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MINIMAL SEMIGROUP TOPOLOGIES ON THE SEMIGROUP OF MATRIX UNITS

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We describe minimal topologies in some class of semigroup topologies on semigroups of matrix units.

Key words: topological semigroup, minimal semigroup topology, semigroups of matrix units

1. Introduction, motivation, and main definitions

In this paper all topological spaces are assumed to be Hausdorff.

A *topological semigroup* is a topological space endowed with a continuous semigroup operation. If S is a semigroup and τ is a topology on S such that (S,τ) is a topological semigroup, then we shall call τ *semigroup topology* on S. A *semitopological semigroup* is a topological space together with a separately continuous semigroup operation. A topological semigroup (S,τ) is said to be *minimal* if no semigroup topology on S is strictly contained in τ. If (S,τ) is a minimal topological semigroup, then τ is called a *minimal semigroup topology.*

Let λ be a nonempty set. By B_λ we denote the set $(\lambda\times\lambda)\cup\{0\}$ endowed with the following semigroup operation:

$$(\alpha,\beta)\cdot(\gamma,\delta)=\begin{cases}(\alpha,\delta), & \text{if } \beta=\gamma;\\ 0, & \text{if } \beta\neq\gamma;\end{cases}$$

and $(\alpha,\beta)\cdot 0=0\cdot(\alpha,\beta)=0\cdot 0=0$, for each $\alpha,\beta,\gamma,\delta\in\lambda$. The semigroup B_λ is called the *semigroup of $\lambda\times\lambda$-matrix units.* The semitopological semigroups of matrix units were investigated in [4].

A *directed graph* (or just *digraph*) D consists of a nonempty set $V(D)$ of elements called *vertices* and a set $A(D)$ of ordered pairs of vertices called *arcs*. We call $V(D)$ the *vertex set* and $A(D)$ the *arc set* of D. The *order* (resp. *size*) of D is the cardinality of the

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2020 Mathematics Subject Classification: 22A15 vertex (resp. arc) set of D. For any arc (u, v) the first vertex u is its *tail* and the second vertex v is its *head*. The head and tail of an arc are its *end-vertices*. If a tail and a head of arc coincide, then this arc is called a *loop*. A vertex v of D is a *source* (resp. *sink*) if v is not a head(resp. tail) of any arc of D. A vertex v of D is *isolated* if v is not a head and tail of any arc of D. A digraph H is a *subdigraph* of a digraph D if $V(H) \subseteq V(D)$, $A(H) \subseteq A(D)$. If every arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say that H is induced by $V(H)$ and call H an induced subdigraph of D.

A *walk* in a digraph D is an alternating sequence

$$W = x_1 a_1 x_2 a_2 x_3 \dots x_{k-1} a_{k-1} x_k$$

of vertices x_i and arcs a_j of D such that $a_i = (x_i, x_{i+1})$ is x_i for every $i = 1, 2, \dots, k-1$. The *length* of a walk is the number of its arcs. When the arcs of W are defined from the context or simply unimportant, we will denote W by $x_1 x_2 \dots x_k$. If the vertices of W are distinct, W is a *path*. If the vertices $x_1, x_2, \dots, x_{k-1}$ are distinct and $x_1 = x_k$, W is a *cycle*. A walk (path, cycle) W is a *Hamilton* (or *Hamiltonian*) walk (path, cycle) if W contains all vertices of D.

Let $\{D_i\}_{i\in I}$ be a family of digraphs. The digraph $\left(\bigsqcup\limits_{i\in I} V(D_i), \bigsqcup\limits_{i\in I} A(D_i)\right)$ is called the *disjoint union* of this family and is denoted by $\bigoplus\limits_{i\in I} D_i$. If D is a digraph and $\mathcal{R}$ is an equivalence relation on $V(D)$, then the *quotient digraph* $D/\mathcal{R}$ has the vertex set $V/\mathcal{R}$ and the arc set $\{([a]_{\mathcal{R}}, [b]_{\mathcal{R}}) \mid (a, b) \in A(D)\}$.

The concept of minimal topological groups was introduced independently in the early 1970's by Doitchinov [3] and Stephenson [7]. Both authors were motivated by the theory of minimal topological spaces, which was well understood at that time. More than 20 years earlier L. Nachbin [6] had studied minimality in the context of division rings, and B. Banaschewski [1] investigated minimality in the more general setting of topological algebras. Two minimal semigroups topologies on topological semigroups of matrix units were described by Gutik and Pavlyk in [4]. The lattice of weak topologies was investigated in [2].

2. Compositional families

If (B_λ, τ) is a semitopological semigroup, then any nonzero element of B_λ is an isolated point of (B_λ, τ), see [4, Lemma 2]. It follows that if λ is infinite, then the topological space (B_λ, τ) is discrete. Later λ is assumed to be infinite. Also the following lemma holds.

Lemma 1. *Let (B_λ, τ) be a semitopological semigroup. If A is a closed subset of (B_λ, τ) which does not contain the zero 0, then any subset of A is closed.*

For any $A \subseteq B_\lambda$ and any $\alpha, \beta \in \lambda$ we denote

$$ {}_{\alpha}A_{\beta} = \{(\beta, \gamma) \mid (\alpha, \gamma) \in A\}; $$
$$ {}^{\alpha}_{\beta}A = \{(\gamma, \beta) \mid (\gamma, \alpha) \in A\}. $$

Lemma 2. *Let τ be a topology on B_λ and any nonzero element of B_λ is isolated in (B_λ,τ). The semigroup operation is continuous on $(B_\lambda\times B_\lambda)\setminus\{(0,0)\}$ if and only if the sets ${}_\alpha A_\beta$ and ${}^\alpha_\beta A$ are closed for all $\alpha,\beta\in\lambda$ and any closed subset A of (B_λ,τ) which does not contain the zero* 0.

Proof. ($\Rightarrow$) Let A be a closed subset of (B_λ,τ) which does not contain the zero 0. By the continuity of operation, the maps $\lambda_{(\alpha,\beta)}\colon B_\lambda\to B_\lambda$ and $\rho_{(\beta,\alpha)}\colon B_\lambda\to B_\lambda$ defined by the formulas $\lambda_{(\alpha,\beta)}(x)=(\alpha,\beta)\cdot x$ and $\rho_{(\beta,\alpha)}(x)=x\cdot(\beta,\alpha)$ for each $x\in B_\lambda$, are continuous. Therefore the sets

$$ {}_\alpha A_\beta=(\lambda_{(\alpha,\beta)})^{-1}(A)$$

and

$$ {}^\alpha_\beta A=(\rho_{(\beta,\alpha)})^{-1}(A)$$

are closed in the topological space (B_λ,τ).

($\Leftarrow$) Since every nonzero point of B_λ is isolated, it suffices to we check the continuity of the semigroup operation only in the cases of $(\alpha,\beta)\cdot 0$ and $0\cdot(\alpha,\beta)$ for each $\alpha,\beta\in B_\lambda$. If U is any open neighborhood of the zero 0 and $A=B_\lambda\setminus U$, then the sets ${}_\alpha A_\beta$ and ${}^\alpha_\beta A$ are closed. Denote by V and W the neighborhoods of zero $B_\lambda\setminus{}_\alpha A_\beta$ and $B_\lambda\setminus{}^\alpha_\beta A$, respectively. Simple calculations show that $\{(\alpha,\beta)\}\cdot V\subseteq U$ and $W\cdot\{(\alpha,\beta)\}\subseteq U$.

Corollary 1. *Let τ be a topology on B_λ. The pair (B_λ,τ) is a semitopological semigroup if and only if the sets ${}_\alpha A_\beta$ and ${}^\alpha_\beta A$ are closed for all $\alpha,\beta\in\lambda$ and any closed subset A of (B_λ,τ) such that $0\notin A$.*

For any nonempty subsets A and B of λ we shall call $A\times B$ by a *rectangle* of B_λ.

Definition 1. A nonempty family $\mathcal{F}$ of rectangles of B_λ is called compositional if for any $A\times B\in\mathcal{F}$ there exists $C\subset\lambda$ such that $A\times(\lambda\setminus C)\in\mathcal{F}$ and $C\times B\in\mathcal{F}$.

Lemma 3. *If τ is a semigroup topology on B_λ, then the family of all closed rectangles of (B_λ,τ) is compositional.*

Proof. Since every point of (B_λ,τ) is a closed subset, the family of all closed rectangles of B_λ is not empty. Let A and B be any subsets of λ such that $A\times B$ is a closed rectangle of B_λ. Since $B_\lambda\setminus A\times B$ is an open neighborhood of 0 and $0\cdot 0=0$, the continuity of the semigroup operation in (B_λ,τ) implies that there exist open neighborhoods U and V of 0 such that $U\cdot V\subseteq B_\lambda\setminus A\times B$. It follows that for each $\alpha\in A$ and $\beta\in B$ there is no $\gamma\in\lambda$ such that both $(\alpha,\gamma)\in U$ and $(\gamma,\beta)\in V$. Put

$$C=\{\gamma\in\lambda\mid \exists\alpha\in A \text{ such that } (\alpha,\gamma)\in U\}.$$

Consider possible cases.

(1) If C is empty, then $\lambda\setminus C=\lambda$. The definition of the set C implies that U does not contain $A\times(\lambda\setminus C)$. Thus $A\times\lambda$ is a closed subset of the topological space (B_λ,τ). Fix any $\varphi\in\lambda$. By Lemma 1, the set $A\times(\lambda\setminus\{\varphi\})$ is closed. Since $\{\varphi\}\times B={}_\alpha(A\times B)_\varphi$, Lemma 2 implies that the set $\{\varphi\}\times B$ is closed, where α is an element of A.

(2) If $C=\lambda$, then the set $\lambda\setminus C$ is empty. The definition of the set C implies that V does not contain $C\times B$. Thus $\lambda\times B$ is a closed subset of the topological space (B_λ,τ). Fix any $\varphi\in\lambda$. Applying 1, we conclude that the set $(\lambda\setminus\{\varphi\})\times B$ is closed. Since $A\times\{\varphi\}={}^{\beta}_{\varphi}(A\times B)$, Lemma 2 implies that the set $A\times\{\varphi\}$ is closed, where β is an element of B.

(3) Suppose that C is not empty and does not equals λ. It follows $A\times(\lambda\setminus C)\subset B_\lambda\setminus U$ and $C\times B\subset B_\lambda\setminus V$, so by Lemma 1, $A\times(\lambda\setminus C)$ and $C\times B$ are closed rectangles of B_λ.

Hence the family of all closed rectangles of (B_λ,τ) is compositional.

Let $\mathcal{F}$ be a compositional family. Denote

$$\mathcal{C}=\mathcal{F}\cup\{{}_{\alpha}A_{\beta},{}^{\alpha}_{\beta}A \mid A\in\mathcal{F}\text{ and }\alpha,\beta\in\lambda\}\cup\{\{(\alpha,\beta)\}\mid\alpha,\beta\in\lambda\}$$

and

$$P_{\mathcal{F}}=\{B_\lambda\setminus B\mid B\in\mathcal{C}\}\cup\{\{(\alpha,\beta)\}\mid\alpha,\beta\in\lambda\}\cup\{\varnothing\}.$$

Proposition 1. *For every compositional family $\mathcal{F}$ the topology generated by the subbase $P_{\mathcal{F}}$ is the smallest semigroup topology on B_λ such that elements of $\mathcal{F}$ are closed.*

Proof. Let τ be the topology generated by $P_{\mathcal{F}}$. Since each nonzero point in (B_λ,τ) is clopen subset, the topological space (B_λ,τ) is Hausdorff.

First we shall show that the topology τ is semigroup. For any $A\in\mathcal{C}$ the sets ${}_{\alpha}A_{\beta}$ and ${}^{\alpha}_{\beta}A$ are elements of $\mathcal{C}$. Since

$$\begin{aligned}
{}_{\alpha}\Big(\bigcup_{i\in I}A_i\Big)_{\beta}&=\bigcup_{\iota\in I}{}_{\alpha}(A_i)_{\beta},\\
{}^{\alpha}_{\beta}\Big(\bigcup_{i\in I}A_i\Big)&=\bigcup_{\iota\in I}{}^{\alpha}_{\beta}(A_i),\\
{}_{\alpha}\Big(\bigcap_{i\in I}A_i\Big)_{\beta}&=\bigcap_{\iota\in I}{}_{\alpha}(A_i)_{\beta},\\
{}^{\alpha}_{\beta}\Big(\bigcap_{i\in I}A_i\Big)&=\bigcap_{\iota\in I}{}^{\alpha}_{\beta}(A_i),
\end{aligned}$$

for arbitrary family $\{A_i\}_{i\in I}$ subsets of $B_\lambda\setminus\{0\}$, by Lemma 2, the semigroup operation is continuous on $(B_\lambda\times B_\lambda)\setminus\{(0,0)\}$.

The continuity of the operation in the point $(0,0)$ can be verify only for elements of the subbase. Let U be an open neighborhood of the zero 0 such that $U\in P_{\mathcal{F}}$. Consider possible cases.

(1) If $B_\lambda\setminus U=A\times B\in\mathcal{F}$, then there exists $C\subset\lambda$ such that $A\times(\lambda\setminus C)$ and $C\times B$ are closed subsets of (B_λ,τ). Thus $B_\lambda\setminus(A\times(\lambda\setminus C))$ and $B_\lambda\setminus(C\times B)$ are open neighborhoods of the zero 0 and

$$(B_\lambda\setminus(A\times(\lambda\setminus C)))\cdot(B_\lambda\setminus(C\times B))\subseteq B_\lambda\setminus(A\times B)=U.$$

(2) If $B_\lambda\setminus U={}_{\alpha}(A\times B)_{\beta}=\{\beta\}\times B$ for some $\alpha\in A,\beta\in\lambda$ and $A\times B\in\mathcal{F}$, then there exists $C\subset\lambda$ such that the sets $A\times(\lambda\setminus C)$ and $C\times B$ are closed

in (B_λ,τ). The set ${}_\alpha(A\times(\lambda\setminus C))_\beta=\{\beta\}\times(\lambda\setminus C)$ is closed in (B_λ,τ). Hence $B_\lambda\setminus(\{\beta\}\times(\lambda\setminus C))$ and $B_\lambda\setminus(C\times B)$ are open neighborhoods of the zero 0 and

$$(B_\lambda\setminus(\{\beta\}\times(\lambda\setminus C)))\cdot(B_\lambda\setminus(C\times B))\subseteq B_\lambda\setminus(\{\beta\}\times B)=U.$$

(3) In the case $B_\lambda\setminus U={}^\alpha_\beta(A\times B)=A\times\{\beta\}$ the proof of the continuity of the semigroup operation is similar to the proof of the case (2).

(4) Suppose that $B_\lambda\setminus U=\{(\alpha,\beta)\}$ for some $(\alpha,\beta)\in B_\lambda\setminus\{0\}$. Since family $\mathcal{C}$ is not empty, there exists $A\times B\in\mathcal{C}$. It follows that there exists $C\subset\lambda$ such that the sets $A\times(\lambda\setminus C)$ and $C\times B$ are closed in (B_λ,τ). Fix any $\varphi\in A$ and $\psi\in B$. Consequently, the sets ${}_\varphi(A\times(\lambda\setminus C))_\alpha$ and ${}^\psi_\beta(C\times B)$ are closed in (B_λ,τ). Thus $B_\lambda\setminus{}_\varphi(A\times(\lambda\setminus C))_\alpha$ and $B_\lambda\setminus{}^\psi_\beta(C\times B)$ are open neighborhoods of the zero 0 and

$$(B_\lambda\setminus{}_\varphi(A\times(\lambda\setminus C))_\alpha)\cdot(B_\lambda\setminus{}^\psi_\beta(C\times B))\subseteq B_\lambda\setminus\{(\alpha,\beta)\}=U.$$

Consequently, the semigroup operation is continuous.

Let τ_1 be a semigroup topology on B_λ such that elements of $\mathcal{F}$ are closed in (B_λ,τ_1). It follows that all nonzero points are closed. By Lemma 2, elements of

$$\{{}_\alpha(A\times B)_\beta,{}^\alpha_\beta(A\times B)\mid A\in\mathcal{F}\quad\text{and}\quad\alpha,\beta\in\lambda\}$$

are closed in (B_λ,τ_1) and hence their complements are open. Since every nonzero point of (B_λ,τ_1) is isolated, elements of $P_\mathcal{F}$ are open in the topological space (B_λ,τ_1). Therefore the topology generated by the subbase $P_\mathcal{F}$ is the smallest semigroup topology on B_λ such that elements of $\mathcal{F}$ are closed.

The topology generated by the subbase $P_\mathcal{F}$ is called *the topology generated by the compositional family* $\mathcal{F}$ and denoted by $\tau_\mathcal{F}$.

Proposition 1 and Lemma 3 imply the following corollary.

Corollary 2. *Every minimal semigroup topology on B_λ is generated by some compositional family.*

For arbitrary sets A and B we denote

$$A\subseteq^* B \text{ if a set } A\setminus B \text{ is finite;}$$
$$A=^* B \text{ if a set } A\triangle B \text{ is finite.}$$

Remark 1. Note that there are semigroup topologies on B_λ such that not generated by compositional families. In particular, a topology generated by the base

$$\mathcal{B}=\{\{(\alpha,\alpha)\mid\alpha\in A\}\cup\{0\}\mid A=^*\lambda\}\cup\{\{(\alpha,\beta)\}\mid\alpha,\beta\in\lambda\}.$$

Remark 2. Observe that a semigroup topology on B_λ can be generated by distinct compositional families. For example, for arbitrary sets $A\subset\lambda$ and $B\subset A$ the following compositional families $\mathcal{F}_1=\{A\times(\lambda\setminus A)\}$ and $\mathcal{F}_2=\{A\times(\lambda\setminus A),B\times(\lambda\setminus A)\}$ generate the same semigroup topology on B_λ.

Let τ be a semigroup topology on B_λ generated by some compositional family. By $\mathrm{Com}(\tau)$ denote the set of all compositional families such that generate the topology τ.

Proposition 2. *Let τ_1 and τ_2 be semigroup topologies on B_λ generated by compositional families. A topology τ_1 is weaker than a topology τ_2 if and only if there exist compositional families $\mathcal{F}_1 \in \mathrm{Com}(\tau_1)$ and $\mathcal{F}_2 \in \mathrm{Com}(\tau_2)$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$.*

Proof. ($\Rightarrow$) If $\mathcal{F}_1 \in \mathrm{Com}(\tau_1)$ and $\mathcal{F} \in \mathrm{Com}(\tau_2)$, then the family $\mathcal{F}_2 = \mathcal{F}_1 \cup \mathcal{F}$ is compositional and $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Since the topology τ_1 is weaker than the topology τ_2, any element of $\mathcal{F}_1$ is a closed set in (B_λ, τ_2). Therefore, the family $\mathcal{F}_2$ generate the topology τ_2.

($\Leftarrow$) It follows from $\mathcal{F}_1 \subseteq \mathcal{F}_2$ that every closed set in (B_λ, τ_1) is closed in (B_λ, τ_2). Hence the topology τ_1 is weaker than the topology τ_2.

Lemma 4. *Let τ be a semigroup topology on B_λ. If the set $A \times B$ is closed in (B_λ, τ) and $C =^* A$, $D =^* B$, then the set $C \times D$ is closed in (B_λ, τ).*

Proof. By Lemma 1, the set $(A \cap C) \times (B \cap D)$ is closed in (B_λ, τ). Lemma 2 implies that the sets $(A \cap C) \times \{\alpha\}$ and $\{\beta\} \times (B \cap D)$ are closed for all $\alpha \in D \setminus B, \beta \in C \setminus A$. Since the sets $D \setminus B$ and $C \setminus A$ are finite, the sets $(A \cap C) \times (D \setminus B)$ and $(C \setminus A) \times (B \cap D)$ are closed. The set $(C \setminus A) \times (D \setminus B)$ is finite and therefore closed. Consequently, the set $C \times D$ is closed in (B_λ, τ) since $C \times D$ is an union of closed sets $(A \cap C) \times (B \cap D)$, $(A \cap C) \times (D \setminus B)$, $(C \setminus A) \times (B \cap D)$ and $(C \setminus A) \times (D \setminus B)$.

3. Compositional digraphs

A compositional family $\mathcal{F}$ can be represented in the form of a digraph with loops $D(\mathcal{F})$. The vertex set of the digraph is the set

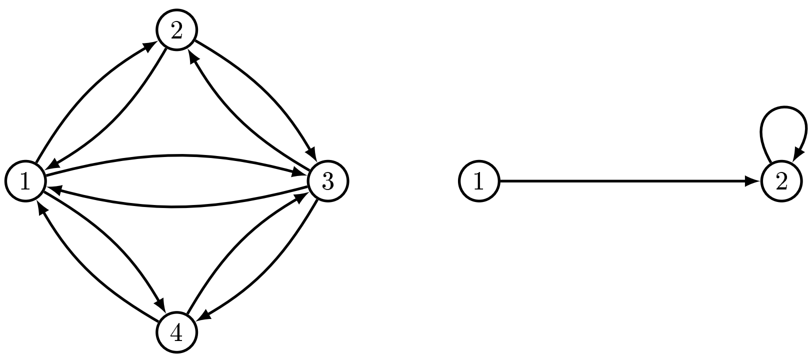
$$\{A \mid A \times (\lambda \setminus B) \in \mathcal{F} \text{ or } B \times (\lambda \setminus A) \in \mathcal{F}\}$$

and $(A, B) \in A(D(\mathcal{F}))$ if and only if $A \times (\lambda \setminus B) \in \mathcal{F}$.

Definition 2. A digraph without isolated vertices D is called *compositional* if for all $(u, v) \in A(D)$ there exists $w \in V(D)$ such that $(u, w) \in A(D)$ and $(w, v) \in A(D)$.

Remark 3. For any compositional family $\mathcal{F}$ the digraph $D(\mathcal{F})$ is compositional and any compositional digraph D such that vertices of D are subsets of λ determines some compositional family.

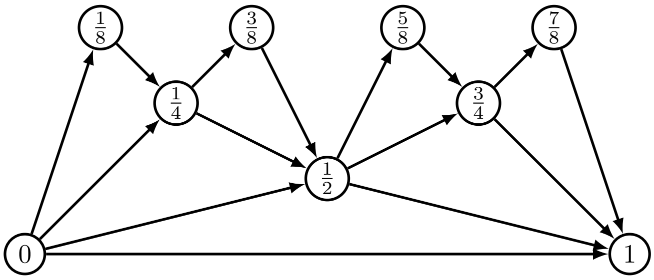
Example 1. The following digraphs are compositional.



Example 2. Let $\mathbb{Z}[\frac{1}{2}]$ be the set of all dyadic rationals in $[0,1]$. The digraph $\mathcal{U}$ with $V(\mathcal{U}) = \mathbb{Z}[\frac{1}{2}]$ and

$$A(\mathcal{U}) = \left\{(v,u) \mid v = \frac{k}{2^n} \text{ and } u = v + \frac{1}{2^m} \text{ for some } m \geqslant n\right\}$$

is compositional. Indeed, if (u,v) is an arc of $\mathcal{U}$, then $(u, \frac{u+v}{2})$ and $(\frac{u+v}{2}, v)$ are arcs of $\mathcal{U}$.



For each $i \in \mathbb{N}$ by $\mathcal{U}_i$ denoted the subdigraph of $\mathcal{U}$ induced by

$$V(\mathcal{U}_i) = \left\{u \in \mathbb{Z}[\tfrac{1}{2}] \mid u = \tfrac{k}{2^n} \text{ for any } n \leqslant i\right\}.$$

Proposition 3. *There exists a Hamiltonian path in $\mathcal{U}_i$ for each $i \in \mathbb{N}$.*

Proof. Consider the sequence of vertices $W = \left\{0, \frac{1}{2^i}, \frac{2}{2^i}, \frac{3}{2^i}, \ldots, \frac{2^i-1}{2^i}, 1\right\}$. Observe that $v_{j+1} - v_j = \frac{1}{2^i}$ for arbitrary $v_j, v_{j+1} \in W$. Thus (v_j, v_{j+1}) is an arc of $\mathcal{U}_i$ and therefore this sequence is a path. Since the path contains all vertices of $\mathcal{U}_i$, W is a Hamiltonian path.

Proposition 3 implies the following corollary.

Corollary 3. *Arbitrary quotient digraph of $\mathcal{U}$ has a finite cycle.*

Proposition 4. *Let $\{D_i\}_{i\in I}$ be a collection of compositional digraphs. Arbitrary quotient digraph of the digraph $D = \bigoplus_{i\in I} D_i$ is compositional.*

Proof. Let $\mathcal{R}$ be an equivalence relation on $V(D)$ and $([u],[v]) \in A(D/\mathcal{R})$. It follows that there exist $u', v' \in V(D)$ such that $(v', u') \in A(D)$. Thus (v', u') is an arc of D_j for some $j \in I$. Since digraph D_j is compositional, there exist arcs (u', w) and (w, v') of D_j. Therefore $([u'],[w]), ([w],[v']) \in A(D/\mathcal{R})$. Hence the quotient digraph $D/\mathcal{R}$ is compositional.

Lemma 5. *Let D be a compositional digraph and (u,v) be an arc of D. There exists an equivalence relation $\mathcal{R}$ on $V(\mathcal{U})$ such that $\mathcal{U}/\mathcal{R}$ is isomorphic to a subdigraph of D which contains (u,v).*

Proof. For each $w \in V(D)$ define a set C_w of vertices of $\mathcal{U}$. First put that $0 \in C_u$ and $1 \in C_v$. If $x \in C_r$, $y \in C_t$ and (r,t) is an arc of D, then $\frac{x+y}{2} \in C_s$, where s is a vertex of D such that $(r,s), (s,t) \in A(D)$.

Nonempty elements of $\{C_w\}_{w\in V(D)}$ provide a partition of $V(\mathcal{U})$ which determines an equivalence relation $\mathcal{R}$ on $V(\mathcal{U})$. Consider the map $f : \mathcal{U}/\mathcal{R} \to D$ defined by the formula $f(C_w) = w$ for any $w \in V(D)$ such that $C_w \neq \varnothing$. Since sets C_u and C_v is not empty, $u, v \in f(\mathcal{U}/\mathcal{R})$. Consequently, $f(\mathcal{U}/\mathcal{R})$ is a subdigraph of D which contains (u, v). Observe that $f : \mathcal{U}/\mathcal{R} \to f(\mathcal{U}/\mathcal{R})$ is bijective.

Let us show by induction that if (u, v) is an arc of $\mathcal{U}$ and $u \in C_r$, $v \in C_s$, then (r, s) is an arc of D. For $\mathcal{U}_0$ this statement holds. Suppose that the statement holds for $\mathcal{U}_n$ and $(u, v) \in A(\mathcal{U}_{n+1}) \setminus A(\mathcal{U}_n)$. If $u \notin V(\mathcal{U}_{n+1})$, then $(2u - v, v) \in A(\mathcal{U}_n)$. It follows that $2u - v \in C_t$, $v \in C_s$ and (t, s) is an arc of D. The definition of C_i implies that $(r, s) \in A(D)$ and $u \in C_r$. If $v \notin V(\mathcal{U}_{n+1})$, then $(u, 2v - u) \in A(\mathcal{U}_n)$. It follows that $u \in C_t$, $v2v - u \in C_s$ and (t, s) is an arc of D. Consequently, $(t, r) \in V(D)$ and $v \in C_r$.

If (C_r, C_t) is an arc of $\mathcal{U}/\mathcal{R}$, then there exist $u \in C_r$ and $v \in C_t$ such that (u, v) is an arc of $\mathcal{U}$. It follows that (r, t) is an arc of D. The digraph H with vertex set $f(\mathcal{U}/\mathcal{R})$ and arc set

$$\{(r, t) \in A(D) \mid (f^{-1}(r), f^{-1}(t)) \in A(\mathcal{U}/\mathcal{R})\}$$

is a subdigraph of the digraph D. Hence $f : \mathcal{U}/\mathcal{R} \to H$ is an isomorphism.

Proposition 5. *Any compositional digraph D is isomorphic to quotient digraph of $\bigoplus_{a\in A(D)} \mathcal{U}^a$, where $\mathcal{U}^a$ is an isomorphic copy of $\mathcal{U}$.*

Proof. By lemma 5, for any digraph $\mathcal{U}^a$ there exist an equivalence relation $\mathcal{R}_a$ and an isomorphism f_a between $\mathcal{U}^a/\mathcal{R}_a$ and the subdigraph of D which contains the arc a.

Define the equivalence relation $\mathcal{R}$ on $\bigoplus_{a\in A(D)} \mathcal{U}^a$ in the following way:

$$u\mathcal{R}v \text{ if and only if } f_b([u]_{\mathcal{R}_b}) = f_c([v]_{\mathcal{R}_c})$$

for all $u \in V(\mathcal{U}^b)$ and $v \in V(\mathcal{U}^c)$.

Now define the map $f : \left(\bigoplus_{a\in A(D)} \mathcal{U}^a\right)/\mathcal{R} \to D$ by the formula $f([w]_{\mathcal{R}}) = f_b([w]_{\mathcal{R}_b})$, where $w \in V(\mathcal{U}^b)$. Let us show that f is an isomorphism.

Since every vertex s of D is a head or a tail of some arc b of D, there exists $w \in V(\mathcal{U}^b)$ such that $f_b([w]_{\mathcal{R}_b}) = s$. Thus $f([w]_{\mathcal{R}}) = s$ and hence f is surjective. If $f([u]_{\mathcal{R}}) = f([v]_{\mathcal{R}})$, then $f_b([u]_{\mathcal{R}_b}) = f_c([v]_{\mathcal{R}_c})$, where $u \in V(\mathcal{U}^b)$ and $v \in V(\mathcal{U}^c)$. Therefore $u\mathcal{R}v$ and hence $[u]_{\mathcal{R}} = [v]_{\mathcal{R}}$. It follows that f is injective.

Let $([u]_{\mathcal{R}}, [v]_{\mathcal{R}})$ be a vertex of $\left(\bigoplus_{a\in A(D)} \mathcal{U}^a\right)/\mathcal{R}$. It follows that there exist vertices $r \in [u]_{\mathcal{R}}$ and $t \in [v]_{\mathcal{R}}$ of $\bigoplus_{a\in A(D)} \mathcal{U}^a$ such that (r, t) is an arc of $\bigoplus_{a\in A(D)} \mathcal{U}^a$. Consequently, $r, t \in V(\mathcal{U}^a)$ for some $a \in V(D)$ and therefore $([r]_{\mathcal{R}_a}, [t]_{\mathcal{R}_a})$ is an arc of $\mathcal{U}_a/\mathcal{R}_a$. Since f_a is a isomorphism, $(f_a([r]_{\mathcal{R}_a}), f_a([t]_{\mathcal{R}_a}))$ is an arc of D. Hence $(f([u]_{\mathcal{R}}), f([v]_{\mathcal{R}}))$ is an arc of D.

Let $(f([u]_{\mathcal{R}}), f([v]_{\mathcal{R}}))$ be an arc of D. It follows that $(f_a([u]_{\mathcal{R}_a}), f_a([v]_{\mathcal{R}_a}))$ is an arc of D, where $a = (f([u]_{\mathcal{R}}), f([v]_{\mathcal{R}}))$. Since f_a is an isomorphism, $([u]_{\mathcal{R}_a}, [v]_{\mathcal{R}_a})$ is an arc of $\mathcal{U}_a/\mathcal{R}_a$. Thus $([u]_{\mathcal{R}}, [v]_{\mathcal{R}})$ is an arc of $\mathcal{U}/\mathcal{R}$ and hence f is an isomorphism.

Corollary 3 and Proposition 5 imply the following corollaries.

Corollary 4. *Any compositional digraph either has a finite cycle or contains an isomorphic copy of* $\mathcal{U}$ *as subdigraph.*

Corollary 5. *Any finite compositional digraph has a finite cycle.*

4. Main result

The following proposition is a generalization of [4, Theorem 5].

Proposition 6. *If* $A \subset \lambda$, *then the semigroup topology* τ *generated by the compositional family* $\{A \times (\lambda \setminus A)\}$ *is minimal.*

Proof. Assume that τ_1 is a weaker topology than the topology τ. Let $B \times (\lambda \setminus C)$ be a closed set in (B_λ, τ_1). By Lemma 3, there exists $D \subseteq \lambda$ such that the sets $B \times (\lambda \setminus D)$ and $D \times (\lambda \setminus C)$ are closed in (B_λ, τ_1). Since τ_1 is weaker than τ, $D \subseteq^* A$ and $(\lambda \setminus D) \subseteq^* (\lambda \setminus A)$. Therefore $D =^* A$. Applying Lemma 3, we conclude that there exists $F \subseteq \lambda$ such that the sets $D \times (\lambda \setminus F)$ and $F \times (\lambda \setminus C)$ are closed in (B_λ, τ_1). Hence $F \subseteq^* A$ and $(\lambda \setminus F) \subseteq^* (\lambda \setminus A)$ and then $F =^* A$. By Lemma 4, the set $A \times (\lambda \setminus A)$ is closed in the topological space (B_λ, τ_1). The obtained contradiction implies that τ is a minimal semigroup topology on B_λ.

Proposition 7. *Let* $\mathcal{F}$ *be a compositional family. If there exists a finite subdigraph* H *of the digraph* $D(\mathcal{F})$ *which does not contain sink or source, then there exists* $A \subseteq \lambda$ *such that the set* $A \times (\lambda \setminus A)$ *is closed in the topological space* $(B_\lambda, \tau_{\mathcal{F}})$.

Proof. Let $V(D) = \{A_1, \ldots, A_n\}$ and H does not contain sink. For each $A_i \in V(H)$ there exits $A_j \in V(H)$ such that $A_i \times (\lambda \setminus A_j)$. Observe that

$$\lambda \setminus (A_1 \cup \ldots \cup A_n) = (\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n).$$

If $(\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n) = \varnothing$, then $A_1 \cup \ldots \cup A_n = \lambda$ and hence, by Lemmas 1 and 2, the set $(\lambda \setminus \{\alpha\}) \times \{\alpha\}$ is closed in the topological space $(B_\lambda, \tau_{\mathcal{F}})$ for each $\alpha \in \lambda$. Let $(\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n) \neq \varnothing$. By Lemma 1, the set $A_i \times ((\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n))$ is closed for any $A_i \in V(H)$. Hence

$$A_1 \cup \ldots \cup A_n \times ((\lambda \setminus A_1) \cap \ldots \cap (\lambda \setminus A_n)) = \bigcup_{i=1}^{n} A_i \times \lambda \setminus \left(\bigcup_{i=1}^{n} A_i \right)$$

is closed in the topological space $(B_\lambda, \tau_{\mathcal{F}})$. The case with source is proved similarly.

Propositions 6 and 7 imply the following corollary.

Corollary 6. *If the digraph* $D(\mathcal{F})$ *has a finite cycle, then* $\mathcal{F}$ *is a singleton or generates a nonminimal topology.*

Corollaries 3 and 6 imply the following theorem.

Theorem 1. *Let τ be a semigroup topology on B_λ generated by a compositional family $\mathcal{F}$ such that $D(\mathcal{F})$ does not contain subdigraph isomorphic to $\mathcal{U}$. The topology τ is minimal if and only if τ is generated by a singleton compositional family.*

Problem 1. *Is there a minimal semigroup topology on B_λ generated by a composition family $\mathcal{F}$ such that $D(\mathcal{F})$ is isomorphic to $\mathcal{U}$?*

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References

1. B. Banaschewski, *Minimal topological algebras*, Math. Ann. **211** (1974), no. 2, 107–114. DOI: 10.1007/BF01344165
2. S. Bardyla and O. Gutik, *On the lattice of weak topologies on the bicyclic monoid with adjoined zero*, Algebra Discrete Math. **30** (2020), no. 1, 26–43. DOI: 10.12958/adm1459
3. D. Doïtchinov, *Produits de groupes topologiques minimaux*, Bull. Sci. Math., II. Ser. **97** (1972), 59–64.
4. O. V. Gutik and K. P. Pavlyk, *On topological semigroups of matrix units*, Semigroup Forum **71** (2005), no. 3, 389–400. DOI: 10.1007/s00233-005-0530-0
5. J. Bang-Jensen and G. Gutin, *Digraphs. Theory, algorithms and applications*, Springer Monographs in Mathematics. Springer, London, 2nd edition, 2009. DOI: 10.1007/978-1-84800-998-1
6. L. Nachbin, *On strictly minimal topological division rings*, Bull. Amer. Math. Soc. **55** (1949), 1128–1136. DOI: 10.1090/S0002-9904-1949-09339-4
7. R. M. Stephenson, Jr., *Minimal topological groups*, Math. Ann. **192** (1971), 193–195. DOI: 10.1007/BF02052870

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МІНІМАЛЬНІ НАПІВГРУПОВІ ТОПОЛОГІЇ НА НАПІВГРУПІ МАТРИЧНИХ ОДИНИЦЬ

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Ми описали мінімальні топології в деякому класі напівгрупових топологій на напівгрупі матричних одиниць.

Ключові слова: топологічна напівгрупа, мінімальна напівгрупова топологія, напівгрупа матричних одиниць.