.$$

Remark that the definition of $\bar{I}_s(X)$ does not depend on the choice of base point.

Lemma 3. *The subspaces $\bar{I}_\beta(X), \bar{I}_\omega(X), \bar{I}_a(X), \bar{I}_s(X)$ are l.c-spaces for all metric spaces X.*

Proof. It will be enough to check the closeness of these subspaces with respect to their convex combination. Let $A_1, \dots, A_n \in \bar{I}_c(X)$, then $\bigcup_{i=1}^n t_i A_i \in \bar{I}(X)$ and the support of this set will be compact as finite union of compact supports.

Similarly one can prove the closeness with respect to their convex combination for the other subspaces. □

Corollary 5. $\bar{I}_\beta(X), \bar{I}_\omega(X), \bar{I}_a(X), \bar{I}_s(X) \in AR(Metric)$.

4. Remarks

As we already noted, the obtained results can be used for topological characterization of some non-complete spaces of idempotent measures. In particular, we conjecture that the space $\bar{I}_\beta(X)$ is homeomorphic to the pseudo-boundary $B(Q)$ of the Hilbert cube Q, for any locally compact non-compact separable metric space X.

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ПРОСТОРИ ІДЕМПОТЕНТНИХ МІР ТА ЇХНІ АБСОЛЮТНІ РЕТРАКТИ

Юрій МАРКО

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Мета статті — визначити AR-властивості для деяких некомпактних просторів ідемпотентних мір (тобто, аналоги ймовірнісних мір у ідемпотентній математиці).

Ключові слова: ідемпотента міра, абсолютний ретракт, опуклість.