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HILBERT POLYNOMIALS FOR THE ALGEBRA OF INVARIANTS OF BINARY d-FORM

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We consider one of the fundamental problems of classical invariant theory – the research of Hilbert polynomials for the algebra of invariants of the Lie group SL_2. Form of the Hilbert polynomials gives us important information about the structure of the algebra. Besides, the coefficients and the degree of the Hilbert polynomial play an important role in Algebraic Geometry. It is well-known that the Hilbert function of the algebra SL_n-invariants is quasi-polynomial. The Cayley-Sylvester formula for calculation of values of the Hilbert function for algebra of covariants of binary $d-$form $\mathcal{C}_d = \mathbb{C}[V_d \oplus \mathbb{C}^2]^{SL_2}$ (here V_d is the $d+1-$dimensional space of binary forms of degree d) was obtained by Sylvester. Then it was generalized to algebra of invariants of binary $d-$form $\mathcal{I}_d = \mathbb{C}[V_d]^{SL_2}$. But the Cayley-Sylvester formula is not expressed in terms of polynomials. In our article we consider the problem of computing the Hilbert polynomials for the algebra of invariants of binary $d-$form.

Key words: classical invariant theory, invariants, Hilbert function, Hilbert polynomials, Poincaré series.

1. Introduction

Let V_d be $d+1-$dimensional module of binary forms. Denote by $\mathcal{I}_d = \mathbb{K}[V_d]^{SL_2}$ algebra of polynomial SL_2-invariant functions on V_d, where SL_2 is the group of 2×2 matrices with complex coefficients and determinant one. Algebra $\mathcal{I}_d$ is called *the algebra of invariants of binary $d-$form.* It is well-known that the algebra of invariants of binary $d-$form is finitely generated and graded

$$\mathcal{I}_d = (\mathcal{I}_d)_0 \oplus (\mathcal{I}_d)_1 \oplus \ldots \oplus (\mathcal{I}_d)_n \oplus \ldots,$$

2020 Mathematics Subject Classification: 13A50 here $(\mathcal{I}_d)_n$ is a vector $\mathbb{C}-$space of invariants of degree n. Dimension of the vector space $(\mathcal{I}_d)_n$ is called *the Hilbert function* of the algebra $\mathcal{I}_d$. It is defined as a function of the variable n:

$$\mathcal{H}(\mathcal{I}_d, n) = \dim(\mathcal{I}_d)_n.$$

The formal power series

$$\mathcal{P}(\mathcal{I}_d, z) = \sum_{n=0}^{\infty} \mathcal{H}(\mathcal{I}_d, n)\, z^n$$

are called *the Poincaré series of the algebra* $\mathcal{I}_d$. The following formula for computation of the Poincaré series $\mathcal{P}(\mathcal{I}_d, z)$ for the algebra of invariants of binary $d-$form was found by T. Springer in [12].

$$(1) \qquad \mathcal{P}(\mathcal{I}_d, z) = \sum_{0 \leqslant k < d/2} \varphi_{d-2\,k}\left(\frac{(-1)^k z^{k(k+1)}(1-z^2)}{(z^2, z^2)_k\,(z^2, z^2)_{d-k}}\right) = \sum_{0 \leqslant k < d/2} \varphi_{d-2\,k}\left(f_k(z^2)\right).$$

Another proof of this formula was obtained by Leonid Bedratyuk in [5].

It is well-known that the Hilbert function of arbitrary finitely generated graded $\mathbb{C}$-algebra is quasi-polynomial (starting from some n), see [8, 11, 13]. Since, the algebra of invariants $\mathcal{I}_d$ is finitely generated, we have

$$\mathcal{H}(\mathcal{I}_d, n) = h_0(n) n^r + h_1(n) n^{r-1} + \ldots,$$

where $h_k(n)$ is some periodic function with values in $\mathbb{Q}$. The quasi-polynomial $\mathcal{H}(\mathcal{I}_d, n)$ is called *the Hilbert polynomials* of algebra $\mathcal{I}_d$.

There exists classical Cayley-Sylvester formula for calculation of values of Hilbert function of $\mathcal{I}_d$:

$$\mathcal{H}(\mathcal{I}_d, n) = \omega_d(n, 0) - \omega_d(n, 2),$$

where $\omega_d(n, k)$ is the number non-negative integer solutions of the system

$$\begin{cases} \alpha_1 + 2\alpha_2 + \ldots + d\alpha_d = \dfrac{dn-k}{2}, \\ \alpha_1 + \alpha_2 + \ldots + \alpha_d = n. \end{cases}$$

Also, see [9, 12], we have

$$\mathcal{H}(\mathcal{I}_d, n) = \left[q^{\frac{nd}{2}}\right]\left(\frac{\left(1-q^{d+1}\right)\left(1-q^{d+2}\right)\ldots\left(1-q^{d+n}\right)}{\left(1-q^2\right)\left(1-q^3\right)\ldots\left(1-q^n\right)}\right),$$

where $\left[q^{\frac{nd}{2}}\right]$ denotes the coefficient of $q^{\frac{nd}{2}}$. The generalizations of these formulas to the algebra

$$\mathcal{I}_{\mathbf{d}} = \mathbb{K}[V_{d_1} \oplus V_{d_2} \oplus \ldots \oplus V_{d_m}]^{SL_2}$$

was obtained in [2, 3, 1, 4]. It is not assumed that the d_k are pairwise distinct.

However, all these results are combinatorial formulas. They are not expressed in terms of Hilbert polynomials in n. Note that, it is hard to calculate for those formulas even for small values of d_k and n. Although, Maple-procedure for computing of the Hilbert polynomials of the algebras of SL_2- invariants for small values of d was being offered in [6].

In [10] author have expressed the Hilbert polynomials of algebras

$$\mathbb{K}[\underbrace{V_1 \oplus V_1 \oplus \ldots \oplus V_1}_{m \text{ times}}]^{SL_2} \quad \text{and} \quad \mathbb{K}[\underbrace{V_2 \oplus V_2 \oplus \ldots \oplus V_2}_{m \text{ times}}]^{SL_2}$$

in terms quasi-polynomial. Author also presented them in the form Narayana numbers and generalized hypergeometric series.

In the present paper we obtain the following formula for computation of the Hilbert polynomials of the algebra $\mathcal{I}_d$:

$$\mathcal{H}(\mathcal{I}_d, n) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\oint_{\mathbb{S}^1} \sum_{k=0}^{\frac{d-1}{2}} (-1)^{k+1} \dfrac{z^{\frac{k(k+1)}{2} - \left(\frac{d}{2} - k\right)n - 1}}{(z, z)_k (z^2, z)_{d-k-1}} dz, & \text{if } dn = 0 \pmod 2, \\ 0, & \text{if } dn = 1 \pmod 2, \end{cases}$$

here $(a, q)_n = (1 - a)(1 - a\, q) \cdots (1 - a\, q^{n-1})$ is q-shifted factorial.

We use formula for Poincaré series of the algebra of invariants $\mathcal{I}_d$ of binary $d-$form $\mathcal{I}_d$ to obtain the Hilbert polynomials of the algebra.

2. Background and notation

Consider the $\mathbb{C}$-algebra $\mathbb{C}[[z]]$ of formal power series. For each $k \in \mathbb{N}$ define $\mathbb{C}$-linear function φ_k in the following way

$$\varphi_k\big(z^m\big) := \begin{cases} z^{\frac{m}{k}}, & \text{if } m = 0 \pmod k, \\ 0, & \text{if } m \neq 0 \pmod k, \\ 1, & \text{if } m = 0. \end{cases}$$

It is clear that for arbitrary series

$$A = a_0 + a_1 z + a_2 z^2 + \cdots$$

we obtain

$$\varphi_k(A) = a_0 + a_k z + a_{2\,k} z^2 + \cdots + a_{s\,k} z^s + \cdots .$$

The following identity is hold, see [5]:

$$\varphi_k(f(z)) = \frac{1}{k} \sum_{j=1}^{k} f(\zeta_k^j z) \Bigg|_{z^k = z}, \quad \text{where } \zeta_k = e^{2\pi i/k}.$$

In this section we investigate properties of the function φ_k.

Lemma 1. *The expansion of function* $\varphi_k\left(\dfrac{1}{(1 - \zeta_s^j z)^m}\right)$ *into a Taylor series at* $z = 0$ *is*

$$\varphi_k\left(\frac{1}{(1 - \zeta_s^j z)^m}\right) = \sum_{n=0}^{\infty} \binom{m + nk - 1}{nk} \zeta_s^{jkn} z^n.$$

Proof. Expand function $\dfrac{1}{(1 - \zeta_s^j z)^m}$ into a Taylor series at $z = 0$. We have

$$\varphi_k\left(\frac{1}{(1 - \zeta_s^j z)^m}\right) = \varphi_k\left(\sum_{n=0}^{\infty} \binom{m + n - 1}{n} \zeta_s^{jn} z^n\right) = \sum_{n=0}^{\infty} \binom{m + n - 1}{n} \zeta_s^{jn} \varphi_k(z^n).$$

To conclude the proof, it remains to use the definition of the function $\varphi_k(z^n)$. □

Lemma 2. *Suppose $f \in \mathbb{C}[[z]]$. The following identity holds*

$$\varphi_{mk}\left(f(z^k)\right) = \varphi_m\left(f(z)\right),$$

for all integer $k, m > 0$.

Proof. Since $f \in \mathbb{C}[[z]]$, it follows that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We have

$$\varphi_m\left(f(z)\right) = \varphi_m\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n \varphi_m(z^n).$$

By definition of the function $\varphi_m(z)$, we obtain

$$\varphi_m\left(f(z)\right) = \sum_{n=0}^{\infty} a_{nm} z^n.$$

A similar argument yields

$$\varphi_{mk}\left(f(z^k)\right) = \varphi_{mk}\left(\sum_{n=0}^{\infty} a_n z^{kn}\right) = \sum_{n=0}^{\infty} a_{nm} z^n.$$

□

Lemma 3. *Let us $f = \sum_{n=0}^{\infty} a_n z^n$. Suppose k is odd number; then $\varphi_k\left(f(z^2)\right)$ is even function:*

$$\varphi_k\left(f(z^2)\right) = \sum_{n=0}^{\infty} a_{\left[\frac{nk}{2}\right]} \cos^2 \frac{n\pi}{2} z^n,$$

where $[x]$ is integer part of a number x.

Proof.

$$\varphi_{2m+1}\left(f(z^2)\right) = \varphi_{2m+1}\left(\sum_{n=0}^{\infty} a_n z^{2n}\right) = \sum_{n=0}^{\infty} a_{n(2m+1)} z^{2n} = \sum_{n=0}^{\infty} a_{\left[\frac{n(2m+1)}{2}\right]} \cos^2 \frac{n\pi}{2} z^n.$$

□

Combining Lemmas 1,2,3, we obtain the following Lemma.

Lemma 4. *The following formula holds*

$$\varphi_{d-2k}\left(\frac{1}{(1-\zeta_s^j z^2)^m}\right) = \sum_{n=0}^{\infty} \binom{m + [(d/2-k)n] - 1}{m-1} \zeta_s^{j[(d/2-k)n]} \cos^2 \frac{dn\pi}{2} z^n.$$

3. Hilbert polynomials

Theorem 1. *The Hilbert polynomials of the algebra of invariants of binary d–form is given by*

$$\mathcal{H}(\mathcal{I}_d,n)=\frac{\cos^2\frac{\pi nd}{2}}{2\pi i}\oint_{\mathbb{S}^1}\sum_{k=0}^{\frac{d-1}{2}}(-1)^{k+1}\frac{z^{\frac{k(k+1)}{2}-\left(\frac{d}{2}-k\right)n-1}}{(z,z)_k(z^2,z)_{d-k-1}}dz.$$

Proof. The proof is similar to that of proposition 6.2 in [7].

Let's find a partial fraction decomposition for the rational function

$$f_k(z)=\frac{(-1)^k z^{k(k+1)/2}(1-z)}{(z,z)_k\,(z,z)_{d-k}}=$$
$$=\frac{(-1)^k z^{k(k+1)/2}}{((1-z)(1-z^2)\dots(1-z^k))\cdot((1-z^2)(1-z^3)\dots(1-z^{d-k}))}$$

as $k<\frac{d}{2}$ over $\mathbb{C}$. Clearly, a partial fraction decomposition for the rational function $f_k(z)$ is possible. Since $k<\frac{d}{2}$, we obtain $k<d-k$. Using residue method, we have

$$f_k(z)=\sum_{s=1}^{d-k}\sum_{\substack{j=1\\(j,s)=1}}^{s}\sum_{t=1}^{u}\frac{\frac{(-\zeta_s^{-j})^{u-t}}{(u-t)!}\lim\limits_{z\to\zeta_s^{-j}}\frac{d^{u-t}\left(f_k(z)(1-\zeta_s^jz)^u\right)}{dz^{u-t}}}{(1-\zeta_s^jz)^t},$$

where

$$u=\begin{cases} d-1, & \text{as } s=1,\\ \left[\frac{k}{s}\right]+\left[\frac{d-k}{s}\right], & \text{as } 2\leqslant s\leqslant k,\\ \left[\frac{d-k}{s}\right], & \text{as } k+1\leqslant s\leqslant d-k,\end{cases}$$

and (j,s) is the greatest common divisor of two integeres j and s.

If we combine formula (1) with Lemma 4, we get

$$\mathcal{P}(\mathcal{I}_d,z)=$$
$$=\sum_{0\leqslant k<d/2}\sum_{s=1}^{d-k}\sum_{\substack{j=1\\(j,s)=1}}^{s}\sum_{t=1}^{u}\frac{(-\zeta_s^{-j})^{u-t}}{(u-t)!}\lim_{z\to\zeta_s^{-j}}\frac{d^{u-t}\left(f_k(z)(1-\zeta_s^jz)^u\right)}{dz^{u-t}}\varphi_{d-2k}\left(\frac{1}{(1-\zeta_s^jz^2)^t}\right)=$$
$$=\sum_{n=0}^{\infty}\sum_{0\leqslant k<d/2}\sum_{s=1}^{d-k}\sum_{\substack{j=1\\(j,s)=1}}^{s}\sum_{t=1}^{u}\frac{(-\zeta_s^{-j})^{u-t}}{(u-t)!}\times$$
$$\times\lim_{z\to\zeta_s^{-j}}\frac{d^{u-t}\left(f_k(z)(1-\zeta_s^jz)^u\right)}{dz^{u-t}}\binom{t+[(d/2-k)n]-1}{t-1}\left(\zeta_s^j\right)^{[(\frac{d}{2}-k)n]}\cos^2\frac{dn\pi}{2}z^n.$$

By definition of Poincaré series and Hilbert function, so that

$$\mathcal{H}(\mathcal{I}_d,n)=\cos^2\frac{dn\pi}{2}\sum_{0\leqslant k<d/2}\sum_{s=1}^{d-k}\sum_{\substack{j=1\\(j,s)=1}}^{s}\sum_{t=1}^{u}\frac{(-\zeta_s^{-j})^{u-t}}{(u-t)!}\times$$
$$\times\lim_{z\to\zeta_s^{-j}}\frac{d^{u-t}\left(f_k(z)(1-\zeta_s^jz)^u\right)}{dz^{u-t}}\binom{t+[(d/2-k)n]-1}{[(d/2-k)n]}\left(\zeta_s^j\right)^{[(\frac{d}{2}-k)n]}.$$

It now follows that

$$\mathcal{H}(\mathcal{I}_d,n)=\cos^2\frac{dn\pi}{2}\sum_{0\leqslant k<d/2}\sum_{s=1}^{d-k}\sum_{\substack{j=1\\(j,s)=1}}^{s}\sum_{t=0}^{u-1}(-1)^{u-1}(\zeta_s^{-j})^u\frac{1}{t!(u-t-1)!}\times$$
$$\times\lim_{z\to\zeta_s^{-j}}\left(\frac{d^{u-t-1}\left(f_k(z)(1-\zeta_s^j z)^u\right)}{dz^{u-t-1}}\frac{d^t\left(\frac{1}{z}\right)^{[(d/2-k)n]+1}}{dz^t}\right).$$

Using general Leibniz rule, we obtain

$$\mathcal{H}(\mathcal{I}_d,n)=\cos^2\frac{dn\pi}{2}\sum_{0\leqslant k<d/2}(-1)^{k+1}\sum_{s=1}^{d-k}\sum_{\substack{j=1\\(j,s)=1}}^{s}\frac{(-\zeta_s^{-j})^u}{(u-1)!}\times$$
$$\times\lim_{z\to\zeta_s^{-j}}\frac{d^{u-1}\dfrac{\left(1-\zeta_s^j z\right)^u(1-z)z^{\frac{k(k+1)}{2}-\left[\left(\frac{d}{2}-k\right)n\right]-1}}{(z,z)_k(z,z)_{d-k}}}{dz^{u-1}}.$$

By definition, we have

$$(2)\quad \mathcal{H}(\mathcal{I}_d,n)=\cos^2\frac{dn\pi}{2}\sum_{0\leqslant k<d/2}\sum_{s=1}^{d-k}\sum_{\substack{j=1\\(j,s)=1}}^{s}Res\left((-1)^{k+1}\frac{z^{\frac{k(k+1)}{2}-\left[\left(\frac{d}{2}-k\right)n\right]-1}}{(z,z)_k(z^2,z)_{d-k-1}},z{=}\zeta_s^j\right).$$

The application of Cauchy's residue theorem yields

$$\mathcal{H}(\mathcal{I}_d,n)=\frac{\cos^2\frac{\pi nd}{2}}{2\pi i}\sum_{k=0}^{\frac{d-1}{2}}\oint_{\mathbb{S}^1}(-1)^{k+1}\frac{z^{\frac{k(k+1)}{2}-\left(\frac{d}{2}-k\right)n-1}}{(z,z)_k(z^2,z)_{d-k-1}}dz.$$

This completes the proof of Theorem 1. □

Combining Lemma 4 and formula 1, we obtain

Corollary 1. *If d is an odd number, then the Poincaré series $\mathcal{P}(\mathcal{I}_d,z)$ for the algebra of invariants of binary $d-$form is an even function.*

Corollary 2. *The Hilbert polynomials of the algebra of invariants of binary $d-$form is given by*

$$\mathcal{H}(\mathcal{I}_d,n)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\oint_{\mathbb{S}^1}\sum_{k=0}^{\frac{d-1}{2}}(-1)^{k+1}\frac{z^{\frac{k(k+1)}{2}-\left(\frac{d}{2}-k\right)n-1}}{(z,z)_k(z^2,z)_{d-k-1}}dz, & \text{if } dn=0 \pmod 2,\\ 0, & \text{if } dn=1 \pmod 2.\end{cases}$$

4. Calculating Hilbert polynomials of the algebra $\mathcal{I}_d$

Example 1. Let's find Hilbert polynomials of the algebra of invariants of bynary $d-$form, as $d=4$. Using formula (1), we have

$$\mathcal{H}(\mathcal{I}_4,n)=\cos^2\frac{4\pi n}{2}\sum_{0\leqslant k<2}\sum_{s=1}^{4-k}\sum_{\substack{j=1\\(j,s)=1}}^{s}Res\left((-1)^{k+1}\frac{z^{\frac{k(k+1)}{2}-[(2-k)n]-1}}{(z,z)_k(z^2,z)_{3-k}},z=\zeta_s^j\right).$$

Since n is an integer number, we obtain

$$\begin{aligned}\mathcal{H}(\mathcal{I}_4, n) = & -Res\left(\frac{z^{-2n-1}}{(1-z^2)(1-z^3)(1-z^4)}, z{=}1\right) - \\ & - Res\left(\frac{z^{-2n-1}}{(1-z^2)(1-z^3)(1-z^4)}, z{=}-1\right) - \\ & - Res\left(\frac{z^{-2n-1}}{(1-z^2)(1-z^3)(1-z^4)}, z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) - \\ & - Res\left(\frac{z^{-2n-1}}{(1-z^2)(1-z^3)(1-z^4)}, z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) - \\ & - Res\left(\frac{z^{-2n-1}}{(1-z^2)(1-z^3)(1-z^4)}, z = i\right) - \\ & - Res\left(\frac{z^{-2n-1}}{(1-z^2)(1-z^3)(1-z^4)}, z = -i\right) + \\ & + Res\left(\frac{z^{-2n}}{(1-z)(1-z^2)(1-z^3)}, z = 1\right) + \\ & + Res\left(\frac{z^{-2n}}{(1-z)(1-z^2)(1-z^3)}, z = -1\right) + \\ & + Res\left(\frac{z^{-2n}}{(1-z)(1-z^2)(1-z^3)}, z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \\ & + Res\left(\frac{z^{-2n}}{(1-z)(1-z2)(1-z^3)}, z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right).\end{aligned}$$

Computing the residue, we get

$$\begin{aligned}\mathcal{H}(\mathcal{I}_4, n) = & \frac{5}{12} + \frac{1}{6}\,n + \frac{(-1)^n}{4} + \frac{1}{9}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right)^{-2\,n} - \\ & - \frac{1}{18}\,i\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{-n}\sqrt{3} + \frac{1}{18}\,i\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{-n}\sqrt{3} + \\ & + \frac{1}{9}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{-2\,n} + \frac{1}{18}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{-n} + \frac{1}{18}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{-n}.\end{aligned}$$

Now if we recall $e^{\frac{\pi}{3}i} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and de Moivre's formula, we get the following formula (see [6])

$$\mathcal{H}(\mathcal{I}_4, n) = \frac{1}{6}n + \frac{1}{4}\cos(\pi n) + \frac{1}{3}\cos\frac{2\pi n}{3} - \frac{1}{9}\sqrt{3}\sin\frac{2\pi n}{3} + \frac{5}{12}.$$

Example 2. Let us Hilbert polynomials of the algebra of invariants of bynary $d-$form, as $d = 8$.

The following formula for $\mathcal{H}(\mathcal{I}_8, n)$ can be obtained in a way analogous to that used in Example 1.

$$\mathcal{H}(\mathcal{I}_8, n) = \frac{1}{120960}n^5 + \frac{1}{5376}n^4 + \frac{16975873}{576}n^3 + \left(\frac{839}{26880} + \frac{\cos \pi n}{256}\right) n^2 +$$
$$+\left(\frac{99797}{725760} + \frac{371}{6912}\cos(\pi n) + \frac{1}{27}\cos\frac{2\pi n}{3} - \frac{\sqrt{3}}{243}\sin\frac{2\pi n}{3} - \frac{2\sqrt{3}}{243}\sin\left(\frac{\pi n}{3}+\frac{\pi}{3}\right)\right) n +$$
$$+\frac{1}{32}\cos\frac{\pi n}{2} + \frac{1}{32}\sin\frac{\pi n}{2} + \frac{33401}{124416}\cos(\pi n) + \frac{41}{243}\cos\frac{2\pi n}{3} - \frac{\sqrt{3}}{27}\sin\frac{2\pi n}{3} -$$
$$-\frac{14}{243}\cos\frac{\pi n}{3} - \frac{2\sqrt{3}}{243}\sin\left(\frac{\pi n}{3}\right) + \left(-\frac{1}{2} + \frac{7\sqrt{5}}{100}\right)\cos\frac{2\pi n}{5} +$$
$$+\left(\frac{\sqrt{5+\sqrt{5}}}{10} + \frac{3\sqrt{2}\sqrt{\sqrt{5}-1}}{20} - \frac{\sqrt{2}\sqrt{1+\sqrt{5}}}{20}\right)\sin\frac{2\pi n}{5} +$$
$$+\left(-\frac{1}{2} - \frac{7\sqrt{5}}{100}\right)\cos\frac{4\pi n}{5} + \left(\frac{\sqrt{5-\sqrt{5}}}{10} + \frac{\sqrt{2}\sqrt{\sqrt{5}-1}}{20} + \frac{3\sqrt{2}\sqrt{1+\sqrt{5}}}{20}\right)\sin\frac{4\pi n}{5} +$$
$$+\frac{1}{9}\left(\cos\frac{2\pi n}{7} + \cos\frac{4\pi n}{7} + \cos\frac{6\pi n}{7} - \cos\frac{2\pi(n+2)}{7} - \cos\frac{4\pi(n+2)}{7} - \cos\frac{6\pi(n+2)}{7}\right) +$$
$$+\frac{1072613}{4354560}.$$

Note that last expression is a quasi-polynomial.

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МНОГОЧЛЕНИ ГІЛЬБЕРТА АЛГЕБРИ ІНВАРІАНТІВ БІНАРНОЇ d-ФОРМИ

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Розглядаємо одну з фундаментальних проблем класичної конструктивної теорії інваріантів – дослідження функцій Гільберта алгебр інваріантів групи Лі SL_2. Її вигляд має важливу інформацію необхідну для обчислення мінімальних породжуючих систем цих алгебр. Відомо, що починаючи з деякого i функції Гільберта алгебр SL_n-інваріантів є квазімногочленами. Формула Келлі-Сільвестра для обчислень функцій Гільберта алгебри коваріантів бінарної $d-$форми $\mathcal{C}_d = \mathbb{C}[V_d \oplus \mathbb{C}^2]^{SL_2}$ (тут V_d- комплексний $d+1-$ вимірний векторний простір бінарних форм степеня d) була запропонована ще Сільвестром. Пізніше узагальнена для алгебри інваріантів бінарної $d-$форми $\mathcal{I}_d = \mathbb{C}[V_d]^{SL_2}$. Проте ці формули не виражають функції Гільберта як многочлен від i. Ми розглядаємо задачу обчислення в явній формі многочленів Гільберта алгебри інваріантів $\mathcal{I}_d$ бінарної $d-$форми.

Ключові слова: класична теорія інваріантів, інваріанти, функція Гільберта, многочлени Гільберта, ряди Пуанкаре.