MINIMAL SEMIGROUP TOPOLOGIES ON THE SEMIGROUP
OF MATRIX UNITS

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We describe minimal topologies in some class of semigroup topologies on
semigroups of matrix units.

Key words: topological semigroup, minimal semigroup topology, semigroups of matrix units

1. INTRODUCTION, MOTIVATION, AND MAIN DEFINITIONS

In this paper all topological spaces are assumed to be Hausdorff.

A topological semigroup is a topological space endowed with a continuous semigroup
operation. If S is a semigroup and τ is a topology on S such that (S, τ) is a topological
semigroup, then we shall call τ semigroup topology on S. A semitopological semigroup
is a topological space together with a separately continuous semigroup operation. A
topological semigroup (S, τ) is said to be minimal if no semigroup topology on S is
strictly contained in τ. If (S, τ) is a minimal topological semigroup, then τ is called a
minimal semigroup topology.

Let λ be a nonempty set. By $B_\lambda$ we denote the set $(\lambda \times \lambda) \cup \{0\}$ endowed
with the following semigroup operation:

$$(\alpha, \beta) \cdot (\gamma, \delta) = \begin{cases} 
(\alpha, \delta), & \text{if } \beta = \gamma; \\
0, & \text{if } \beta \neq \gamma;
\end{cases}$$

and $(\alpha, \beta) \cdot 0 = 0 \cdot (\alpha, \beta) = 0 \cdot 0 = 0$, for each $\alpha, \beta, \gamma, \delta \in \lambda$. The semigroup $B_\lambda$ is called the
semigroup of $\lambda \times \lambda$-matrix units. The semitopological semigroups of matrix units were
investigated in [3].

A directed graph (or just digraph) $D$ consists of a nonempty set $V(D)$ of elements
called vertices and a set $A(D)$ of ordered pairs of vertices called arcs. We call $V(D)$ the
vertex set and $A(D)$ the arc set of $D$. The order (resp. size) of $D$ is the cardinality of the
Lemma 1. Let $B_\lambda$ be a semitopological semigroup. If $A$ is a closed subset of $(B_\lambda, \tau)$ which does not contain the zero $0$, then any subset of $A$ is closed.

For any $A \subseteq B_\lambda$ and any $\alpha, \beta \in \lambda$ we denote

$$\alpha A_\beta = \{(\beta, \gamma) \mid (\alpha, \gamma) \in A\};$$
$$\beta A_\gamma = \{(\gamma, \beta) \mid (\gamma, \alpha) \in A\}.$$
Lemma 2. Let τ be a topology on $B_\lambda$ and any nonzero element of $B_\lambda$ is isolated in $(B_\lambda, \tau)$. The semigroup operation is continuous on $(B_\lambda \times B_\lambda) \setminus \{(0, 0)\}$ if and only if the sets $\alpha A_\beta$ and $\beta A_\gamma$ are closed for all $\alpha, \beta \in \lambda$ and any closed subset $A$ of $(B_\lambda, \tau)$ which does not contain the zero $0$.

Proof. ($\Rightarrow$) Let $A$ be a closed subset of $(B_\lambda, \tau)$ which does not contain the zero $0$. By the continuity of operation, the maps $\lambda_{(\alpha, \beta)} : B_\lambda \to B_\lambda$ and $\rho_{(\beta, \alpha)} : B_\lambda \to B_\lambda$ defined by the formulas $\lambda_{(\alpha, \beta)}(x) = (\alpha, \beta) \cdot x$ and $\rho_{(\beta, \alpha)}(x) = x \cdot (\beta, \alpha)$ for each $x \in B_\lambda$ are continuous. Therefore the sets

$$\alpha A_\beta = (\lambda_{(\alpha, \beta)})^{-1}(A) \quad \text{and} \quad \beta A_\gamma = (\rho_{(\beta, \alpha)})^{-1}(A)$$

are closed in the topological space $(B_\lambda, \tau)$.

($\Leftarrow$) Since every nonzero point of $B_\lambda$ is isolated, it suffices to check the continuity of the semigroup operation only in the cases of $(\alpha, \beta) \cdot 0$ and $0 \cdot (\alpha, \beta)$ for each $\alpha, \beta \in B_\lambda$.

If $U$ is any open neighborhood of the zero $0$ and $A = B_\lambda \setminus U$, then the sets $\alpha A_\beta$ and $\beta A_\gamma$ are closed. Denote by $V$ and $W$ the neighborhoods of zero $B_\lambda \setminus \alpha A_\beta$ and $B_\lambda \setminus \beta A_\gamma$, respectively. Simple calculations show that $\{(\alpha, \beta)\} \cdot V \subseteq U$ and $W \cdot \{(\alpha, \beta)\} \subseteq U$.

Corollary 1. Let τ be a topology on $B_\lambda$. The pair $(B_\lambda, \tau)$ is a semitopological semigroup if and only if the sets $\alpha A_\beta$ and $\beta A_\gamma$ are closed for all $\alpha, \beta \in \lambda$ and any closed subset $A$ of $(B_\lambda, \tau)$ such that $0 \notin A$.

For any nonempty subsets $A$ and $B$ of $\lambda$ we shall call $A \times B$ by a rectangle of $B_\lambda$.

Definition 1. A nonempty family $F$ of rectangles of $B_\lambda$ is called compositional if for any $A \times B \in F$ there exists $C \in \lambda \setminus \lambda$ such that $A \times (\lambda \setminus C) \in F$ and $C \times B \in F$.

Lemma 3. If $\tau$ is a semigroup topology on $B_\lambda$, then the family of all closed rectangles of $(B_\lambda, \tau)$ is compositional.

Proof. Since every point of $(B_\lambda, \tau)$ is a closed subset, the family of all closed rectangles of $B_\lambda$ is not empty. Let $A$ and $B$ be any subsets of $\lambda$ such that $A \times B$ is a closed rectangle of $B_\lambda$. Since $B_\lambda \setminus A \times B$ is an open neighborhood of $0$ and $0 \cdot 0 = 0$, the continuity of the semigroup operation in $(B_\lambda, \tau)$ implies that there exist open neighborhoods $U$ and $V$ of $0$ such that $U \cdot V \subseteq B_\lambda \setminus A \times B$. It follows that for each $\alpha \in A$ and $\beta \in B$ there is no $\gamma \in \lambda$ such that both $(\alpha, \gamma) \in U$ and $(\gamma, \beta) \in V$. Put

$$C = \{ \gamma \in \lambda \mid \exists \alpha \in A \text{ such that } (\alpha, \gamma) \in U \}.$$ 

Consider possible cases.

1. If $C$ is empty, then $\lambda \setminus C = \lambda$. The definition of the set $C$ implies that $U$ does not contain $A \times (\lambda \setminus C)$. Thus $A \times \lambda$ is a closed subset of the topological space $(B_\lambda, \tau)$. Fix any $\varphi \in \lambda$. By Lemma 1, the set $A \times (\lambda \setminus \{\varphi\})$ is closed. Since $\{\varphi\} \times B = \alpha (A \times B)_{\varphi}$, Lemma 2 implies that the set $\{\varphi\} \times B$ is closed, where $\alpha$ is an element of $A$. 


Let \( \alpha \) and \( c \) be open subset, the topological space \((B_{\lambda}, \tau)\). We conclude that the set \( A \times \varnothing \) is closed.

Proposition 1. For every compositional family \( F \) the topology generated by the subbase \( P_F \) is the smallest semigroup topology on \( B_{\lambda} \) such that elements of \( F \) are closed.

Proof. Let \( \tau \) be the topology generated by \( P_F \). Since each nonzero point in \( (B_{\lambda}, \tau) \) is closed subset, the topological space \( (B_{\lambda}, \tau) \) is Hausdorff.

First we shall show that the topology \( \tau \) is semigroup. For any \( A \in C \) the sets \( \alpha A \beta \) and \( \beta A \alpha \) are elements of \( C \).

\[
\begin{align*}
\alpha (\bigcup_{i \in I} A_i) & = \bigcup_{i \in I} \alpha (A_i), \\
\beta (\bigcup_{i \in I} A_i) & = \bigcup_{i \in I} \beta (A_i), \\
\alpha (\bigcap_{i \in I} A_i) & = \bigcap_{i \in I} \alpha (A_i), \\
\beta (\bigcap_{i \in I} A_i) & = \bigcap_{i \in I} \beta (A_i),
\end{align*}
\]

for arbitrary family \( \{A_i\}_{i \in I} \) subsets of \( B_{\lambda} \setminus \{0\} \), by Lemma 2 the semigroup operation is continuous on \((B_{\lambda} \times B_{\lambda}) \setminus \{(0,0)\}\).

The continuity of the operation in the point \((0,0)\) can be verify only for elements of the subbase. Let \( U \) be an open neighborhood of the zero \( 0 \) such that \( U \subseteq P_F \). Consider possible cases.

1. If \( B_{\lambda} \setminus U = A \times B \in F \), then there exists \( C \subseteq \lambda \) such that \( A \times (\lambda \setminus C) \) and \( C \times B \) are closed subsets of \( (B_{\lambda}, \tau) \). Thus \( B_{\lambda} \setminus (A \times (\lambda \setminus C)) \) and \( B_{\lambda} \setminus (C \times B) \) are open neighborhoods of the zero \( 0 \) and

\[
(B_{\lambda} \setminus (A \times (\lambda \setminus C))) \cdot (B_{\lambda} \setminus (C \times B)) \subseteq B_{\lambda} \setminus (A \times B) = U.
\]

2. If \( B_{\lambda} \setminus U = \alpha (A \times B) \beta = \{\beta\} \times B \) for some \( \alpha \in A, \beta \in \lambda \) and \( A \times B \in F \), then there exists \( C \subseteq \lambda \) such that the sets \( A \times (\lambda \setminus C) \) and \( C \times B \) are closed
Remark 2. For arbitrary sets \( A \subset \lambda \) and \( B \subset A \) the following compositional families \( \mathcal{F}_1 = \{ A \times (\lambda \setminus A) \} \) and \( \mathcal{F}_2 = \{ A \times (\lambda \setminus A), B \times (\lambda \setminus A) \} \) generate the same semigroup topology on \( B_\lambda \).

Let \( \tau \) be a semigroup topology on \( B_\lambda \) generated by some compositional family. By \( \text{Com}(\tau) \) denote the set of all compositional families such that generate the topology \( \tau \).
**Proposition 2.** Let $\tau_1$ and $\tau_2$ be semigroup topologies on $B_\lambda$ generated by compositional families. A topology $\tau_1$ is weaker than a topology $\tau_2$ if and only if there exist compositional families $F_1 \subseteq \text{Com}(\tau_1)$ and $F_2 \subseteq \text{Com}(\tau_2)$ such that $F_1 \subseteq F_2$.

**Proof.** ($\Rightarrow$) If $F_1 \subseteq \text{Com}(\tau_1)$ and $F \subseteq \text{Com}(\tau_2)$, then the family $F_2 = F \cup F$ is compositional and $F_1 \subseteq F_2$. Since the topology $\tau_1$ is weaker than the topology $\tau_2$, any element of $F_1$ is closed in $(B_\lambda, \tau_2)$. Therefore, the family $F_2$ generates the topology $\tau_2$.

($\Leftarrow$) It follows from $F_1 \subseteq F_2$ that every closed set in $(B_\lambda, \tau_1)$ is closed in $(B_\lambda, \tau_2)$. Hence the topology $\tau_1$ is weaker than the topology $\tau_2$.

**Lemma 4.** Let $\tau$ be a semigroup topology on $B_\lambda$. If the set $A \times B$ is closed in $(B_\lambda, \tau)$ and $C =^* A$, $D =^* B$, then the set $C \times D$ is closed in $(B_\lambda, \tau)$.

**Proof.** By Lemma 1, the set $(A \cap C) \times (B \cap D)$ is closed in $(B_\lambda, \tau)$. Lemma 2 implies that the sets $(A \cap C) \times \{A\}$ and $(B \cap D) \times \{B\}$ are closed for all $\alpha \in B \setminus C$, $\beta \in C \setminus A$. Since the sets $D \setminus B$ and $C \setminus A$ are finite, the sets $(A \cap C) \times (D \setminus B)$ and $(C \setminus A) \times (B \cap D)$ are closed. The set $(C \setminus A) \times (D \setminus B)$ is finite and therefore closed. Consequently, the set $C \times D$ is closed in $(B_\lambda, \tau)$ since $C \times D$ is the union of closed sets $(A \cap C) \times (B \cap D)$, $(A \cap C) \times (D \setminus B)$, $(C \setminus A) \times (B \cap D)$ and $(C \setminus A) \times (D \setminus B)$.

### 3. Compositional digraphs

A compositional family $\mathcal{F}$ can be represented in the form of a digraph with loops $D(\mathcal{F})$. The vertex set of the digraph is the set

$\{A \mid A \times (\lambda \setminus B) \in \mathcal{F} \text{ or } B \times (\lambda \setminus A) \in \mathcal{F}\}$

and $(A, B) \in A(D(\mathcal{F}))$ if and only if $A \times (\lambda \setminus B) \in \mathcal{F}$.

**Definition 2.** A digraph without isolated vertices $D$ is called compositional if for all $(u, v) \in A(D)$ there exists $w \in V(D)$ such that $(u, w) \in A(D)$ and $(w, v) \in A(D)$.

**Remark 3.** For any compositional family $\mathcal{F}$ the digraph $D(\mathcal{F})$ is compositional and any compositional digraph $D$ such that vertices of $D$ are subsets of $\lambda$ determines some compositional family.

**Example 1.** The following digraphs are compositional.
Example 2. Let $\mathbb{Z} \left[ \frac{1}{2} \right]$ be the set of all dyadic rationals in $[0,1]$. The digraph $U$ with $V(U) = \mathbb{Z} \left[ \frac{1}{2} \right]$ and
\[
A(U) = \left\{ (v,u) \mid v = \frac{k}{2^n} \text{ and } u = v + \frac{1}{2^m} \text{ for some } m \geq n \right\}
\]
is compositional. Indeed, if $(u,v)$ is an arc of $U$, then $(\frac{u+v}{2},v)$ and $(\frac{u+v}{2},v)$ are arcs of $U$.

For each $i \in \mathbb{N}$ by $U_i$ denoted the subdigraph of $U$ induced by
\[
V(U_i) = \{ u \in \mathbb{Z} \left[ \frac{1}{2} \right] \mid u = \frac{k}{2^n} \text{ for any } n \leq i \}.
\]

Proposition 3. There exists a Hamiltonian path in $U_i$ for each $i \in \mathbb{N}$.

Proof. Consider the sequence of vertices $W = \{ 0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^{i-1}}, 1 \}$. Observe that $v_{j+1} - v_j = \frac{1}{2^j}$ for arbitrary $v_j, v_{j+1} \in W$. Thus $(v_j, v_{j+1})$ is an arc of $U_i$ and therefore this sequence is a path. Since the path contains all vertices of $U_i$, $W$ is a Hamiltonian path.

Proposition 3 implies the following corollary.

Corollary 3. Arbitrary quotient digraph of $U$ has a finite cycle.

Proposition 4. Let $\{D_i\}_{i \in I}$ be a collection of compositional digraphs. Arbitrary quotient digraph of the digraph $D = \bigoplus_{i \in I} D_i$ is compositional.

Proof. Let $R$ be an equivalence relation on $V(D)$ and $([u],[v]) \in A(D/R)$. It follows that there exist $u', v' \in V(D)$ such that $(v', u') \in A(D)$. Thus $(v', u')$ is an arc of $D_j$ for some $j \in I$. Since digraph $D_j$ is compositional, there exist arcs $(u', w)$ and $(w, v')$ of $D_j$. Therefore $([u'], [w]), ([w], [v']) \in A(D/R)$. Hence the quotient digraph $D/R$ is compositional.

Lemma 5. Let $D$ be a compositional digraph and $(u, v)$ be an arc of $D$. There exists an equivalence relation $R$ on $V(D)$ such that $U/R$ is isomorphic to a subdigraph of $D$ which contains $(u, v)$.

Proof. For each $w \in V(D)$ define a set $C_w$ of vertices of $U$. First put that $0 \in C_u$ and $1 \in C_v$. If $x \in C_r$, $y \in C_t$ and $(r, t)$ is an arc of $D$, then $\frac{s+t}{2} \in C_s$, where $s$ is a vertex of $D$ such that $(r, s), (s, t) \in A(D)$. 

0
\[ \frac{1}{2} \]
\[ \frac{1}{4} \]
\[ \frac{1}{8} \]
\[
\frac{1}{16} \]
\[ 2 \]
\[ \frac{5}{2} \]
\[ \frac{5}{4} \]
\[ \frac{5}{8} \]
\[ \frac{5}{16} \]
\[ 3 \]
\[ \frac{7}{4} \]
\[ \frac{7}{8} \]
\[ \frac{7}{16} \]
\[ 4 \]
\[ \frac{9}{8} \]
\[ \frac{9}{16} \]
\[ 5 \]
\[ \frac{11}{16} \]
\[ 6 \]
\[ \frac{13}{16} \]
\[ 7 \]
\[ \frac{15}{16} \]
\[ 8 \]
Nonempty elements of \( \{C_w\}_{w \in V(D)} \) provide a partition of \( V(U) \) which determines an equivalence relation \( \mathcal{R} \) on \( V(U) \). Consider the map \( f : U/R \to D \) defined by the formula \( f(C_w) = w \) for any \( w \in V(D) \) such that \( C_w \neq \emptyset \). Since sets \( C_u \) and \( C_v \) is not empty, \( u,v \in f(U/R) \). Consequently, \( f(U/R) \) is a subdigraph of \( D \) which contains \( (u,v) \). Observe that \( f : U/R \to f(U/R) \) is bijective.

Let us show by induction that if \( (u,v) \) is an arc of \( U \) and \( u \in C_r, v \in C_s \), then \( (r,s) \) is an arc of \( D \). For \( U_0 \) this statement holds. Suppose that the statement holds for \( U_n \) and \( (u,v) \in A(U_n+1) \setminus A(U_n) \). If \( u \notin V(U_n+1) \), then \( 2u - v \in A(U_n) \). It follows that \( 2u - v \in C_t, v \in C_s \) and \( (r,s) \) is an arc of \( D \). The definition of \( C_t \) implies that \( (r,s) \in A(D) \) and \( u \in C_r \). If \( v \notin V(U_n+1) \), then \( u, 2v - u \in A(U_n) \). It follows that \( u \in C_s, v \in C_s \) and \( (r,s) \) is an arc of \( D \). Consequently, \( (r,s) \in A(D) \).

If \( (C_r,C_s) \) is an arc of \( U/R \), then there exist \( u \in C_r \) and \( v \in C_s \) such that \( (u,v) \) is an arc of \( U \). It follows that \( (r,s) \) is an arc of \( D \). The digraph \( H \) with vertex set \( f(U/R) \) and arc set

\[
\{(r,t) \in A(D) \mid (f^{-1}(r), f^{-1}(t)) \in A(U/R)\}
\]

is a subdigraph of the digraph \( D \). Hence \( f : U/R \to H \) is an isomorphism.

**Proposition 5.** Any compositional digraph \( D \) is isomorphic to quotient digraph of \( \bigoplus_{a \in A(D)} U^a \), where \( U^a \) is an isomorphic copy of \( U \).

**Proof.** By lemma 5 for any digraph \( U^a \) there exist an equivalence relation \( \mathcal{R}_a \) and an isomorphism \( f_a \) between \( U^a/\mathcal{R}_a \) and the subdigraph of \( D \) which contains the arc \( a \).

Define the equivalence relation \( \mathcal{R} \) on \( \bigoplus_{a \in A(D)} U^a \) in the following way:

\[
uRv \text{ if and only if } f_a([u]_{\mathcal{R}_a}) = f_c([v]_{\mathcal{R}_c})
\]

for all \( u \in V(U^a) \) and \( v \in V(U^c) \).

Now define the map \( f : \left( \bigoplus_{a \in A(D)} U^a \right) / \mathcal{R} \to D \) by the formula \( f([u]_{\mathcal{R}_a}) = f_a([u]_{\mathcal{R}_a}) \), where \( w \in V(U^a) \). Let us show that \( f \) is an isomorphism.

Since every vertex \( s \) of \( D \) is a head or a tail of some arc of \( D \), there exists \( w \in V(U^a) \) such that \( f_a([u]_{\mathcal{R}_a}) = s \). Thus \( f([v]_{\mathcal{R}_a}) = s \) and hence \( f \) is surjective. If \( f([u]_{\mathcal{R}_a}) = f([v]_{\mathcal{R}_a}) \), then \( f_a([u]_{\mathcal{R}_a}) = f_c([v]_{\mathcal{R}_c}) \), hence \( [u]_{\mathcal{R}_a} = [v]_{\mathcal{R}_c} \). Therefore \( uRv \) and hence \( [u]_{\mathcal{R}_a} = [v]_{\mathcal{R}_c} \). It follows that \( f \) is injective.

Let \( ([u]_{\mathcal{R}_a}, [v]_{\mathcal{R}_b}) \) be a vertex of \( \left( \bigoplus_{a \in A(D)} U^a \right) / \mathcal{R} \). It follows that there exist vertices \( r \in [u]_{\mathcal{R}_a} \) and \( t \in [v]_{\mathcal{R}_b} \) of \( \bigoplus_{a \in A(D)} U^a \) such that \( (r,t) \) is an arc of \( \bigoplus_{a \in A(D)} U^a \). Consequently, \( r, t \in V(U^a) \) for some \( a \in V(D) \) and therefore \( ([r]_{\mathcal{R}_a}, [t]_{\mathcal{R}_a}) \) is an arc of \( U_a/\mathcal{R}_a \). Since \( f_a \) is a isomorphism, \( (f_a([r]_{\mathcal{R}_a}), f_a([t]_{\mathcal{R}_a})) \) is an arc of \( D \). Hence \( (f([u]_{\mathcal{R}_a}), f([v]_{\mathcal{R}_b})) \) is an arc of \( D \).
Let \((f([u]_R), f([v]_R))\) be an arc of \(D\). It follows that \((f_a([u]_{R_a}), f_a([v]_{R_a}))\) is an arc of \(D\), where \(a = (f([u]_R), f([v]_R))\). Since \(f_a\) is an isomorphism, \((\lambda [u]_{R_a}, \lambda [v]_{R_a})\) is an arc of \(U_a/R_a\). Thus \((\lambda [u]_R, \lambda [v]_R)\) is an arc of \(U/R\) and hence \(f\) is an isomorphism.

Corollary 3 and Proposition 5 imply the following corollary.

**Corollary 4.** Any compositional digraph either has a finite cycle or contains an isomorphic copy of \(U\) as subdigraph.

**Corollary 5.** Any finite compositional digraph has a finite cycle.

4. **Main result**

The following proposition is a generalization of [4] Theorem 5.

**Proposition 6.** If \(A \subseteq \lambda\), then the semigroup topology \(\tau\) generated by the compositional family \(\{A \times (\lambda \setminus A)\}\) is minimal.

**Proof.** Assume that \(\tau_1\) is a weaker topology than the topology \(\tau\). Let \(B \times (\lambda \setminus C)\) be a closed set in \((B_\lambda, \tau_1)\). By Lemma 3 there exists \(D \subseteq \lambda\) such that the sets \(B \times (\lambda \setminus D)\) and \(D \times (\lambda \setminus C)\) are closed in \((B_\lambda, \tau_1)\). Since \(\tau_1\) is weaker than \(\tau\), \(D \subseteq^* \lambda\) and \((\lambda \setminus D) \subseteq^* (\lambda \setminus A)\).

Therefore \(D =^* A\). Applying Lemma 3 we conclude that there exists \(F \subseteq \lambda\) such that the sets \(D \times (\lambda \setminus F)\) and \(F \times (\lambda \setminus C)\) are closed in \((B_\lambda, \tau_1)\). Hence \(F \subseteq^* \lambda\) and \((\lambda \setminus F) \subseteq^* (\lambda \setminus A)\) and then \(F =^* A\). By Lemma 4 the set \(A \times (\lambda \setminus A)\) is closed in the topological space \((B_\lambda, \tau_1)\). The obtained contradiction implies that \(\tau\) is a minimal semigroup topology on \(B_\lambda\).

**Proposition 7.** Let \(F\) be a compositional family. If there exists a finite subdigraph \(H\) of the digraph \(D(F)\) which does not contain sink or source, then there exists \(A \subseteq \lambda\) such that the set \(A \times (\lambda \setminus A)\) is closed in the topological space \((B_\lambda, \tau_F)\).

**Proof.** Let \(V(D) = \{A_1, \ldots, A_n\}\) and \(H\) does not contain sink. For each \(A_i \in V(H)\) there exist \(A_j \in V(H)\) such that \(A_i \times (\lambda \setminus A_j)\). Observe that

\[
A \setminus \big(\Lambda \cup \cdots \cup \Lambda_n\big) = \big(\Lambda \setminus \Lambda_1\big) \cap \cdots \cap \big(\Lambda \setminus \Lambda_n\big).
\]

If \((\Lambda \setminus \Lambda_1) \cap \cdots \cap (\Lambda \setminus \Lambda_n) \neq \emptyset\), then \(\Lambda_1 \cup \cdots \cup \Lambda_n = \lambda\) and hence, by Lemmas 1 and 2 the set \((\Lambda \setminus \Lambda) \times (\Lambda)\) is closed in the topological space \((B_\lambda, \tau_F)\) for each \(\alpha \in \lambda\). Let \((\Lambda \setminus \Lambda_1) \cap \cdots \cap (\Lambda \setminus \Lambda_n) \neq \emptyset\). By Lemma 4 the set \(\Lambda \times ((\Lambda \setminus \Lambda_1) \cap \cdots \cap (\Lambda \setminus \Lambda_n))\) is closed for any \(A_i \in V(H)\). Hence

\[
A_1 \cup \cdots \cup A_n \times ((\Lambda \setminus \Lambda_1) \cap \cdots \cap (\Lambda \setminus \Lambda_n)) = \bigcup_{i=1}^{n} A_i \times \Lambda \setminus \bigcup_{i=1}^{n} A_i
\]

is closed in the topological space \((B_\lambda, \tau_F)\). The case with source is proved similarly.

Propositions 6 and 7 imply the following corollary.

**Corollary 6.** If the digraph \(D(F)\) has a finite cycle, then \(F\) is a singleton or generates a nonminimal topology.

Corollaries 3 and 6 imply the following theorem.
Theorem 1. Let $\tau$ be a semigroup topology on $B_\lambda$ generated by a compositional family $F$ such that $D(F)$ does not contain subdigraph isomorphic to $U$. The topology $\tau$ is minimal if and only if $\tau$ is generated by a singleton compositional family.

Problem 1. Is there a minimal semigroup topology on $B_\lambda$ generated by a compositional family $F$ such that $D(F)$ is isomorphic to $U$?

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МІНІМАЛЬНІ НАПІВГРУПОВІ ТОПОЛОГІЇ НА НАПІВГРУПІ МАТРИЧНИХ ОДИНЦІВ

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Ми описали мінімальні топології в кожному класі напівгрупових топологій на напівгрупі матричних одиниць.

Ключові слова: топологічна напівгрупа, мінімальна напівгрупова топологія, напівгрупа матричних одиниць.