*ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2021. Випуск 92. С. 34–50*
*Visnyk of the Lviv Univ. Series Mech. Math. 2021. Issue 92. P. 34–50*
http://publications.lnu.edu.ua/bulletins/index.php/mmf
doi: http://dx.doi.org/10.30970/vmm.2021.92.034-050

УДК 512.53

# ON THE SEMIGROUP $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ WHICH IS GENERATED BY THE FAMILY $\mathscr{F}$ OF ATOMIC SUBSETS OF $\omega$

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We study the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, which is introduced in [O. Gutik and M. Mykhalenych, *On some generalization of the bicyclic monoid*, Visnyk Lviv. Univ. Ser. Mech.-Mat. **90** (2020), 5–19], in the case when the family $\mathscr{F}$ of subsets of cardinality $\leqslant 1$ in $\omega$. We show that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the subsemigroup $\mathscr{B}_{\omega}^{\Uparrow}(\boldsymbol{F}_{\min})$ of the Brandt $\omega$-extension of the semilattice $\boldsymbol{F}_{\min}$ and describe all shift-continuous feebly compact $T_1$-topologies on the semigroup $\mathscr{B}_{\omega}^{\Uparrow}(\boldsymbol{F}_{\min})$. In particulary we prove that every shift-continuous feebly compact $T_1$-topology $\tau$ on $\mathscr{B}_{\omega}^{\Uparrow}(\boldsymbol{F}_{\min})$ is compact and moreover in this case the space $(\mathscr{B}_{\omega}^{\Uparrow}(\boldsymbol{F}_{\min}),\tau)$ is homeomorphic to the one-point Alexandroff compactification of the discrete countable space $\mathfrak{D}(\omega)$. We study the closure of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ in a semitopological semigroup. In particularly we show that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is algebraically complete in the class of Hausdorff semitopological inverse semigroups with continuous inversion, and a Hausdorff topological inverse semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is closed in any Hausdorff topological semigroup if and only if the band $E(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ is compact.

*Key words:* semitopological semigroup, topological semigroup, bicyclic monoid, inverse semigroup, feebly compact, compact, Brandt $\omega$-extension, closure.

## 1. Introduction, motivation and main definitions

We shall follow the terminology of [2, 3, 4, 5, 19]. By $\omega$ we denote the set of all non-negative integers.

Let $\mathscr{P}(\omega)$ be the family of all subsets of $\omega$. For any $F\in\mathscr{P}(\omega)$ and $n,m\in\omega$ we put

$$n-m+F=\{n-m+k\colon k\in F\}$$

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*2020 Mathematics Subject Classification*: 22A15

This definition implies that $n-m+F=\varnothing$ if $F=\varnothing$. A subfamily $\mathscr{F}\subseteq\mathscr{P}(\omega)$ is called *$\omega$-closed* if $F_1\cap(-n+F_2)\in\mathscr{F}$ for all $n\in\omega$ and $F_1,F_2\in\mathscr{F}$.

A semigroup $S$ is called *inverse* if for any element $x\in S$ there exists a unique $x^{-1}\in S$ (called the *inverse of* $x$) such that $xx^{-1}x=x$ and $x^{-1}xx^{-1}=x^{-1}$. If $S$ is an inverse semigroup, then the function $\operatorname{inv}\colon S\to S$ which assigns to every element $x$ of $S$ its inverse element $x^{-1}$ is called the *inversion*.

If $S$ is a semigroup, then we shall denote the subset of all idempotents in $S$ by $E(S)$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* $S$). The semigroup operation of $S$ determines the following partial order $\preccurlyeq$ on $E(S)$: $e\preccurlyeq f$ if and only if $ef=fe=e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. By $(\omega,\min)$ or $\omega_{\min}$ we denote the set $\omega$ with the semilattice operation $x\cdot y=\min\{x,y\}$.

If $S$ is an inverse semigroup then the semigroup operation on $S$ determines the following partial order $\preccurlyeq$ on $S$: $s\preccurlyeq t$ if and only if there exists $e\in E(S)$ such that $s=te$. This order is called the *natural partial order* on $S$ [22].

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements $p$ and $q$ subjected only to the condition $pq=1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^kp^l\cdot q^mp^n=q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial $E$-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [3].

On the set $\boldsymbol{B}_{\omega}=\omega\times\omega$ we define a semigroup operation "$\cdot$" in the following way

$$(i_1,j_1)\cdot(i_2,j_2)=\begin{cases}(i_1-j_1+i_2,j_2), & \text{if } j_1<i_2;\\ (i_1,j_2), & \text{if } j_1=i_2;\\ (i_1,j_1-i_2+j_2), & \text{if } j_1>i_2.\end{cases}$$

It is well known that the semigroup $\boldsymbol{B}_{\omega}$ is isomorphic to the bicyclic monoid by the mapping $\mathfrak{h}\colon\mathscr{C}(p,q)\to\boldsymbol{B}_{\omega}$, $q^kp^l\mapsto(k,l)$ (see: [3, Section 1.12] or [18, Exercise IV.1.11$(ii)$]).

A *topological* (*semitopological*) *semigroup* is a topological space together with a continuous (separately continuous) semigroup operation. If $S$ is a semigroup and $\tau$ is a topology on $S$ such that $(S,\tau)$ is a topological semigroup, then we shall call $\tau$ a *semigroup topology* on $S$, and if $\tau$ is a topology on $S$ such that $(S,\tau)$ is a semitopological semigroup, then we shall call $\tau$ a *shift-continuous topology* on $S$. An inverse topological semigroup with the continuous inversion is called a *topological inverse semigroup*. If $S$ is an inverse semigroup and $\tau$ is a topology on $S$ such that $(S,\tau)$ is a topological inverse semigroup, then we shall call $\tau$ a *semigroup inverse topology* on $S$.

Next we shall describe the construction which is introduced in [9].

Let $\boldsymbol{B}_{\omega}$ be the bicyclic monoid and $\mathscr{F}$ be an $\omega$-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_{\omega}\times\mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1,j_1,F_1)\cdot(i_2,j_2,F_2)=\begin{cases}(i_1-j_1+i_2,j_2,(j_1-i_2+F_1)\cap F_2), & \text{if } j_1<i_2;\\ (i_1,j_2,F_1\cap F_2), & \text{if } j_1=i_2;\\ (i_1,j_1-i_2+j_2,F_1\cap(i_2-j_1+F_2)), & \text{if } j_1>i_2.\end{cases}$$

By [9], if the family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ is $\omega$-closed, then $(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)$ is a semigroup. Moreover, if an $\omega$-closed family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ contains the empty set $\varnothing$, then the set

$$\boldsymbol{I}=\{(i,j,\varnothing)\colon i,j\in\omega\}$$

is an ideal of the semigroup $(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)$. For any $\omega$-closed family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_{\omega}^{\mathscr{F}}=\left\{\begin{array}{ll}(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)/\boldsymbol{I}, & \text{if } \varnothing\in\mathscr{F};\\ (\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot), & \text{if } \varnothing\notin\mathscr{F}\end{array}\right.$$

is defined in [9]. The semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in [9] that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its set of idempotents are described. The criteria of simplicity, 0-simplicity, bisimplicity, 0-bisimplicity of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and when $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ has the identity, is isomorphic to the bicyclic semigroup or the countable semigroup of matrix units are given. In particular in [9] it is proved that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega\times\omega$-matrix units if and only if $\mathscr{F}$ consists of sets of cardinality $\leqslant 1$ in $\omega$.

Let $\mathscr{F}$ be some family of cardinality $\leqslant 1$ in $\omega$. In this case we shall say that $\mathscr{F}$ is the *family of atomic subsets* of $\omega$. It is obvious that if $\mathscr{F}=\{\varnothing\}$ then the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is trivial and hence in this paper we assume that the family $\mathscr{F}$ contains at least one singleton subset of $\omega$. It is obvious that in this case $\mathscr{F}$ is an $\omega$-closed subfamily of $\mathscr{P}(\omega)$ and hence $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is an inverse semigroup with zero. Later by $\mathbf{0}$ we denote the zero of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and by $(i,j,\{k\})$ a non-zero element of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for some $i,j\in\omega$, $\{k\}\in\mathscr{F}$.

We put $\boldsymbol{F}=\bigcup\mathscr{F}$. Since the semilattice $(\omega,\min)$ is linearly ordered, the set $\boldsymbol{F}$ with the binary operation $xy=\min\{x,y\}$ is a subsemilattice of $(\omega,\min)$ and later by $\boldsymbol{F}_{\min}$ we shall denote the set $\boldsymbol{F}$ with the semilattice operation inherited from $(\omega,\min)$.

We need the following construction from [6].

Let $S$ be a semigroup with zero and $\lambda\geqslant 1$ be a cardinal. On the set $B_{\lambda}(S)=(\lambda\times S\times\lambda)\sqcup\{\mathscr{O}\}$ we define a semigroup operation as follows

$$(\alpha,s,\beta)\cdot(\gamma,t,\delta)=\left\{\begin{array}{ll}(\alpha,st,\delta), & \text{if } \beta=\gamma;\\ \mathscr{O}, & \text{if } \beta\neq\gamma\end{array}\right.$$

and

$$(\alpha,s,\beta)\cdot\mathscr{O}=\mathscr{O}\cdot(\alpha,s,\beta)=\mathscr{O}\cdot\mathscr{O}=\mathscr{O},$$

for all $\alpha,\beta,\gamma,\delta\in\lambda$ and $s,t\in S$. The semigroup $\mathscr{B}_{\lambda}(S)$ is called the *Brandt $\lambda$-extension of the semigroup* $S$ [6]. Algebraic properties of $\mathscr{B}_{\lambda}(S)$ and its generalization the Brandt $\lambda^0$-extension $\mathscr{B}_{\lambda}^{0}(S)$ are studied in [6, 7, 10, 12].

In this paper we study the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for a family $\mathscr{F}$ of atomic subsets of $\omega$. We show that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the subsemigroup $\mathscr{B}_{\omega}^{\Vdash}(\boldsymbol{F}_{\min})$ of the Brandt $\omega$-extension of the semilattice $\boldsymbol{F}_{\min}$ and describe all shift-continuous feebly compact $T_1$-topologies on the semigroup $\mathscr{B}_{\omega}^{\Vdash}(\boldsymbol{F}_{\min})$. In particular, we prove that every shift-continuous feebly compact $T_1$-topology $\tau$ on $\mathscr{B}_{\omega}^{\Vdash}(\boldsymbol{F}_{\min})$ is compact and moreover in this case the space $(\mathscr{B}_{\omega}^{\Vdash}(\boldsymbol{F}_{\min}),\tau)$ is homeomorphic to the one-point Alexandroff compactification of the discrete countable space $\mathfrak{D}(\omega)$. We study the closure of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ in a semitopological semigroup.

In particularly we show that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is algebraically complete in the class of Hausdorff semitopological inverse semigroups with continuous inversion, and a Hausdorff topological inverse semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is closed in any Hausdorff topological semigroup if and only if the band $E(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ is compact.

Later in this paper we assume that $\mathscr{F}$ is a non-trivial family of atomic subsets of $\omega$, i.e., $\mathscr{F}$ contains at least one nontrivial singleton subset of $\omega$.

## 2. Algebraic properties of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$

Proposition 2 of [9] implies the following proposition which describing the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.

**Proposition 1.** *Let $(i_1, j_1, \{k_1\})$ and $(i_2, j_2, \{k_2\})$ be non-zero elements of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. Then $(i_1, j_1, \{k_1\}) \preccurlyeq (i_2, j_2, \{k_2\})$ if and only if*

$$k_2 - k_1 = i_1 - i_2 = j_1 - j_2 = p$$

*for some $p \in \omega$.*

Since the set $\omega$ is well ordered by the usual order we enumerate the set $\boldsymbol{F} = \{k_i : i \in \omega\}$ in the following way $k_0 < k_1 < \cdots < k_n < k_{n+1} < \cdots$. It is obvious that the set $\boldsymbol{F}$ is finite if and only if $\boldsymbol{F}$ contains the maximum.

Proposition 1 implies the structure of maximal chains in $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with the respect to its natural partial order

**Corollary 1.** *Let $i, j$ be arbitrary elements of $\omega$. Then in the case when the set $\boldsymbol{F}$ is infinite then the following finite series*

$$\begin{aligned}
&\mathbf{0} \preccurlyeq (i, j, \{k_0\});\\
&\mathbf{0} \preccurlyeq (i + k_1 - k_0, j + k_1 - k_0, \{k_0\}) \preccurlyeq (i, j, \{k_1\});\\
&\mathbf{0} \preccurlyeq (i + k_1 - k_0, j + k_1 - k_0, \{k_0\}) \preccurlyeq (i + k_2 - k_1, j + k_2 - k_1, \{k_1\}) \preccurlyeq (i, j, \{k_2\});\\
&\qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
&\mathbf{0} \preccurlyeq (i + k_1 - k_0, j + k_1 - k_0, \{k_0\}) \preccurlyeq (i + k_2 - k_1, j + k_2 - k_1, \{k_1\}) \preccurlyeq \cdots \preccurlyeq\\
&\qquad \preccurlyeq (i + k_{n+1} - k_n, j + k_{n+1} - k_n, \{k_n\}) \preccurlyeq (i, j, \{k_{n+1}\});\\
&\qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{aligned}$$

*describes maximal chains in the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and in the case when the set $\boldsymbol{F}$ is finite and contains maximum $k_n$ then the following finite series*

$$\begin{aligned}
&\mathbf{0} \preccurlyeq (i, j, \{k_0\});\\
&\mathbf{0} \preccurlyeq (i + k_1 - k_0, j + k_1 - k_0, \{k_0\}) \preccurlyeq (i, j, \{k_1\});\\
&\mathbf{0} \preccurlyeq (i + k_1 - k_0, j + k_1 - k_0, \{k_0\}) \preccurlyeq (i + k_2 - k_1, j + k_2 - k_1, \{k_1\}) \preccurlyeq (i, j, \{k_2\});\\
&\qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
&\mathbf{0} \preccurlyeq (i + k_1 - k_0, j + k_1 - k_0, \{k_0\}) \preccurlyeq (i + k_2 - k_1, j + k_2 - k_1, \{k_1\}) \preccurlyeq \cdots \preccurlyeq\\
&\qquad \preccurlyeq (i + k_n - k_{n-1}, j + k_n - k_{n-1}, \{k_n\}) \preccurlyeq (i, j, \{k_n\})
\end{aligned}$$

*describes maximal chains in the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.*

We define a map $\mathfrak{f}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}} \to \mathscr{B}_{\omega}(\boldsymbol{F}_{\min})$ by the formulae

$$\mathfrak{f}(i,j,\{k\}) = (i+k,k,j+k) \qquad \text{and} \qquad (\mathbf{0})\mathfrak{f} = \mathscr{O}, \tag{1}$$

for $i,j\in\omega$ and $\{k\}\in\mathscr{F}\setminus\{\varnothing\}$.

**Proposition 2.** *The map $\mathfrak{f}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}} \to \mathscr{B}_{\omega}(\boldsymbol{F}_{\min})$ is an isomorphic embedding.*

*Proof.* It is obvious that the map $\mathfrak{f}$ which is defined by formulae (1) is injective.

For arbitrary $(i_1,j_1,\{k_1\}),(i_2,j_2,\{k_2\})\in\boldsymbol{B}_{\omega}^{\mathscr{F}}$ we have that

$$\begin{aligned}
&\mathfrak{f}((i_1,j_1,\{k_1\})\cdot(i_2,j_2,\{k_2\})) =\\
&\qquad=\begin{cases}
\mathfrak{f}(i_1-j_1+i_2,j_2,\{k_2\}), & \text{if } j_1<i_2 \text{ and } j_1+k_1=i_2+k_2;\\
\mathfrak{f}(i_1,j_2,\{k_1\}), & \text{if } j_1=i_2 \text{ and } k_1=k_2;\\
\mathfrak{f}(i_1,j_1-i_2+j_2,\{k_1\}), & \text{if } j_1>i_2 \text{ and } j_1+k_1=i_2+k_2;\\
\mathfrak{f}(\mathbf{0}), & \text{if } j_1+k_1\neq i_2+k_2
\end{cases} =\\
&\qquad=\begin{cases}
(i_1-j_1+i_2+k_2,k_2,j_2+k_2), & \text{if } j_1<i_2 \text{ and } j_1+k_1=i_2+k_2;\\
(i_1+k_1,k_1,j_2+k_1), & \text{if } j_1=i_2 \text{ and } k_1=k_2;\\
(i_1+k_1,k_1,j_1-i_2+j_2+k_1), & \text{if } j_1>i_2 \text{ and } j_1+k_1=i_2+k_2;\\
\mathscr{O}, & \text{if } j_1+k_1\neq i_2+k_2
\end{cases} =\\
&\qquad=\begin{cases}
(i_1+k_1,k_2,j_2+k_2), & \text{if } j_1<i_2 \text{ and } j_1+k_1=i_2+k_2;\\
(i_1+k_1,k_1,j_2+k_2), & \text{if } j_1=i_2 \text{ and } k_1=k_2;\\
(i_1+k_1,k_1,j_2+k_2), & \text{if } j_1>i_2 \text{ and } j_1+k_1=i_2+k_2;\\
\mathscr{O}, & \text{if } j_1+k_1\neq i_2+k_2,
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{f}((i_1,j_1,\{k_1\})\cdot\mathfrak{f}(i_2,j_2,\{k_2\})) = (i_1+k_1,k_1,j_1+k_1)\cdot(i_2+k_2,k_2,j_2+k_2) =\\
&\qquad=\begin{cases}
(i_1+k_1,\min\{k_1,k_2\},j_2+k_2), & \text{if } j_1+k_1=i_2+k_2;\\
\mathscr{O}, & \text{if } j_1+k_1\neq i_2+k_2
\end{cases} =\\
&\qquad=\begin{cases}
(i_1+k_1,k_2,j_2+k_2), & \text{if } k_2<k_1 \text{ and } j_1+k_1=i_2+k_2;\\
(i_1+k_1,k_1,j_2+k_2), & \text{if } k_2=k_1 \text{ and } k_1=k_2;\\
(i_1+k_1,k_1,j_2+k_2), & \text{if } k_2>k_1 \text{ and } j_1+k_1=i_2+k_2;\\
\mathscr{O}, & \text{if } j_1+k_1\neq i_2+k_2,
\end{cases} =\\
&\qquad=\begin{cases}
(i_1+k_1,k_2,j_2+k_2), & \text{if } j_1<i_2 \text{ and } j_1+k_1=i_2+k_2;\\
(i_1+k_1,k_1,j_2+k_2), & \text{if } j_1=i_2 \text{ and } k_1=k_2;\\
(i_1+k_1,k_1,j_2+k_2), & \text{if } j_1>i_2 \text{ and } j_1+k_1=i_2+k_2;\\
\mathscr{O}, & \text{if } j_1+k_1\neq i_2+k_2.
\end{cases}
\end{aligned}$$

Since $\mathbf{0}$ and $\mathscr{O}$ are the zeros of the semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and $\mathscr{B}_{\omega}(\boldsymbol{F}_{\min})$, respectively, the above equalities imply that the map $\mathfrak{f}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}} \to \mathscr{B}_{\omega}(\boldsymbol{F}_{\min})$ is a homomorphism. This completes the proof of the proposition. □

Next we define

$$\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min}) = \{\mathscr{O}\}\cup\left\{(i+k,k,j+k)\in\mathscr{B}_{\omega}(\boldsymbol{F}_{\min})\setminus\{\mathscr{O}\} : (i,j,\{k\})\in\boldsymbol{B}_{\omega}^{\mathscr{F}}\right\}.$$

Proposition 2 implies

**Theorem 1.** *Let $\mathscr{F}^*$ be any family of atomic subsets of $\omega$. Then the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$ by the mapping $\mathfrak{f}$.*

**Proposition 3.** *Let $\mathscr{F}^*$ be any family of subsets of $\omega$ which contains a non-empty set, and $k_0 = \min\bigcup\mathscr{F}^*$. Then the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^*}$ is isomorphic to the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_0^*}$ where*

$$\mathscr{F}_0^* = \{-k_0 + F \colon F \in \mathscr{F}^*\}.$$

*Proof.* Since the set $\omega$ with the usual order $\leqslant$ is well ordered, the number $k_0$ is well defined. This implies that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_0^*}$ is well defined, because $F \subseteq \{n \in \omega \colon n \geqslant k_0\}$ for any $F \in \mathscr{F}^*$. Without loss of generality we may assume that $\varnothing \in \mathscr{F}^*$, which implies that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^*}$ has zero $\boldsymbol{0}$, and hence the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_0^*}$ has zero $\boldsymbol{0}$, too.

We define the map $\mathfrak{h}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}^*} \to \boldsymbol{B}_{\omega}^{\mathscr{F}_0^*}$ in the following way

$$\mathfrak{h}(i,j,\{k\}) = (i-k_0, j-k_0, \{k-k_0\}) \qquad \text{and} \qquad (\boldsymbol{0})\mathfrak{h} = \boldsymbol{0} \tag{2}$$

for $i,j \in \omega$ and $\{k\} \in \mathscr{F}^* \setminus \{\varnothing\}$. It is obvious that such defined map $\mathfrak{h}$ is bijective.

For arbitrary $(i_1,j_1,\{k_1\}),(i_2,j_2,\{k_2\}) \in \boldsymbol{B}_{\omega}^{\mathscr{F}^*}$ we have that

$$\begin{aligned}
&\mathfrak{h}((i_1,j_1,\{k_1\})\cdot(i_2,j_2,\{k_2\})) = \\
&\quad= \begin{cases}
\mathfrak{h}(i_1-j_1+i_2,j_2,\{k_2\}), & \text{if } j_1<i_2 \text{ and } j_1+k_1=i_2+k_2;\\
\mathfrak{h}(i_1,j_2,\{k_1\}), & \text{if } j_1=i_2 \text{ and } k_1=k_2;\\
\mathfrak{h}(i_1,j_1-i_2+j_2,\{k_1\}), & \text{if } j_1>i_2 \text{ and } j_1+k_1=i_2+k_2;\\
\mathfrak{h}(\boldsymbol{0}), & \text{if } j_1+k_1\neq i_2+k_2
\end{cases} = \\
&\quad= \begin{cases}
(i_1-j_1+i_2-k_0,j_2-k_0,\{k_2-k_0\}), & \text{if } j_1<i_2 \text{ and } j_1+k_1=i_2+k_2;\\
(i_1-k_0,j_2-k_0,\{k_1-k_0\}), & \text{if } j_1=i_2 \text{ and } k_1=k_2;\\
(i_1-k_0,j_1-i_2+j_2-k_0,\{k_1-k_0\}), & \text{if } j_1>i_2 \text{ and } j_1+k_1=i_2+k_2;\\
\boldsymbol{0}, & \text{if } j_1+k_1\neq i_2+k_2
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{h}(i_1,j_1,\{k_1\})\cdot\mathfrak{h}(i_2,j_2,\{k_2\}) = \\
&= (i_1-k_0,j_1-k_0,\{k_1-k_0\})\cdot(i_2-k_0,j_2-k_0,\{k_2-k_0\}) = \\
&= \begin{cases}
(i_1-k_0-(j_1-k_0)+i_2-k_0,j_2-k_0,\{k_2-k_0\}), & \text{if } j_1-k_0<i_2-k_0 \text{ and}\\
 & \quad j_1-k_0+k_1-k_0=i_2-k_0+k_2-k_0;\\
(i_1-k_0,j_2-k_0,\{k_1-k_0\}), & \text{if } j_1-k_0=i_2-k_0 \text{ and}\\
 & \quad k_1-k_0=k_2-k_0;\\
(i_1-k_0,j_1-k_0-(i_2-k_0)+j_2-k_0,\{k_1-k_0\}), & \text{if } j_1-k_0>i_2-k_0 \text{ and}\\
 & \quad j_1-k_0+k_1-k_0=i_2-k_0+k_2-k_0;\\
\boldsymbol{0}, & \text{if } j_1-k_0+k_1-k_0\neq i_2-k_0+k_2-k_0
\end{cases}\\
&= \begin{cases}
(i_1-j_1+i_2-k_0,j_2-k_0,\{k_2-k_0\}), & \text{if } j_1<i_2 \text{ and } j_1+k_1=i_2+k_2;\\
(i_1-k_0,j_2-k_0,\{k_1-k_0\}), & \text{if } j_1=i_2 \text{ and } k_1=k_2;\\
(i_1-k_0,j_1-i_2+j_2-k_0,\{k_1-k_0\}), & \text{if } j_1>i_2 \text{ and } j_1+k_1=i_2+k_2;\\
\boldsymbol{0}, & \text{if } j_1+k_1\neq i_2+k_2.
\end{cases}
\end{aligned}$$

Since $\boldsymbol{0}$ is the zero of both semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}^*}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}_0^*}$, the above equalities imply that such defined map $\mathfrak{h}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}^*} \to \boldsymbol{B}_{\omega}^{\mathscr{F}_0^*}$ is a homomorphism. $\square$

**Theorem 2.** *Let $\mathscr{F}^1$ and $\mathscr{F}^2$ be some families of atomic subsets of $\omega$. Then the semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}^1}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^2}$ are isomorphic if and only if there exists an integer $n$ such that*

$$\mathscr{F}^1 = \left\{n + F \colon F \in \mathscr{F}^2\right\}.$$

*Proof.* The implication ($\Leftarrow$) follows from Proposition 3.

($\Rightarrow$) Put $\boldsymbol{F}^1 = \bigcup \mathscr{F}^1$ and $\boldsymbol{F}^2 = \bigcup \mathscr{F}^2$. By Proposition 3. without loss of generality we may assume that $0 \in \boldsymbol{F}^1 \cap \boldsymbol{F}^2$, i.e., $\{0\} \in \mathscr{F}^1$ and $\{0\} \in \mathscr{F}^2$.

Suppose to the contrary that the semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}^1}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^2}$ are isomorphic but $\mathscr{F}^1 \neq \mathscr{F}^2$. Since $\mathscr{F}^1$ and $\mathscr{F}^2$ are some families of atomic subsets of $\omega$, we get that $\boldsymbol{F}^1 \neq \boldsymbol{F}^2$. Hence without loss of generality we may assume that there exists the minimum positive integer $m$ of the set $\boldsymbol{F}^1$ such that $m \notin \boldsymbol{F}^2$. Put

$$\widetilde{\boldsymbol{F}} = \left\{k \in \boldsymbol{F}^2 \colon k < m\right\}.$$

We enumerate the set $\widetilde{\boldsymbol{F}} = \{k_0, k_1, \ldots, k_n\}$ in the following way

$$k_0 = 0 < k_1 < \cdots < k_n.$$

Then we have that $\widetilde{\boldsymbol{F}} \subset \boldsymbol{F}^1$.

By Lemma 2 of [9] a non-zero element $(i, j, \{k\})$ of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^1}$ (or $\boldsymbol{B}_{\omega}^{\mathscr{F}^2}$) is an idempotent if and only if $i = j$. This and Corollary 1 imply the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^1}$ contains exactly $m - k_n$ distinct chains (or a chain) of idempotents of the length $k_n + 2$, but the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^1}$ contains at least $m - k_n + 1$ distinct chains of idempotents of the length $k_n + 2$. This contradicts that the semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}^1}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^2}$ are isomorphic. The obtained contradiction implies the implication. □

For any $i, j \in \omega$ we denote

$$\boldsymbol{F}_{\min}^{(i,j)_{\Gamma}} = \left\{(i, k, j) \colon (i, k, j) \in \mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})\right\}$$

and

$$\omega_{\min}^{(i,j)} = \left\{(i, k, j) \colon (i, k, j) \in \mathscr{B}_{\omega}(\omega_{\min})\right\},$$

where by $\omega_{\min}$ we denote the semilattice $(\omega, \min)$.

**Lemma 1.** *In the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ both equations $A \cdot X = B$ and $X \cdot A = B$ have only finitely many solutions for $B \neq \mathbf{0}$.*

*Proof.* We show that the equation $A \cdot X = B$ has finitely many solutions for $B \neq \mathscr{O}$ in the semigroup $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$. In the case of the equation $X \cdot A = B$ the proof is similar.

We denote

$$A = (i_A, k_A, j_A), \qquad X = (i_X, k_X, j_X) \qquad \text{and} \qquad B = (i_B, k_B, j_B),$$

where $(i_X, k_X, j_X)$ is a variable, $(i_A, k_A, j_A)$ and $(i_B, k_B, j_B)$ are constants of the equation

$$(i_A, k_A, j_A) \cdot (i_X, k_X, j_X) = (i_B, k_B, j_B). \tag{3}$$

First we establish the solution of equation (3) in the Brandt $\omega$-extension $\mathscr{B}_{\omega}(\omega_{\min})$ of the semilattice $\omega_{\min}$. The semigroup operation in $\mathscr{B}_{\omega}(\omega_{\min})$ implies that equation (3) has a non-empty set of solutions if and only if $k_B \preccurlyeq k_A$ in $\omega_{\min}$ and $i_A = i_B$. Hence we have that the set of solutions of (3) is a subset of $\omega_{\min}^{(j_A, j_B)}$. This implies that the set of solutions of equation (3) is a subset of $\boldsymbol{F}_{\min}^{(j_A,j_B)_{\Gamma}}$. This and Theorem 1 imply the statement of the lemma. □

## 3. On topogizations of the semigroup $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$

By Proposition 3 for any family $\mathscr{F}$ of atomic subsets of $\omega$ the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_0}$ where $\mathscr{F}_0$ is a family of atomic subsets of $\omega$ such that $0 \in \bigcup \mathscr{F}_0$. Hence later we shall assume that $0 \in \boldsymbol{F}$, i.e., $(i,0,i) \in \mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$ for any $i,j \in \omega$.

**Proposition 4.** *Let $\tau$ be a shift-continuous $T_1$-topology on the semigroup $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$. Then every non-zero element of $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$ is an isolated point in $\left(\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min}),\tau\right)$.*

*Proof.* Fix arbitrary $i,j \in \omega$. Since

$$(i,0,i)\cdot(i,0,j)\cdot(j,0,j)=(i,0,j)$$

the assumption of the proposition implies that for any open neighbourhood $W_{(i,0,j)} \not\ni \mathscr{O}$ of the point $(i,0,j)$ there exists its open neighbourhood $V_{(i,0,j)}$ in the topological space $\left(\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min}),\tau\right)$ such that

$$(i,0,i)\cdot V_{(i,0,j)}\cdot(j,0,j)\subseteq W_{(i,0,j)}.$$

The definition of the semigroup operation on $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$ implies that $V_{(i,0,j)} \subseteq \boldsymbol{F}_{\min}^{(i,j)_{\Gamma}}$. Then $\boldsymbol{F}_{\min}^{(i,j)_{\Gamma}}$ is an open subset of the set $\left(\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min}),\tau\right)$ because it is the full preimage of $V_{(i,0,j)}$ under the mapping

$$\mathfrak{h}\colon \mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})\to\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min}),\ x\mapsto(i,0,i)\cdot x\cdot(j,0,j).$$

By Corollary 1 the set $\boldsymbol{F}_{\min}^{(i,j)_{\Gamma}}$ is finite, which implies the statement of the proposition. □

Next we shall show that the semigroup $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$ admits a compact shift-continuous Hausdorff topology.

**Example 1.** A topology $\tau_{\mathrm{Ac}}$ on the semigroup $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$ is defined as follows:

a) all nonzero elements of $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$ are isolated points in $\left(\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min}),\tau_{\mathrm{Ac}}\right)$;
b) the family

$$\mathscr{B}_{\mathrm{Ac}}(\mathscr{O})=\Big\{U_{(i_1,j_1),\ldots,(i_n,j_n)}=\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})\setminus\Big(\boldsymbol{F}_{\min}^{(i_1,j_1)_{\Gamma}}\cup\cdots\cup\boldsymbol{F}_{\min}^{(i_n,j_n)_{\Gamma}}\Big)\colon n,i_1,j_1,\ldots,i_n,j_n\in\omega\Big\}$$

is the base of the topology $\tau_{\mathrm{Ac}}$ at the point $\mathscr{O}\in\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$.

Corollary 1 implies that the set $\boldsymbol{F}_{\min}^{(i,j)_{\Gamma}}$ is finite for any $i,j \in \omega$ which implies that the topological space $\left(\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min}),\tau_{\mathrm{Ac}}\right)$ is homeomorphic to the one-point Alexandroff compactification of the discrete space $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})\setminus\{\mathscr{O}\}$.

**Proposition 5.** $\left(\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min}),\tau_{\mathrm{Ac}}\right)$ *is a Hausdorff compact semitopological semigroup with continuous inversion.*

*Proof.* It is obvious that the topology $\tau_{\mathrm{Ac}}$ is Hausdorff and compact.

Fix any $U_{(i_1,j_1),\ldots,(i_n,j_n)} \in \mathscr{B}_{\mathrm{Ac}}(\mathscr{O})$ and $(i,k,j),(l,m,p) \in \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}) \setminus \{\mathscr{O}\}$. Put

$$\boldsymbol{K} = \{i, i_1, \ldots, i_n, j, j_1, \ldots, j_n\} \qquad \text{and} \qquad U_{\boldsymbol{K}} = \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}) \setminus \bigcup_{x,y\in\boldsymbol{K}} \boldsymbol{F}_{\min}^{(x,y)\Rsh}.$$

Then we have that $U_{\mathbf{K}} \in \mathscr{B}_{\mathrm{Ac}}(\mathscr{O})$ and the following conditions hold

$$U_{\boldsymbol{K}} \cdot \{(i,k,j)\} \subseteq U_{(i_1,j_1),\ldots,(i_n,j_n)},$$
$$\{(i,k,j)\} \cdot U_{\boldsymbol{K}} \subseteq U_{(i_1,j_1),\ldots,(i_n,j_n)},$$
$$\{\mathscr{O}\} \cdot \{(i,k,j)\} = \{(i,k,j)\} \cdot \{\mathscr{O}\} = \{\mathscr{O}\} \subseteq U_{(i_1,j_1),\ldots,(i_n,j_n)},$$
$$\{\mathscr{O}\} \cdot U_{(i_1,j_1),\ldots,(i_n,j_n)} = U_{(i_1,j_1),\ldots,(i_n,j_n)} \cdot \{\mathscr{O}\} = \{\mathscr{O}\} \subseteq U_{(i_1,j_1),\ldots,(i_n,j_n)},$$
$$\{(i,k,j)\} \cdot \{(l,m,p)\} = \{\mathscr{O}\} \subseteq U_{(i_1,j_1),\ldots,(i_n,j_n)}, \quad \text{if} \quad j \neq l,$$
$$\{(i,k,j)\} \cdot \{(l,m,p)\} = \{(i,\min\{k,m\},p)\}, \quad \text{if} \quad j = l,$$
$$\left(U_{(j_1,i_1),\ldots,(j_n,i_n)}\right)^{-1} \subseteq U_{(i_1,j_1),\ldots,(i_n,j_n)}$$

Therefore, $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau_{\mathrm{Ac}}\right)$ is a semitopological inverse semigroup with continuous inversion. □

We recall that a topological space $X$ is said to be

- *perfectly normal* if $X$ is normal and and every closed subset of $X$ is a $G_\delta$-set;
- *scattered* if $X$ does not contain a non-empty dense-in-itself subspace;
- *hereditarily disconnected* (or *totally disconnected*) if $X$ does not contain any connected subsets of cardinality larger than one;
- *compact* if each open cover of $X$ has a finite subcover;
- *countably compact* if each open countable cover of $X$ has a finite subcover;
- *$H$-closed* if $X$ is a closed subspace of every Hausdorff topological space containing $X$;
- *infra $H$-closed* provided that any continuous image of $X$ into any first countable Hausdorff space is closed (see [15]);
- *feebly compact* if each locally finite open cover of $X$ is finite [1];
- *d-feebly compact* (or *DFCC*) if every discrete family of open subsets in $X$ is finite (see [17]);
- *pseudocompact* if $X$ is Tychonoff and each continuous real-valued function on $X$ is bounded;
- *$Y$-compact* for some topological space $Y$, if the image $f(X)$ is compact for any continuous map $f\colon X \to Y$.

The relations between above defined compact-like spaces are presented at the diagram in [14].

**Lemma 2.** *Every shift-continuous $T_1$-topology $\tau$ on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ is regular.*

*Proof.* By Proposition 5 every non-zero element of the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ is an isolated point in the space $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau\right)$. Hence every open neighbourhood $V(\mathscr{O})$ of the zero $\mathscr{O}$ is a closed subset in $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau\right)$, which implies that the topological space $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau\right)$ is regular. □

Since in any countable $T_1$-space $X$ every open subset of $X$ is a $F_\sigma$-set, Theorem 1.5.17 from [5] and Lemma 2 imply the following corollary.

**Corollary 2.** *Let $\tau$ be a shift-continuous $T_1$-topology on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$. Then $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)$ is a perfectly normal, scattered, hereditarily disconnected space.*

By $\mathfrak{D}(\omega)$ we denote the infinite countable discrete space and by $\mathbb{R}$ the set of all real numbers with the usual topology.

**Theorem 3.** *Let $\tau$ be a shift-continuous $T_1$-topology on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$. Then the following statements are equivalent:*

(*i*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)$ *is compact;*
(*ii*) $\tau=\tau_{\mathrm{Ac}}$*;*
(*iii*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)$ *is $H$-closed;*
(*iv*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)$ *is feebly compact;*
(*v*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)$ *is infra $H$-closed;*
(*vi*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)$ *is d-feebly compact;*
(*vii*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)$ *is pseudocompact;*
(*viii*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)$ *is $\mathbb{R}$-compact;*
(*ix*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)$ *is $\mathfrak{D}(\omega)$-compact.*

*Proof.* Implications $(ii)\Rightarrow(i)\Rightarrow(iii)\Rightarrow(iv)\Rightarrow(v)\Rightarrow(viii)\Rightarrow(ix)$ and $(i)\Rightarrow(vii)\Rightarrow(iv)\Rightarrow(vi)$ are trivial (see the diagram in [14]). By Lemma 2 we get implications $(vi)\Rightarrow(iv)$ and $(iii)\Rightarrow(i)$.

$(ix)\Rightarrow(i)$ Suppose to the contrary that there exists a shift-continuous $T_1$-topology $\tau$ on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ such that $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)$ is a $\mathfrak{D}(\omega)$-compact non-compact space. Then there exists an open cover $\mathscr{U}=\{U_\alpha\}$ of $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)$ which does not contain a finite subcover. Fix $U_{\alpha_0}\in\mathscr{U}$ such that $\mathscr{O}\in U_{\alpha_0}$. Since the space $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)$ is not compact the set $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})\setminus U_{\alpha_0}$ is infinite. We enumerate the set $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})\setminus U_{\alpha_0}$, i.e., put $\{\boldsymbol{x}_i\colon i\in\omega\}=\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})\setminus U_{\alpha_0}$. We identify $\mathfrak{D}(\omega)$ with $\omega$ and define a map $\mathfrak{f}\colon\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}),\tau\right)\to\mathfrak{D}(\omega)$ by the formula

$$\mathfrak{f}(\boldsymbol{x})=\begin{cases}0, & \text{if } \boldsymbol{x}\in U_{\alpha_0};\\ i, & \text{if } \boldsymbol{x}=\boldsymbol{x}_i.\end{cases}$$

Proposition 4 implies that such defined map $\mathfrak{f}$ is continuous. Also, the image $\mathfrak{f}(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$ is not a compact subset of $\mathfrak{D}(\omega)$, which contradicts the assumption. □

*Remark* 1. (1) By Proposition 4 of [9] the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ contains an isomorphic copy of the semigroup of $\omega\times\omega$-matrix units. Then Theorem 5 from [11] implies that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ does not embed into a countably compact Hausdorff topological semigroup.

(2) A Hausdorff topological semigroup $S$ is called $\Gamma$-*compact* if for every $x\in S$ the closure of the set $\{x,x^2,x^3,\ldots\}$ is compact in $S$ (see [16]). The semigroup operation $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ implies that either $a\cdot a=a$ or $a\cdot a=\mathscr{O}$ for any $a\in\boldsymbol{B}_{\omega}^{\mathscr{F}}$. Hence the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with any Hausdorff semigroup topology is $\Gamma$-compact.

## 4. On the closure of $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ in a (semi)topological semigroup

**Lemma 3.** *Let $S$ be a dense subsemigroup of a $T_1$-semitopological semigroup $T$ and $0$ be the zero of $S$. Then the element $0$ is the zero of $T$.*

*Proof.* Suppose to the contrary that there exists $a \in T \setminus S$ such that $0 \cdot a = b \neq 0$. Then for every open neighbourhood $U(b) \not\ni 0$ in $T$ there exists an open neighbourhood $V(a) \not\ni 0$ of the point $a$ in $T$ such that $0 \cdot V(a) \subseteq U(b)$. But $|V(a) \cap S| \geqslant \omega$, and hence $0 \in 0 \cdot V(a) \subseteq U(b)$. This contradicts the choice of the neighbourhood $U(b)$. Therefore $0 \cdot a = 0$ for all $a \in T \setminus S$.

The proof of the equality $a \cdot 0 = 0$ is similar. □

**Theorem 4.** *Let $T$ be a $T_1$-semitopological semigroup which contains the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ as a dense proper subsemigroup. Then $I = \left(T \setminus \boldsymbol{B}_{\omega}^{\mathscr{F}}\right) \cup \{\mathbf{0}\}$ is an ideal of $T$.*

*Proof.* Lemma 3 implies that $\mathbf{0}$ is the zero of the semigroup $T$. Since $T$ is a $T_1$-topological space, the set $\boldsymbol{B}_{\omega}^{\mathscr{F}} \setminus \{\mathbf{0}\}$ is dense in $T$. By Lemma 3 [13], $\boldsymbol{B}_{\omega}^{\mathscr{F}} \setminus \{\mathbf{0}\}$ is an open subspace of $T$.

Fix an arbitrary non-zero element $y \in I$. If $x \cdot y = z \notin I$ for some $x \in \boldsymbol{B}_{\omega}^{\mathscr{F}} \setminus \{\mathbf{0}\}$ then there exists an open neighbourhood $U(y)$ of the point $y$ in the space $T$ such that

$$\{x\} \cdot U(y) = \{z\} \subset \boldsymbol{B}_{\omega}^{\mathscr{F}} \setminus \{\mathbf{0}\}.$$

By Lemma 1 the open neighbourhood $U(y)$ should contain finitely many elements of the set $\boldsymbol{B}_{\omega}^{\mathscr{F}} \setminus \{\mathbf{0}\}$ which contradicts our assumption. Hence $x \cdot y \in I$ for all $x \in \boldsymbol{B}_{\omega}^{\mathscr{F}} \setminus \{\mathbf{0}\}$ and $y \in I$. The proof of the statement that $y \cdot x \in I$ for all $x \in \boldsymbol{B}_{\omega}^{\mathscr{F}} \setminus \{\mathbf{0}\}$ and $y \in I$ is similar.

Suppose to the contrary that $x \cdot y = w \notin I$ for some non-zero elements $x, y \in I$. Then $w \in \boldsymbol{B}_{\omega}^{\mathscr{F}} \setminus \{\mathbf{0}\}$ and the separate continuity of the semigroup operation in $T$ yields open neighbourhoods $U(x)$ and $U(y)$ of the points $x$ and $y$ in the space $T$, respectively, such that $\{x\} \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$. Since both neighbourhoods $U(x)$ and $U(y)$ contain infinitely many elements of the set $\boldsymbol{B}_{\omega}^{\mathscr{F}} \setminus \{\mathbf{0}\}$, equalities $\{x\} \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$ do not hold, because $\{x\} \cdot \left(U(y) \cap \boldsymbol{B}_{\omega}^{\mathscr{F}} \setminus \{\mathbf{0}\}\right) \subseteq I$. The obtained contradiction implies that $x \cdot y \in I$. □

A subset $D$ of a semigroup $S$ is said to be $\omega$-unstable if $D$ is infinite and $aB \cup Ba \nsubseteq D$ for any $a \in D$ and any infinite subset $B \subseteq D$.

**Definition 1** ([8]). An *ideal series* (see, for example, [3, 4]) for a semigroup $S$ is a chain of ideals

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = S.$$

We call the ideal series *tight* if $I_0$ is a finite set and $D_k = I_k \setminus I_{k-1}$ is an $\omega$-unstable subset for each $k = 1, \ldots, n$.

**Lemma 4.** *The ideal series $I_0 = \{\mathscr{O}\} \subset I_1 = \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ is tight for the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$.*

*Proof.* Fix any infinite subset $D \subseteq \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}) \setminus \{\mathscr{O}\}$ and any element $a \in \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}) \setminus \{\mathscr{O}\}$. Since the set $D$ is infinite and the set $\boldsymbol{F}_{\min}^{(i,j)_{\Rsh}}$ is finite for any $i, j \in \omega$, at least one of the following conditions holds:

($i$) there exist infinitely many $i_n \in \omega$ such that $(i_n, k_n, j_n) \in D$ for some $j_n \in \omega$ and $k_n \in \boldsymbol{F}_{\min}$;
($ii$) there exist infinitely many $j_n \in \omega$ such that $(i_n, k_n, j_n) \in D$ for some $i_n \in \omega$ and $k_n \in \boldsymbol{F}_{\min}$.

Both above conditions and the semigroup operation of $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ imply that $\mathscr{O} \in (i,k,j) \cdot D \cup D \cdot (i,k,j)$, which completes the proof of the lemma. □

Let $\mathfrak{S}$ be a class of semitopological semigroups. A semigroup $S \in \mathfrak{S}$ is called $\mathfrak{S}$-*closed*, if $S$ is a closed subsemigroup of any semitopological semigroup $T \in \mathfrak{S}$ which contains $S$ both as a subsemigroup and as a topological space. $\mathscr{HTS}$-closed topological semigroups, where $\mathscr{HTS}$ is the class of Hausdorff topological semigroups, are introduced by Stepp in [20], and there they were called *maximal semigroups*. An algebraic semigroup $S$ is called *algebraically complete in* $\mathfrak{S}$, if $S$ with any Hausdorff topology $\tau$ such that $(S,\tau) \in \mathfrak{S}$ is $\mathfrak{S}$-closed.

By Proposition 10 from [8], every inverse semigroup $S$ with a tight ideal series is algebraically complete in the class of Hausdorff semitopological inverse semigroups with continuous inversion. Hence Theorem 1 and Lemma 4 imply the following theorem.

**Theorem 5.** *Let $\mathscr{F}$ be a family of atomic subsets of $\omega$. Then the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is algebraically complete in the class of Hausdorff semitopological inverse semigroups with continuous inversion.*

The following lemma describes the closure of the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ in a $T_1$-topological semigroup.

**Lemma 5.** *Let $S$ be a $T_1$-topological semigroup which contains the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ as a dense subsemigroup. Then the following conditions hold:*
($i$) *if $S \setminus \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}) \neq \varnothing$ then $x^2 = \mathscr{O}$ for all $x \in S \setminus \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$;*
($ii$) $E(S) = E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$.

*Proof.* ($i$) By Lemma 3 the element $\mathscr{O}$ is the zero of the semigroup $S$. Suppose to the contrary that there exists $x \in S \setminus \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ such that $x^2 = y \neq \mathscr{O}$. Since $S$ is a $T_1$-space there exists an open neighbourhood $U(y)$ of the point $y$ in $S$ such that $\mathscr{O} \notin U(y)$. The continuity of the semigroup operation in $S$ implies that there exists an open neighbourhood $V(x)$ of the point $x$ in the space $S$ such that $V(x) \cdot V(x) \subseteq U(y)$. By Corollary 1 the set $\boldsymbol{F}_{\min}^{(i,j)_{\Rsh}}$ is finite for any $i, j \in \omega$. Since the set $V(x) \cap \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ is infinite, the above arguments and the definition of the semigroup operation in $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ imply that $\mathscr{O} \in V(x) \cdot V(x) \subseteq U(y)$, a contradiction.

Statement ($ii$) follows from ($i$). □

**Lemma 6.** *Let $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ be a Hausdorff topological semigroup with the compact band $E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$. If a Hausdorff topological semigroup $S$ contains $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ as a subsemigroup then $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ is a closed subset of $S$.*

*Proof.* Suppose to the contrary that there exists a Hausdorff topological semigroup $S$ which contains $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ as a non-closed subsemigroup. Since the closure of a subsemigroup of $S$ is again a subsemigroup in $S$ (see [2, page 9]), without loss of generality we may assume that $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ is a dense subsemigroup of $S$ and $S \setminus \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}) \neq \varnothing$. By Lemma 3 the element $\mathscr{O}$ is the zero of $S$.

Fix an arbitrary $x \in S \setminus \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$. By Hausdorffness of $S$ there exist open neighbourhoods $U(x)$ and $U(\mathscr{O})$ of the points $x$ and $\mathscr{O}$ in $S$, respectively, such that $U(x) \cap U(\mathscr{O}) = \varnothing$. Since $x \cdot \mathscr{O} = \mathscr{O} \cdot x = \mathscr{O}$, there exist open neighbourhoods $V(x)$ and $V(\mathscr{O})$ of the points $x$ and $\mathscr{O}$ in the space $S$, respectively, such that

$$V(x) \cdot V(\mathscr{O}) \subseteq U(\mathscr{O}), \qquad V(\mathscr{O}) \cdot V(x) \subseteq U(\mathscr{O}), \qquad V(x) \subseteq U(x) \quad \text{and} \quad V(\mathscr{O}) \subseteq U(\mathscr{O}).$$

The compactness of $E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$ and Proposition 4 imply that the set $E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})) \setminus V(\mathscr{O})$ is finite. Also, by Corollary 1 the set $\boldsymbol{F}_{\min}^{(i,j)_{\Rsh}}$ is finite for any $i, j \in \omega$. Since the set $V(x) \cap \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ is infinite, the above arguments and the definition of the semigroup operation in $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ imply that there exists $(i,k,j) \in V(x)$ such that $(i,k,i) \in V(\mathscr{O})$ or $(j,k,j) \in V(\mathscr{O})$. Therefore, we have that at least one of the following conditions holds:

$$(V(x) \cdot V(\mathscr{O})) \cap V(x) \neq \varnothing, \qquad (V(\mathscr{O}) \cdot V(x)) \cap V(x) \neq \varnothing.$$

Since $V(x) \subseteq U(x)$, this contradicts the assumption $U(x) \cap U(\mathscr{O}) = \varnothing$. The obtained contradiction implies the statement of the lemma. □

Later by $\mathscr{HTS}$ we denote the class of all Hausdorff topological semigroups.

The following lemma shows that the converse statement to Lemma 6 is true in the case when $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ is a topological inverse semigroup.

**Lemma 7.** *Let $(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau)$ be a Hausdorff topological inverse semigroup. If $(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau)$ is an $\mathscr{HTS}$-closed topological semigroup then the band $E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$ is compact.*

*Proof.* Suppose to the contrary that there exists a Hausdorff semigroup inverse topology $\tau$ on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ such that $(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau)$ is an $\mathscr{HTS}$-closed topological semigroup and the band $E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$ is not compact. By Proposition 4 every non-zero element of $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ is an isolated point in $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau\right)$ and hence there exists an open neighbourhood $V(\mathscr{O})$ of the zero $\mathscr{O}$ in the space $\left(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau\right)$ such that $M = E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})) \setminus V(\mathscr{O})$ is an infinite subset of the band $E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$. Since the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ is countable, so is the set $M$. Next we enumerate elements of the set $M$ by positive integers:

$$M = \{(i_n, k_n, i_n) \colon n = 1, 2, 3, \ldots\}.$$

By Corollary 1 the set $\boldsymbol{F}_{\min}^{(i,j)_{\Rsh}}$ is finite for any $i, j \in \omega$, and hence without loss of generality we may assume that $i_m < i_n$ for any positive integers $m < n$. Since $(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau)$ is a topological inverse semigroup the maps $\varphi \colon \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}) \to E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$ and $\psi \colon \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}) \to E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$ defined by the formulae $\varphi(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{x}^{-1}$ and $\psi(\boldsymbol{x}) = \boldsymbol{x}^{-1} \cdot \boldsymbol{x}$, respectively, are continuous, and hence $\mathcal{I}_M = \varphi^{-1}(M) \cup \psi^{-1}(M)$ is a closed subset in the topological space $(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau)$.

Let $\boldsymbol{y} \notin \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$. Put $S = \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}) \cup \{\boldsymbol{y}\}$. We extend the semigroup operation from $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ onto $S$ as follows:

$$\boldsymbol{y} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x} = \boldsymbol{x} \cdot \boldsymbol{y} = \mathscr{O}, \qquad \text{for all} \quad \boldsymbol{x} \in \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}).$$

Simple verifications show that so extended binary operation is associative.

We put

$$M_n = \{(i_{2j-1}, k_{2j-1}, i_{2j}) \colon j = n, n+1, n+2, \ldots\}$$

for any positive integer $n$. We define a topology $\tau_S$ on $S$ in the following way:

(i) for every $\boldsymbol{x} \in \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ the bases of topologies $\tau$ and $\tau_S$ at the point $\boldsymbol{x}$ coincide; and

(ii) the family $\mathscr{B} = \{U_n(\boldsymbol{y}) = \{\boldsymbol{y}\} \cup M_n : n = 1, 2, 3, \ldots\}$ is the base of the topology $\tau_S$ at the point $\boldsymbol{y}$.

Since $M_n \subset \mathcal{I}_M$ for any positive integer $n$, $\tau_S$ is a Hausdorff topology on $S$.

For any open neighbourhood $V(\mathscr{O})$ of the zero $\mathscr{O}$ such that $V(\mathscr{O}) \subseteq U(\mathscr{O})$ and any positive integer $n$ we have that

$$V(\mathscr{O}) \cdot U_n(\boldsymbol{y}) = U_n(\boldsymbol{y}) \cdot V(\mathscr{O}) = U_n(\boldsymbol{y}) \cdot U_n(\boldsymbol{y}) = \{\mathscr{O}\} \subseteq V(\mathscr{O}).$$

We remark that the definition of the set $M_n$ implies that for any non-zero element $(i,k,j)$ of the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ there exists the smallest positive integer $n_{(i,k,j)}$ such that

$$(i,k,j) \cdot M_{n_{(i,k,j)}} = M_{n_{(i,k,j)}} \cdot (i,k,j) = \{\mathscr{O}\}.$$

This implies that

$$(i,k,j) \cdot U_{n_{(i,k,j)}}(\boldsymbol{y}) = U_{n_{(i,k,j)}}(\boldsymbol{y}) \cdot (i,k,j) = \{\mathscr{O}\} \subseteq V(\mathscr{O}).$$

Therefore $(S, \tau_S)$ is a Hausdorff topological semigroup which contains $(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau)$ as a proper dense subsemigroup, which contradicts the assumption of the lemma. The obtained contradiction implies that the band $E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$ is compact. □

The proof of Lemma 7 implies Proposition 6, which gives the sufficient conditions on the topological semigroup $(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau)$ to be non-$\mathscr{HTS}$-closed.

**Proposition 6.** *Let $\tau$ be a semigroup topology on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$. Let $\varphi \colon \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}) \to E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$ and $\psi \colon \mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}) \to E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$ be the maps which are defined by the formulae $\varphi(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{x}^{-1}$ and $\psi(\boldsymbol{x}) = \boldsymbol{x}^{-1} \cdot \boldsymbol{x}$. If there exists an open neighbourhood $U(\mathscr{O})$ of zero in $(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau)$ such that*

$$\left(\varphi^{-1}(M) \cup \psi^{-1}(M)\right) \cap U(\mathscr{O}) = \varnothing$$

*for some infinite subset $M$ of the band $E(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}))$, then $(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau)$ is not an $\mathscr{HTS}$-closed topological semigroup.*

Theorem 1 and Lemmas 6, 7 imply

**Theorem 6.** *Let $\mathscr{F}$ be a some family of atomic subsets of $\omega$. Then a Hausdorff topological semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with the compact band is an $\mathscr{HTS}$-closed topological semigroup. Moreover, a Hausdorff topological inverse semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is an $\mathscr{HTS}$-closed topological semigroup if and only the band $E(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ is compact.*

Example 2 and Proposition 7 imply that the converse statement to Lemma 6 (and hence to the first statement of Theorem 1) is not true.

**Example 2.** For any positive integer $n$ we denote

$$U_n(\mathscr{O}) = \{\mathscr{O}\} \cup \bigcup \left\{\boldsymbol{F}_{\min}^{(i,j)\Rsh} : n \leqslant i < j\right\}.$$

We define a topology $\tau_1$ on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ in the following way:

(i) any non-zero element of the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min})$ is an isolated point in $(\mathscr{B}_{\omega}^{\Rsh}(\boldsymbol{F}_{\min}), \tau_1)$;

(*ii*) the family $\mathscr{B}_1(\mathscr{O}) = \{U_n(\mathscr{O})\colon n \in \omega\}$ is the base of the topology $\tau_1$ at the zero $\mathscr{O}$.

It is obvious that $(\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}), \tau_1)$ is a Hausdorff topological space.

**Proposition 7.** *$(\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}), \tau_1)$ is an $\mathscr{H}\mathscr{T}\mathscr{S}$-closed topological semigroup.*

*Proof.* First we show that the semigroup operation is continuous in $(\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}), \tau_1)$. Since every non-zero element of the semigroup $(\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}), \tau_1)$ is an isolated point, it is complete to show that the semigroup operation in $(\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}), \tau_1)$ is continuous at zero. Fix an arbitrary $(i,k,j) \in \mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}) \setminus \{\mathscr{O}\}$. Then for $n = \max\{i,j\}+1$ we have that

$$(i,k,j)\cdot U_n(\mathscr{O}) = U_n(\mathscr{O})\cdot(i,k,j) = \{\mathscr{O}\} \subset U_n(\mathscr{O}).$$

Also for any $n \in \omega$ we have that

$$U_n(\mathscr{O})\cdot U_n(\mathscr{O}) \subseteq U_n(\mathscr{O}).$$

Therefore $(\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}), \tau_1)$ is a topological semigroup.

Suppose to the contrary that there exists a Hausdorff topological semigroup $S$ which contains $(\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}), \tau_1)$ as a non-closed subsemigroup. Since the closure of a subsemigroup in a topological semigroup is a subsemigroup (see [2, page 9]), without loss of generality we can assume that $(\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}), \tau_1)$ is a dense proper subsemigroup of $S$.

Fix an arbitrary $\boldsymbol{x} \in S \setminus \mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min})$. By Lemmas 3 and 5 we have that

$$\boldsymbol{x}\cdot\boldsymbol{x} = \boldsymbol{x}\cdot\mathscr{O} = \mathscr{O}\cdot\boldsymbol{x} = \mathscr{O}.$$

Fix any positive integer $n$. Let $W(\mathscr{O})$ be an open neighbourhood of zero $\mathscr{O}$ in $S$ such that $W(\mathscr{O})\cap\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}) = U_n(\mathscr{O})$. The continuity of the semigroup operation in $S$ implies that there exist open neighbourhoods $V(\boldsymbol{x})$, $V(\mathscr{O})$ and $U(\mathscr{O})$ of the points $\boldsymbol{x}$ and $\mathscr{O}$ in the space $S$, respectively, such that

$$V(\boldsymbol{x})\cdot V(\mathscr{O}) \subseteq U(\mathscr{O}), \quad V(\mathscr{O})\cdot V(\boldsymbol{x}) \subseteq U(\mathscr{O}), \quad V(\boldsymbol{x})\cdot V(\boldsymbol{x}) \subseteq U(\mathscr{O}),$$

$$V(\boldsymbol{x})\cap U(\mathscr{O}) = \varnothing \quad \text{and} \quad V(\mathscr{O}) \subseteq U(\mathscr{O}) \subseteq W(\mathscr{O}).$$

Theorem 9 of [21] implies that $E(\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}))$ is a closed subset of $S$. Hence, we may assume that $V(\boldsymbol{x}) \cap E(\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min})) = \varnothing$, and moreover $U(\mathscr{O})\cap\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}) = U_m(\mathscr{O})$ and $V(\mathscr{O})\cap\mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}) = U_l(\mathscr{O})$ for some positive integers $l$ and $m$ such that $l \geqslant m \geqslant n$.

Then conditions

$$V(\boldsymbol{x})\cdot V(\mathscr{O}) \subseteq U(\mathscr{O}) \qquad \text{and} \qquad V(\boldsymbol{x})\cap U(\mathscr{O}) = \varnothing$$

imply that there exists on open neighbourhood $V_1(\boldsymbol{x}) \subseteq V(\boldsymbol{x})$ of the point $\boldsymbol{x}$ in the space $S$ such that

$$V_1(\boldsymbol{x}) \cap \left(\bigcup\left\{\boldsymbol{F}_{\min}^{(i,s)_{\Rsh}} : s \in \omega\right\}\right) = \varnothing$$

for any non-negative integer $i < m$. This and Theorem 9 of [21] imply that there exists an open neighbourhood $V_2(\boldsymbol{x}) \subseteq V(\boldsymbol{x})$ of the point $\boldsymbol{x}$ in $S$ such that

$$V_2(\boldsymbol{x}) \cap \mathscr{B}_\omega^{\Rsh}(\boldsymbol{F}_{\min}) \subseteq \bigcup\left\{\boldsymbol{F}_{\min}^{(i,j)_{\Rsh}} : i > j,\ i,j \in \omega\right\}.$$

Hence there exists an infinite sequence $\{(i_p,k_p,j_p)\}_{p\in\omega}$ in $V_2(\boldsymbol{x})$ such that the sequence $\{i_p\}_{p\in\omega}$ is increasing and $j_p \leqslant i_p - 1$ for any $p \in \omega$. The definition of the topology $\tau_1$

implies that there exists an element $(i_{p_0}, k_{p_0}, j_{p_0})$ of the sequence $\{(i_p, k_p, j_p)\}_{p\in\omega}$ such that

$$\boldsymbol{F}_{\min}^{(i_{p_0}-1, i_{p_0})_{\Gamma}} \subseteq U_l(\mathscr{O}) \subseteq V(\mathscr{O}).$$

Then we have that

$$\boldsymbol{F}_{\min}^{(i_{p_0}-1, i_{p_0})_{\Gamma}} \cdot (i_{p_0}, k_{p_0}, j_{p_0}) \subseteq \boldsymbol{F}_{\min}^{(i_{p_0}-1, j_{p_0})_{\Gamma}} \nsubseteq U_m(\mathscr{O}),$$

which contradicts the inclusion $V(\mathscr{O}) \cdot V(\boldsymbol{x}) \subseteq U(\mathscr{O})$. The obtained contradiction implies that $\boldsymbol{x}$ is not an accumulation point of $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$ in the topological space $S$, and hence the statement of the proposition holds. □

## Acknowledgements

The authors acknowledge the referee for their comments and suggestions.

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*Стаття: надійшла до редколегії 22.11.2021*
*доопрацьована 25.12.2021*
*прийнята до друку 05.01.2022*

## ПРО НАПІВГРУПУ $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, ПОРОДЖЕНУ СІМ'ЄЮ $\mathscr{F}$ АТОМАРНИХ ПІДМНОЖИН В $\omega$

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Вивчаємо напівгрупу $\boldsymbol{B}_{\omega}^{\mathscr{F}}$, яка побудована в праці [9], у випадку коли сім'я $\mathscr{F}$ складається з порожньої множини та деяких одноелементних підмножин у $\omega$. Доводимо, що напівгрупа $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ ізоморфна піднапівгрупі $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$ $\omega$-розширення Брандта напівґратки $\boldsymbol{F}_{\min}$ й описуємо усі трансляційно неперервні слабко компактні $T_1$-топології на напівгрупі $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$. Зокрема, доводимо, що кожна трансляційно неперервна слабко компактна $T_1$-топологія $\tau$ на $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$ компактна, і більше того, у цьому випадку простір $(\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min}),\tau)$ гомеоморфний одноточковій компактифікації Алєксандрова дискретного зліченного простору $\mathfrak{D}(\omega)$. Вивчаємо замикання напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ в напівтопологічній напівгрупі. Зокрема доводимо, що напівгрупа $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ алгебрично повна в класі гаусдорфових напівтопологічних інверсних напівгруп з неперервною інверсією, і гаусдофова топологічна інверсна напівгрупа $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ є замкненою в кожній гаусдорфовій топологічній напівгрупі тоді і лише тоді, коли в'язка $E(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ компактна.

*Ключові слова:* напівтопологічна напівгрупа, топологічна напівгрупа, біциклічний моноїд, інверсна напівгрупа, слабко комавктний, компактний, $\omega$-розширення Брандта, замикання.