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E-SEPARATED SEMIGROUPS

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A semigroup is called *E-separated* if for any distinct idempotents $x, y \in X$ there exists a homomorphism $h : X \to Y$ to a semilattice Y such that $h(x) \neq h(y)$. Developing results of Putcha and Weissglass, we characterize E-separated semigroups via certain commutativity properties of idempotents of X. Also we characterize E-separated semigroups in the class of π-regular E-semigroups.

Key words: E-central semigroup, the least semilattice congruence, the binary quasiorder.

1. Introduction

In this paper we introduce and study E-separated E-semigroups. A semigroup X is defined to be *E-separated* if for any distinct idempotents $x, y \in X$ there exists a homomorphism $h : X \to Y$ to a semilattice Y such that $h(x) \neq h(y)$. We recall that a *semilattice* is a commutative semigroup of idempotents. An element x of a semigroup X is an *idempotent* if $xx = x$. A semigroup X is called an *E-semigroup* if the set $E(X) \overset{\text{def}}{=} \{x \in X : xx = x\}$ is a subsemigroup of X.

Developing results of Putcha and Weissglass [19], in Theorem 5 we characterize E-separated semigroup via suitable commutativity properties of the idempotents of the semigroup.

In Proposition 8 we prove that the class of E-separated E-semigroups contains all duo semigroups (and hence all commutative semigroups). A semigroup X is called *duo* if $xX = Xx$ for every $x \in X$. It is clear that each commutative semigroup is duo. In Theorem 6 we establish some structural properties of E-separated E-semigroups. In particular, we distinguish a natural subsemigroup $\Updownarrow E(X)$ of X that admits homomorphic

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2020 Mathematics Subject Classification: 20M10, 20M15, 18B40 retractions onto the semilattice $E(X)$ and also on the Clifford part $H(X) \stackrel{\text{def}}{=} \bigcup_{e\in E(X)} H_e$ of X.

In Theorem 7 we characterize E-separated semigroups within the class of π-regular E-semigroups.

The main instrument for studying E-separated semigroups is the binary quasiorder whose properties are discussed in Section 2.

2. Preliminaries

In this section we collect some standard notions that will be used in the paper. We refer to [10] for Fundamentals of Semigroup Theory.

We denote by ω the set of all finite ordinals and by $\mathbb{N} \stackrel{\text{def}}{=} \omega\setminus\{0\}$ the set of all positive integer numbers.

Let X be a semigroup. For an element $x \in X$ let

$$x^{\mathbb{N}} \stackrel{\text{def}}{=} \{x^n : n \in \mathbb{N}\}$$

be the monogenic subsemigroup of X, generated by the element x. For two subsets $A, B \subseteq X$, let $AB \stackrel{\text{def}}{=} \{ab : a \in A,\ b \in B\}$ be the product of A, B in X. For a subset $A \subseteq X$, let

$$\sqrt[\mathbb{N}]{A} \stackrel{\text{def}}{=} \bigcup_{n\in\mathbb{N}} \sqrt[n]{A} \quad \text{where} \quad \sqrt[n]{A} \stackrel{\text{def}}{=} \{x \in X : x^n \in A\}.$$

For an element a of a semigroup X, the set

$$H_a = \{x \in X : (xX^1 = aX^1) \ \wedge\ (X^1x = X^1a)\}$$

is called the *$\mathcal{H}$-class* of a. Here $X^1 = X\cup\{1\}$ where 1 is an element such that $1x = x = x1$ for all $x \in X^1$.

By Corollary 2.2.6 [10], for every idempotent $e \in E(X)$ its $\mathcal{H}$-class H_e coincides with the maximal subgroup of X, containing the idempotent e. The union

$$H(X) = \bigcup_{e\in E(X)} H_e$$

of all maximal subgroups of X is called the *Clifford part* of X (it should be mentioned that $H(X)$ is not necessarily a subsemigroup of X).

For any element $x \in H(X)$, there exists a unique element $x^{-1} \in H(X)$ such that

$$xx^{-1}x = x, \quad x^{-1}xx^{-1} = x^{-1}, \quad \text{and} \quad xx^{-1} = x^{-1}x.$$

The set

$$\sqrt[\mathbb{N}]{H(X)} = \bigcup_{e\in E(X)} \sqrt[\mathbb{N}]{H_e}$$

is called the *eventually Clifford part* of X. Let $\pi : \sqrt[\mathbb{N}]{H(X)} \to E(X)$ be the function assigning to each $x \in \sqrt[\mathbb{N}]{H(X)}$ the unique idempotent $e \in E(X)$ such that $x^{\mathbb{N}} \cap H_e \neq \emptyset$. The following lemma shows that the function π is well-defined.

Lemma 1. *Let x be an element of a semigroup X such that $x^n \in H_e$ for some $n \in \mathbb{N}$ and $e \in E(X)$. Then $x^m \in H_e$ for all $m \geq n$.*

Proof. To derive a contradiction, assume that $x^m \notin H_e$ for some $m \geq n$. We can assume that m is the smallest number such that $m \geq n$ and $x^m \notin H_e$. It follows from $x^n \in H_e$ and $x^m \notin H_e$ that $m > n > 1$ and hence $m-2 \in \mathbb{N}$. The minimality of m ensures that $x^{m-1} \in H_e$. Observe that

$$x^m X^1 \subseteq x^{m-1}X^1 = ex^{m-1}X^1 \subseteq eX^1$$

and

$$eX^1 = x^{2(m-1)}(x^{2(m-1)})^{-1}X^1 \subseteq x^{2(m-1)}X^1 = x^m x^{m-2}X^1 \subseteq x^m X^1.$$

Therefore, $x^m X^1 = eX^1$. By analogy one can prove that $X^1 x^m = X^1 e$. Therefore, $x^m \in H_e$, which contradicts the choice of m. □

A semigroup X is called

- *Clifford* if $X = H(X)$;
- *eventually Clifford* if $X = \sqrt[\mathbb{N}]{H(X)}$.

In fact, the class of (eventually) Clifford semigroups coincides with the class of completely (π-)regular semigroups, considered in [16] (and [7], [11], [17]).

Let us recall that a semigroup X is defined to be

- (*completely*) *regular* if for every $x \in X$ there exists $y \in X$ such that $x = xyx$ (and $xy = yx$);
- (*completely*) *π-regular* if for every $x \in X$ there exist $n \in \mathbb{N}$ and $y \in X$ such that $x^n = x^n y x^n$ (and $x^n y = yx^n$).

Each semilattice X carries the *natural partial order* $\leq$ defined by $x \leq y$ iff

$$xy = y = yx.$$

Let $\mathbb{2}$ denote the set $\{0,1\}$ endowed with the operation of multiplication inherited from the ring $\mathbb{Z}$. It is clear that $\mathbb{2}$ is a two-element semilattice, so it carries the natural partial order, which coincides with the linear order inherited from $\mathbb{Z}$.

For elements x, y of a semigroup X we write $x \lesssim y$ if $\chi(x) \leq \chi(y)$ for every homomorphism $\chi : X \to \mathbb{2}$. The relation $\lesssim$ is a quasiorder, called the *binary quasiorder* on X, see [2]. The obvious order properties of the semilattice $\mathbb{2}$ imply the following (obvious) properties of the binary quasiorder on X.

Proposition 1. *For any semigroup X and any elements $x, y, a \in X$, the following statements hold:*

(1) *if $x \lesssim y$, then $ax \lesssim ay$ and $xa \lesssim ya$;*
(2) *$xy \lesssim yx \lesssim xy$;*
(3) *$x \lesssim x^2 \lesssim x$;*
(4) *$xy \lesssim x$ and $xy \lesssim y$.*

For an element a of a semigroup X and subset $A \subseteq X$, consider the following sets:

$$\Uparrow a \stackrel{\text{def}}{=} \{x \in X : a \lesssim x\}, \quad \Downarrow a \stackrel{\text{def}}{=} \{x \in X : x \lesssim a\}, \quad \text{and} \quad \Updownarrow a \stackrel{\text{def}}{=} \{x \in X : a \lesssim x \lesssim a\},$$

called the *upper $\mathbb{2}$-class*, *lower $\mathbb{2}$-class* and the *$\mathbb{2}$-class* of x, respectively. Proposition 1 implies that those three classes are subsemigroups of X.

The following simple fact follows from the definition of the class $\Updownarrow x$.

Proposition 2. *For every idempotent e of a semigroup X we have $\sqrt[\mathbb{N}]{H_e} \subseteq \Updownarrow e$.*

For two elements x, y of a semigroup X, we write $x \Updownarrow y$ iff ${\Updownarrow}x = {\Updownarrow}y$ iff $\chi(x) = \chi(y)$ for any homomorphism $\chi : X \to \mathbb{2}$. Proposition 1 implies that $\Updownarrow$ is a congruence on X.

We recall that a *congruence* on a semigroup X is an equivalence relation $\approx$ on X such that for any elements $x \approx y$ of X and any $a \in X$ we have $ax \approx ay$ and $xa \approx ya$. For any congruence $\approx$ on a semigroup X, the quotient set $X/_\approx$ has a unique semigroup structure such that the quotient map $X \to X/_\approx$ is a semigroup homomorphism. The semigroup $X/_\approx$ is called the *quotient semigroup* of X by the congruence $\approx$.

A congruence $\approx$ on a semigroup X is called a *semilattice congruence* if the quotient semigroup $X/_\approx$ is a semilattice. Proposition 1 implies that $\Updownarrow$ is a semilattice congruence on X. Moreover, $\Updownarrow$ is equal to the smallest semilattice congruence on X, see [2], [14], [15], [22]. The quotient semigroup $X/_\Updownarrow$ is called the *semilattice reflection* of X. More information on the smallest semilattice congruence and semilattice decompositions of semigroups can be found in [18], [8], [11], [12], [20].

A semigroup X is called $\mathbb{2}$*-trivial* if every homomorphism $h : X \to \mathbb{2}$ is constant. Tamura [22], [23] called $\mathbb{2}$-trivial semigroups *semilattice-indecomposable* (or briefy *s-indecomposable*) semigroups. The following fundamental fact was first proved by Tamura [21] and then reproved by another methods in [25], [14], [15], and [2].

Theorem 1 (Tamura). *For every element x of a semigroup X its $\mathbb{2}$-class ${\Updownarrow}x$ is a $\mathbb{2}$-trivial semigroup.*

The binary quasiorder admits an inner description via prime (co)ideals, which was first noticed by Petrich [15] and Tamura [22].

A subset I of a semigroup X is called

- an *ideal* if $(IX) \cup (XI) \subseteq I$;
- a *prime ideal* if I is an ideal such that $X \setminus I$ is a subsemigroup of X;
- a (*prime*) *coideal* if the complement $X \setminus I$ is a (prime) ideal in X.

According to this definition, the sets $\emptyset$ and X are prime (co)ideals in X.

Observe that a subset A of a semigroup X is a prime coideal in X if and only if its *characteristic function*

$$\chi_A : X \to \mathbb{2}, \quad \chi_A : x \mapsto \chi_A(x) \overset{\text{def}}{=} \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

is a homomorphism. This function characterization of prime coideals implies the following inner description of the $\mathbb{2}$-quasiorder, first noticed by Tamura in [22].

Proposition 3. *For any element x of a semigroup X, its upper $\mathbb{2}$-class ${\Uparrow}x$ coincides with the smallest coideal of X that contains x.*

Corollary 1. *A semigroup X is $\mathbb{2}$-trivial if and only if every nonempty prime ideal in X coincides with X.*

Remark 1. By [1], [9] (see also [5], [6], [3], [4]), $\mathbb{2}$-trivial semigroups can contain non-trivial ideals, in particular, there exist infinite congruence-free (and hence $\mathbb{2}$-trivial) monoids with zero.

The following inner description of the upper $\mathbb{2}$-classes is a modified version of Theorem 3.3 in [15]. Its proof can be found in [2].

Proposition 4. *For any element x of a semigroup X its upper $\mathsf{2}$-class $\Uparrow x$ is equal to the union $\bigcup_{n\in\omega}\Uparrow_n x$, where $\Uparrow_0 x=\{x\}$ and*

$$\Uparrow_{n+1}x \overset{\mathsf{def}}{=} \{y\in X : X^1yX^1\cap(\Uparrow_n x)^2\ne\emptyset\}$$

for $n\in\omega$.

For duo semigroups, Proposition 4 simplifies to the following form, proved in [2].

Proposition 5. *For any element $a\in X$ of a duo semigroup X we have*

$$\Uparrow a=\{x\in X : a^{\mathbb{N}}\cap XxX\ne\emptyset\}.$$

A semigroup X is called *Archimedean* if for any elements $x,y\in X$ there exists $n\in\mathbb{N}$ such that $x^n\in XyX$ for some $a,b\in X$. A standard example of an Archimedean semigroup is the additive semigroup $\mathbb{N}$ of positive integers. For commutative semigroups the following characterization (that can be easily derived from Proposition 5) was obtained by Tamura and Kimura in [24].

Theorem 2. *A duo semigroup X is $\mathsf{2}$-trivial if and only if X is Archimedean.*

For viable semigroups we have another simplification of Proposition 4 due to Putcha and Weissglass [19]. Let us recall that a semigroup X is called *viable* if for any $x,y\in X$ with $\{xy,yx\}\subseteq E(X)$ we have $xy=yx$.

Proposition 6 (Putcha–Weissglass)**.** *If X is a viable semigroup, then for every idempotent $e\in E(X)$ we have*

$$\Uparrow e=\{x\in X : e\in X^1xX^1\}.$$

Proof. Let

$$\Uparrow_1 e \overset{\mathsf{def}}{=} \{x\in X : e\in X^1xX^1\}.$$

By Proposition 4, $\Uparrow_1 e\subseteq\Uparrow e$. The reverse inclusion will follow from the minimality of the prime coideal $\Uparrow e$ as soon as we prove that $\Uparrow_1 e$ is a prime coideal in X. It is clear from the definition that $\Uparrow_1 e$ is a coideal. So, it remains to check that $\Uparrow_1 e$ is a subsemigroup. Given any elements $x,y\in\Uparrow_1 e$, find elements $a,b,c,d\in X^1$ such that $axb=e=cyd$. Then $axbe=ee=e$ and

$$(beax)(beax)=be(axbe)ax=beeax=beax,$$

which means that $beax$ is an idempotent. By the viability of X, $axbe=e=beax$. By analogy we can prove that $ecyd=e=ydec$. Then $beaxydex=ee=e$ and hence $xy\in\Uparrow_1 e$. □

Following Tamura [23], we define a semigroup X to be *unipotent* if X contains a unique idempotent. The following fundamental result was proved by Tamura [23] and reproved by a different method in [2].

Theorem 3 (Tamura, 1982)**.** *For the unique idempotent e of an unipotent $\mathsf{2}$-trivial semigroup X, the maximal group H_e of e in X is an ideal in X.*

An element of a semigroup X is called *central* if it belongs to the *center*

$$Z(X) \stackrel{\text{def}}{=} \{z \in X : \forall x \in X \ \ (zx = xz)\}$$

of the semigroup X.

Corollary 2. *The unique idempotent e of a unipotent $\mathcal{D}$-trivial semigroup X is central in X.*

Proof. Let e be a unique idempotent of the unipotent semigroup X. By Tamura's Theorem 3, the maximal subgroup H_e is an ideal in X. Then for every $x \in X$ we have $xe, ex \in H_e$. Taking into account that xe and ex are elements of the group H_e, we conclude that $ex = exe = xe$. This means that the idempotent e is central in X. □

For any idempotent e of a semigroup X, let

$$\tfrac{H_e}{e} \stackrel{\text{def}}{=} \{x \in X : xe = ex \in H_e\}.$$

The set $\frac{H_e}{e}$ is a subsemigroup of X. Indeed, for any $x, y \in \frac{H_e}{e}$ we have

$$xye = xyee = x(ey)e = (xe)(ye) \in H_eH_e = H_e$$

and

$$exy = eexy = e(xe)y = (ex)(ey) \in H_eH_e = H_e,$$

which implies that $xy \in \frac{H_e}{e}$.

The following theorem nicely complements Theorem 3 and Corollary 2.

Theorem 4. *For any idempotent e we have*

$$\sqrt[\mathbb{N}]{H_e} \subseteq \tfrac{H_e}{e} \subseteq {\Uparrow}e.$$

Доведення. Take any element $x \in \sqrt[\mathbb{N}]{H_e}$. Since $x \in \sqrt[\mathbb{N}]{H_e}$, there exists $n \in \mathbb{N}$ such that $x^n \in H_e$ and hence $x^{2n} \in H_e$. Observe that

$$xeX^1 = xx^nX^1 \subseteq x^nX^1 = eX^1$$

and

$$eX^1 = x^{2n}X^1 \subseteq x^{n+1}X^1 = xeX^1$$

and hence $xeX^1 = eX^1$. By analogy we can prove that $X^1xe = X^1e$. Then $xe \in H_e$ by the definition of the $\mathcal{H}$-class H_e.

By analogy we can prove that $ex \in H_e$. It follows from $xe, ex \in H_e$ that

$$ex = exe = ex \in H_e$$

and hence $x \in \frac{H_e}{e}$.

By Proposition 4,

$$\tfrac{H_e}{e} \subseteq \{x \in X : e \in xH_e \cap H_ex\} \subseteq \{x \in X : e \in X^1xX^1\} \subseteq {\Uparrow}e.$$

□

An idempotent e of a semigroup X is defined to be *viable* if the semigroup $\frac{H_e}{e}$ is a coideal in X.

Proposition 7. *An idempotent e of a semigroup X is viable if and only if $\frac{H_e}{e} = {\Uparrow}e$. In this case H_e is an ideal of the semigroup ${\Uparrow}e$ and $e \in Z({\Uparrow}e)$.*

Proof. If e is viable, then semigroup $\frac{H_e}{e}$ is a prime coideal in X and hence $\Uparrow e \subseteq \frac{H_e}{e}$ as $\Uparrow e$ is the smallest prime coideal containing e, see Proposition 3. Then $\frac{H_e}{e} = \Uparrow e$ by Theorem 4.

If $\frac{H_e}{e} = \Uparrow e$, then e is viable because $\Uparrow e = \frac{H_e}{e}$ is a coideal in X.

Also H_e is an ideal in $\frac{H_e}{e}$ and $e \in Z(\frac{H_e}{e})$ by the definition of $\frac{H_e}{e}$. □

3. Characterizing E-separated semigroups

In this section we find several commutativity properties of semigroups, which are equivalent to the E-separatedness.

Definition 1. A semigroup X is defined to be

- *E-commutative* if $xy = yx$ for any idempotents $x, y \in E(X)$;
- *E-viable* if every idempotent of X is viable;
- *E-central* if for any $e \in E(X)$ and $x \in X$ we have $ex = xe$;
- *$E_{\Uparrow}$-central* if for any $e \in E(X)$ and $x \in \Uparrow e$ we have $ex = xe$;
- *E-hypercentral* if for any $e \in E(X)$ and $x, y \in X$ with $xy = e$ we have $xe = ex$ and $ye = ey$;
- *E-hypocentral* if for any $e \in E(X)$ and $x, y \in X$ with $xy = e$ we have $xe = ex$ or $ye = ey$;
- *E-upcentral* if for any idempotents $e, f \in E(X)$ with $fe = e = ef$ and any $x \in \sqrt[\mathbb{N}]{H_f}$ we have $xe = ex$.

For any semigroup these commutativity properties relate as follows.

$$\begin{array}{ccccccccc}
E\text{-commutative} & \Longleftarrow & E\text{-central} & \Longrightarrow & E_{\Uparrow}\text{-central} & \Longleftrightarrow & E\text{-separated} & \Longrightarrow & E\text{-upcentral} \\
\Downarrow & & \Downarrow & & \Updownarrow & & \Updownarrow & & \\
E\text{-semigroup} & & \text{viable} & \Longleftrightarrow & E\text{-viable} & \Longleftrightarrow & E\text{-hypercentral} & \Longrightarrow & E\text{-hypocentral}
\end{array}$$

Nontrivial equivalences and implications in this diagram are proved in the following theorem.

Theorem 5. *For a semigroup X the following conditions are equivalent:*

(1) *X is E-separated;*
(2) *X is E-viable;*
(3) *X is $E_{\Uparrow}$-central;*
(4) *X is E-hypercentral;*
(5) *X is viable.*

The equivalent conditions (1)–(5) *imply the condition*

(6) *X is E-hypocentral and E-upcentral.*

Proof. We shall prove the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (1) and (4) ⇒ (6).

(1) ⇒ (2) Assume that X is E-separated. To show that X is E-viable, take any $e \in E(X)$ and $x \in \Uparrow e$. Since X is E-separated, the $\mathbb{2}$-class $\Updownarrow e$ of e is unipotent. By Tamura's Theorem 3, the group H_e is an ideal in $\Updownarrow e$. Since $\Updownarrow e$ is an ideal in $\Uparrow e$, the maximal subgroup H_e is an ideal in $\Uparrow e$. Then $xe, ex \in H_e$ and hence $xe = exe = ex \in H_e$

and $x \in \frac{H_e}{e}$. So ${\Uparrow}e \subseteq \frac{H_e}{e}$ and ${\Uparrow}e = \frac{H_e}{e}$ by Theorem 4. Then $\frac{H_e}{e} = {\Uparrow}e$ is a coideal in X and the idempotent e is viable, witnessing that the semigroup X is E-viable.

The implication (2) $\Rightarrow$ (3) follows from Proposition 7.

(3) $\Rightarrow$ (4) Assume that X is $E_{\Uparrow}$-central. To show that X is E-hypercentral, take any idempotent $e \in E(X)$ and any elements $x, y \in X$ with $xy = e$. Proposition 1 ensures that $e \lesssim x$ and $e \lesssim y$ and hence $x, y \in {\Uparrow}e$. Applying the $E_{\Uparrow}$-centrality of X, we conclude that $ex = xe$ and $ey = ye$.

(4) $\Rightarrow$ (5) Assume that X is E-hypercentral. To show that X is viable, take any elements $x, y \in X$ such that $\{xy, yx\} \subseteq E(X)$. The E-hypercentrality of X ensures that

$$xy = xyxy = x(yx)y = (yx)xy = yx(xy) = y(xy)x = yxyx = yx.$$

(5) $\Rightarrow$ (1) To derive a contradiction, assume that X is viable but not E-separated. Then there exist two distinct idempotents $e, f \in E(X)$ such that ${\Uparrow}e = {\Uparrow}f$. By Proposition 6, there are elements $a, b, c, d \in X^1$ such that $e = afb$ and $f = ced$. Observe that $afbe = ee = e$ and

$$(beaf)(beaf) = be(afbe)af = beeaf = beaf$$

and hence $afbe$ and $beaf$ are idempotents. The viability of X ensures that $afbe = beaf$. By analogy we can prove that $eafb = e = fbea$, $cedf = f = dfce$ and $fced = f = edfc$. These equalities imply that $H_e = H_f$ and hence $e = f$ because the group $H_e = H_f$ contains a unique idempotent. But the equality $e = f$ contradicts the choice of the idempotents e, f.

(4) $\Rightarrow$ (6) Assume that X is E-hypercentral. Then X is E-hypocentral. To show that X is E-upcentral, take any idempotents $e, f \in E(X)$ and any element $x \in \sqrt[\mathbb{N}]{H_f}$ such that $fe = e = ef$. By Lemma 1, there exists a number $n \geq 2$ such that $x^n \in H_f$. Let g be the inverse element to x^n in the group H_f. Then

$$e = fe = x^n ge = x(x^{n-1}ge).$$

The E-hypercentrality of X ensures $ex = xe$. □

Remark 2. Viable semigroups were introduced and studied by Putcha and Weissglass who proved in [19, Theorem 6] that a semigroup X is viable if and only if it is E-separated (this is the equivalence (1) $\Leftrightarrow$ (5) in Theorem 5). For another condition (involving $\mathscr{J}$-classes), equivalent to the conditions (1)–(5) of Theorem 5, see Theorem 23.7 in [13].

Example 1. Any semigroup X with left zero multiplication $xy = x$ is E-hypocentral and E-upcentral. If X contains more than one element, then X is not E-hypercentral. This example shows that condition (6) of Theorem 5 is not equivalent to conditions (1)–(5).

Remark 3. By [1], [9], there exists an infinite 0-simple congruence-free monoid X. Being congruence-free, the semigroup X is $\mathfrak{D}$-trivial. On the other hand, X contains at least two central idempotents: 0 and 1. The polycyclic monoids (see [3], [4], [5], [6]) have the similar properties. By Theorem 2.4 in [3], for any cardinal $\lambda \geq 2$ the polycyclic monoid P_λ is congruence-free and hence $\mathfrak{D}$-trivial, but its contains two distinct central idempotents 0 and 1. These examples show that individual central idempotents are not necessarily viable. On the other hand, if all idempotents of a semigroup are central, then all of them are viable, by Theorem 5.

4. E-SEPARATED E-SEMIGROUPS

In this section we establish some results on the structure of E-separated E-semigroups. But first we show that the class of such semigroups contains all duo semigroups and hence all commutative semigroups. Let us recall that a semigroup X is *duo* if $Xx = xX$ for all $x \in X$.

Proposition 8. *Each duo semigroup X is an E-separated E-semigroup.*

Proof. First we show that X is an E-semigroup. Given two idempotents e, f, use the duo property of X to find elements $x, y \in X$ such that $ef = xe$ and $fe = yf$. Then

$$efef = eyff = eyf = efe = xee = xe = ef$$

and hence ef is an idempotent. Therefore, X is an E-semigroup.

Assuming that X is not E-separated, we can find an idempotent $e \in E(X)$ whose $\mathfrak{Q}$-class $\Updownarrow e$ contains an idempotent $f \neq e$. By Proposition 5,

$$e \in XfX = XXf \subseteq Xf = fX$$

and

$$f \in XeX = XXe \subseteq Xe = eX.$$

Then $eX^1 \subseteq fXX^1 \subseteq fX^1$, $fX^1 \subseteq eXX^1 \subseteq eX^1$, $X^1e \subseteq X^1Xf \subseteq X^1f$, and $X^1f \subseteq X^1Xe \subseteq X^1e$, which implies $H_f = H_e$ and hence $f = e$ as the group $H_e = H_f$ contains a unique idempotent. □

The following theorem describing properties of E-separated E-semigroups is the main result of this section. The statements (2), (3) of this theorem hold true for any E-separated semigroup.

Theorem 6. *Any E-separated E-semigroup X has the following properties.*

1. *$E(X)$ is a semilattice.*
2. *For any idempotent $e \in E(X)$ the maximal subgroup $H_e \subseteq X$ is an ideal in the semigroup $\Uparrow e$.*
3. *For any $e \in E(X)$ and $x \in \Uparrow e$ we have $ex = xe \in H_e$;*
4. *For any idempotents $x, y \in E(X)$, the inequality $x \lesssim y$ in X is equivalent to the inequality $x \leq y$ in $E(X)$.*
5. *The map $\pi_{\Updownarrow} : \Updownarrow E(X) \to E(X)$ assigning to each element $x \in \Updownarrow E(X)$ the unique idempotent in the semigroup $\Updownarrow x$ is a well-defined homomorphic retraction of the semigroup $\Updownarrow E(X)$ onto $E(X)$.*
6. *The map $\hbar_{\Updownarrow} : \Updownarrow E(X) \to H(X)$, $\hbar_{\Updownarrow} : x \mapsto x\pi_{\Updownarrow}(x)$, is a well-defined homomorphic retraction of the semigroup $\Updownarrow E(X)$ onto the Clifford part $H(X)$ of X.*
7. *The Clifford part $H(X)$ is a subsemigroup of X.*

Proof. Let $X/_{\Updownarrow}$ be the semilattice reflection of X and $q : X \to X/_{\Updownarrow}$ be the quotient homomorphism.

1. To see that $E(X)$ is a semilattice, take any idempotents $x, y \in E(X)$. Since X is an E-semigroup, the products xy and yx are idempotents. Taking into account that $q : X \to X/_{\Updownarrow}$ is a homomorphism onto the semilattice $X/_{\Updownarrow}$, we conclude that

$$q(xy) = q(x)q(y) = q(y)q(x) = q(yx)$$

and hence ${\Updownarrow}xy = {\Updownarrow}yx$. Since the semigroup X is E-separated, the idempotents xy and yx are equal to the unique idempotent of the unipotent semigroup ${\Updownarrow}xy = {\Updownarrow}yx$ and hence $xy = yx$.

2,3. The statements 2 and 3 follow from Theorem 5 and Proposition 7.

4. Let x, y be two idempotents in X. If $x \le y$, then $x = xy$ and hence

$$h(x) = h(x)h(y) \le h(y)$$

for any homomorphism $h : X \to 2$. Then $x \lesssim y$ by the definition of the quasiorder $\lesssim$. Now assume that $x \lesssim y$. Multiplying this inequality by x from both sides and applying Proposition 1, we obtain $x = xx \lesssim xy \lesssim x$ and hence $xy \in {\Updownarrow}x$. Since X is an E-semigroup, the product xy is an idempotent. Since the semigroup X is E-separated, the semigroup ${\Updownarrow}x$ is unipotent and hence the idempotent $xy \in {\Updownarrow}x$ is equal to the unique idempotent x of ${\Updownarrow}x$. By analogy we can prove that $xy = x$. The equality $xy = x = yx$ means that $x \le y$, by the definition of the partial order $\le$ on the semilattice $E(X)$.

5. Consider the map $\pi_{\Updownarrow} : {\Updownarrow}E(X) \to E(X)$ assigning to each element $x \in X$ the unique idempotent in the unipotent semigroup ${\Updownarrow}x$. It is clear that $\pi_{\Updownarrow}$ is a retraction of ${\Updownarrow}E(X)$ onto $E(X)$. Since $\Updownarrow$ is a semilattice congruence, the quotient semigroup $X/_{\Updownarrow}$ is a semilattice and the quotient map $q : X \to X/_{\Updownarrow}$ is a semigroup homomorphism. By the $\Updownarrow$-unipotence of X, the restriction

$$h \overset{\text{def}}{=} q{\restriction}_{E(X)} : E(X) \to q[E(X)] \subseteq X/_{\Updownarrow}$$

is bijective and hence h is a semigroup isomorphism and so is the inverse function $h^{-1} : q[E(X)] \to E(X)$. Then the function $\pi = h^{-1} \circ q{\restriction}_{{\Updownarrow}E(X)}$ is a semigroup homomorphism, being a composition of two homomorphisms.

6. Since the function $\pi_{\Updownarrow} : {\Updownarrow}E(X) \to E(X)$ is well-defined, so is the function $\hbar_{\Updownarrow} : {\Updownarrow}E(X) \to X$, $\hbar_{\Updownarrow} : x \mapsto x\pi_{\Updownarrow}(x)$. To see that $\hbar_{\Updownarrow}$ is a homomorphism, take any elements $x, y \in {\Updownarrow}E(X)$ and applying Theorem 6(5,3), conclude that

$$\begin{aligned}\hbar_{\Updownarrow}(xy) &= xy\pi_{\Updownarrow}(xy) = \\ &= xy\pi_{\Updownarrow}(x)\pi_{\Updownarrow}(y) = \\ &= x\pi_{\Updownarrow}(x)\pi_{\Updownarrow}(y)y = \\ &= x\pi_{\Updownarrow}(x)y\pi_{\Updownarrow}(y) = \\ &= \hbar_{\Updownarrow}(x)\hbar_{\Updownarrow}(y).\end{aligned}$$

By Theorem 3, for any $e \in E(X)$ and $x \in {\Updownarrow}e$, the group H_e is an ideal in ${\Updownarrow}e$ and hence

$$\hbar_{\Updownarrow}(x) = x\pi_{\Updownarrow}(x) = xe \in H_e \subseteq H(X).$$

If $x \in H(X)$, then $x \in H_e$, and hence $\hbar_{\Updownarrow}(x) = xe = x$. Therefore, $\hbar_{\Updownarrow} : {\Updownarrow}E(X) \to H(X)$ is a well-defined homomorphic retraction of ${\Updownarrow}E(X)$ onto $H(X)$.

7. Since $\hbar_{\Updownarrow} : {\Updownarrow}E(X) \to X$ is a homomorphism, its image $H(X) = \hbar_{\Updownarrow}[{\Updownarrow}E(X)]$ is a subsemigroup of X. □

5. Characterizing E-separated π-regular E-semigroups

In this section we recognize E-separated semigroups among π-regular E-semigroups. We recall that a semigroup X is *π-regular* if for every $x \in X$ there exist $n \in \mathbb{N}$ and $y \in X$ such that $x^n = x^n y x^n$. The class of π-regular semigroups includes all eventually Clifford semigroups (called also completely π-regular semigroups). A semigroup X is *eventually Clifford* if $X = \sqrt[\mathbb{N}]{H(X)}$. For any semigroup X by $\pi : \sqrt[\mathbb{N}]{H(X)} \to E(X)$ we denote the function assigning to each $x \in \sqrt[\mathbb{N}]{H(X)}$ the unique idempotent $e \in E(X)$ such that $x^{\mathbb{N}} \cap H_e \neq \emptyset$.

Proposition 9. *If a semigroup X is E-commutative and E-upcentral, then*

(1) *for every $e, f \in E(X)$ we have $H_e H_f \subseteq H_{ef}$;*
(2) *for every idempotents $e, f \in E(X)$ with $e \le f$ we have*
$$(\sqrt[\mathbb{N}]{H_f} \cdot H_e) \cup (H_e \cdot \sqrt[\mathbb{N}]{H_f}) \subseteq H_e;$$
(3) *for every idempotents $e, f \in E(X)$ and every elements $x \in \sqrt[\mathbb{N}]{H_e}$ and $y \in \sqrt[\mathbb{N}]{H_f}$ we have $(xy)^n ef \in H_{ef}$ for all $n \in \mathbb{N}$;*
(4) *for any $x, y \in \sqrt[\mathbb{N}]{H(X)}$ with $xy \in \sqrt[\mathbb{N}]{H(X)}$ we have $\pi(x)\pi(y) \le \pi(xy)$;*
(5) *for any $e \in E(X)$ and $x \in X$ with $\{xe, ex\} \subseteq \sqrt[\mathbb{N}]{H(X)}$, we have $\pi(xe) = \pi(ex)$;*
(6) *for any $e \in E(X)$ and $x \in \sqrt[\mathbb{N}]{H(X)}$ with $xe \in \sqrt[\mathbb{N}]{H(X)}$ we have $\pi(xe) = \pi(x)e$.*

Proof. 1. Let $u \in H_e$ and $v \in H_f$. The E-upcentrality of X ensures that $efu = uef$ and $efv = vef$. Then $efuv = uefv = uv$, $uvef = uefv = uv$,
$$uvv^{-1}u^{-1} = ufu^{-1} = uefu^{-1} = efuu^{-1} = efe = ef$$
and
$$v^{-1}u^{-1}uv = v^{-1}ev = v^{-1}efv = v^{-1}vef = fef = ef.$$
Hence $uv \in H_{ef}$, witnessing that $H_e H_f \subseteq H_{ef}$.

2. For every $e, f \in E(X)$ with $e \le f$ and every $x \in \sqrt[\mathbb{N}]{H_f}$, we have
$$xe = xfe \in \sqrt[\mathbb{N}]{H_f} fe \subseteq H_f e \subseteq H_{fe} = H_e,$$
see Theorem 4 and Proposition 9(1). By analogy we can prove that $ex \in H_e$.

3. Let $e, f \in E(X)$ and $x \in \sqrt[\mathbb{N}]{H_e}$, $y \in \sqrt[\mathbb{N}]{H_f}$ be any elements. By induction we shall prove that $(xy)^n ef \in H_{ef}$ for every $n \in \mathbb{N}$. For $n = 1$ we have
$$xyef = xefy \in \big(\sqrt[\mathbb{N}]{H_e} \cdot H_e\big) \cdot \big(H_f \cdot \sqrt[\mathbb{N}]{H_f}\big) \subseteq H_e H_f \subseteq H_{ef}$$
by the E-upcentrality of X, Theorem 4 and Proposition 9(1). Assume that for some $n \in \mathbb{N}$ we have proved that $(xy)^n ef \in H_{ef}$. Then
$$(xy)^{n+1} ef = xy(xy)^n ef \in xyH_{ef} = xyefH_{ef} \subseteq H_{ef} H_{ef} = H_{ef}$$
by the inductive assumption and case $n = 1$.

4. Take any elements $x, y \in \sqrt[\mathbb{N}]{H(X)}$ with $xy \in \sqrt[\mathbb{N}]{H(X)}$. Since $xy \in \sqrt[\mathbb{N}]{H(X)}$, there exists $n \in \mathbb{N}$ such that $(xy)^n \in H_{\pi(xy)}$. By Proposition 9(1),
$$(xy)^n \pi(x)\pi(y) \in H_{\pi(xy)} H_{\pi(x)} H_{\pi(y)} \subseteq H_{\pi(xy)\pi(x)\pi(y)}.$$

On the other hand, Proposition 9(3) ensures that

$$(xy)^n\pi(x)\pi(y) \in H_{\pi(x)\pi(y)}.$$

Hence $\pi(xy)\pi(x)\pi(y) = \pi(x)\pi(y)$, which means that $\pi(x)\pi(y) \le \pi(xy)$.

5. Take any elements $e \in E(X)$ and $x \in X$ such that $\{xe, ex\} \subseteq \sqrt[\mathbb{N}]{H(X)}$. By Lemma 1, there exists $n \in \mathbb{N}$ such that $(xe)^n \in H_{\pi(xe)}$ and $(ex)^n \in H_{\pi(ex)}$. Then

$$\begin{aligned} H_{\pi(xe)} \ni (xe)^{n+1} &= x(ex)^n e = \\ &= x(ex)^n\pi(ex)e = \\ &= x(ex)^n e\pi(ex) = \\ &= (xe)^{n+1}\pi(ex) \in H_{\pi(xe)}\pi(ex) \subseteq H_{\pi(xe)\pi(ex)} \end{aligned}$$

and hence $\pi(xe) = \pi(xe)\cdot\pi(ex)$. By analogy we can prove that $\pi(ex) = \pi(ex)\cdot\pi(xe)$. Then

$$\pi(xe) = \pi(xe)\pi(ex) = \pi(ex)\pi(xe) = \pi(ex).$$

6. Take any $e \in E(X)$ and $x \in \sqrt[\mathbb{N}]{H(X)}$ with $xe \in \sqrt[\mathbb{N}]{H(X)}$. Find $n \in \mathbb{N}$ such that $\{(xe)^n, x^n\} \subseteq H(X)$. Let $f \overset{\text{def}}{=} \pi(xe)$ and observe that

$$H_f \ni (xe)^n = (xe)^n e \subseteq H_f e \subseteq H_{fe}$$

implies $f = fe$.

By induction we shall prove that $(xf)^k = (xe)^k f$. For $k = 1$ this follows from $f = ef$. Assume that for some $k \in \mathbb{N}$ we have $(xf)^k = (xe)^k f$. By the inductive assumption and Theorem 4,

$$\begin{aligned} (xf)^{k+1} &= (xf)^k xf = \\ &= (xe)^k fxef = \\ &= (xe)^k\pi(xe)xef = \\ &= (xe)^k xe\pi(xe)f = \\ &= (xe)^{k+1}ff = \\ &= (xe)^{k+1}f. \end{aligned}$$

This completes the inductive step and also the proof of the equality $(xf)^k = (xe)^k f$ for all $k \in \mathbb{N}$.

For $k = n$ we obtain

$$(xf)^n = (xe)^n f \in H_{\pi(xe)}f \subseteq H_{\pi(xe)f} = H_f,$$

which implies $xf \in \sqrt[\mathbb{N}]{H_f}$ and $\pi(xf) = f$.

By induction we shall prove that $(xf)^k = x^k f$. For $k = 1$ this is trivial. Assume that for some $k \in \mathbb{N}$ we have proved that $(xf)^k = x^k f$. By the inductive assumption and Theorem 4,

$$(xf)^{k+1} = (xf)^k xf = x^k fxf = x^k\pi(xf)xf = x^k xf\pi(xf) = x^{k+1}ff = x^{k+1}f.$$

This complete the inductive step and also the proof of the equality $(xf)^k = x^k f$ for all $k \in \mathbb{N}$.

The choice of n ensures that $x^n \in H(X)$ and hence $x^n \in H_{\pi(x)}$ and $x^n = x^n\pi(x)$. By Proposition 9(4), $\pi(x)e \leq \pi(xe) = f$ and hence $\pi(x)e = \pi(x)ef$. Then

$$H_{\pi(x)e} \ni x^n e = x^n\pi(x)e = x^n(\pi(x)ef) = (x^n\pi(x))fe = x^n fe = (xf)^n e \in H_f e \subseteq H_{fe}$$

and finally, $\pi(x)e = fe = f = \pi(xe)$. □

Now we are able to prove the main result of this section.

Theorem 7. *For a π-regular E-semigroup X, the following conditions are equivalent:*
(1) *$\Updownarrow e = \sqrt[\mathbb{N}]{H_e}$ for every $e \in E(X)$;*
(2) *X is E-separated;*
(3) *X is E-upcentral, E-hypocentral, and E-commutative.*

Proof. We shall prove the implications (1) ⇒ (2) ⇒ (3) ⇒ (1). Let $q : X \to X/_{\Updownarrow}$, $q : x \mapsto \Updownarrow x$, be the quotient homomorphism of X onto its semilattice reflection.

(1) ⇒ (2) If $\Updownarrow e = \sqrt[\mathbb{N}]{H_e}$ for every $e \in E(X)$, then for every distinct idempotents $e, f \in E(X)$ we have

$$q(e) = \Updownarrow e = \sqrt[\mathbb{N}]{H_e} \neq \sqrt[\mathbb{N}]{H_f} = \Updownarrow f = q(f),$$

which means that the semigroup X is E-separated.

(2) ⇒ (3) If X is E-separated, then X is $E_{\Uparrow}$-central and E-hypocentral by Theorem 5. To see that X is E-commutative, take any idempotents $x, y \in E(X)$. Since X is an E-semigroup, the products xy, yx are idempotents. By Theorem 5, the E-separated semigroup X is viable and hence $xy = yx$.

(3) ⇒ (1) Assume that a π-regular semigroup X is E-upcentral, E-hypocentral, and E-commutative.

Claim 1. *The semigroup X is eventually Clifford.*

Proof. Take any $x \in X$ and using the π-regularity of X, find $n \in \mathbb{N}$ and $y \in X$ such that $x^n = x^n y x^n$. It follows that $e = x^n y$ and $f = yx^n$ are idempotents. Since X is E-hypocentral, $e = x^n y$ implies $x^n e = ex^n$ or $ey = ye$. If $x^n e = ex^n$, then

$$f = ff = (yx^n)(yx^n) = y(x^n y)x^n = yex^n = yx^n e = fe.$$

If $ey = ye$, then

$$f = ff = (yx^n)(yx^n) = y(x^n y)x^n = yex^n = eyx^n = ef = fe.$$

In both cases we obtain $f = fe$.

On the other hand, by the E-hypocentrality of X, the equality $f = yx^n$ implies $fy = yf$ or $fx^n = x^n f$. If $fy = yf$, then

$$e = ee = x^n yx^n y = x^n fy = x^n yf = ef.$$

If $fx^n = x^n f$, then

$$e = ee = x^n yx^n y = x^n fy = fx^n y = fe = ef.$$

In both cases we obtain $e = ef$. Therefore, $e = ef = f$.

Observe that $eX^1 = x^n yX^1 \subseteq x^n X^1$ and

$$x^n X^1 = x^n yx^n X^1 = ex^n X^1 \subseteq eX^1$$

and hence $eX^1 = x^n X^1$. On the other hand,

$$X^1 x^n = X^1 x^n y x^n \subseteq X^1 y x^n = X^1 f = X^1 e$$

and

$$X^1 e = X^1 f = X^1 y x^n \subseteq X^1 x^n$$

and hence $X^1 e = X^1 x^n$. The equalities $eX^1 = x^n X^1$ and $X^1 e = X^1 x^n$ imply $x^n \in H_e$. Then $x \in \sqrt[\mathbb{N}]{H_e} \subseteq \sqrt[\mathbb{N}]{H(X)}$. □

Since the semigroup X is eventually Clifford, the map $\pi : X \to E(X)$ is well-defined on the whole semigroup $X = \sqrt[\mathbb{N}]{H(X)}$.

Claim 2. *For every $e \in E(X)$, the upper $\mathsf{2}$-set $\Uparrow e$ is equal to the set*

$$\Uparrow_\pi e \overset{\text{def}}{=} \{x \in X : e \le \pi(x)\}.$$

Proof. Given any $x \in \Uparrow_\pi e$, find $n \in \mathbb{N}$ such that $x^n \in H_{\pi(x)}$ and conclude that $e \le \pi(x) \Updownarrow x$ implies $x \in \Uparrow e$, by Proposition 1. Therefore, $\Uparrow_\pi e \subseteq \Uparrow e$. The equality $\Uparrow_\pi e = \Uparrow e$ will follow from the minimality of the prime coideal $\Uparrow e$ as soon as we check that the set $\Uparrow_\pi e$ is a prime coideal in X.

By Proposition 9(4), for every $x, y \in \Uparrow_\pi e$ we have

$$e = ee \le \pi(x)\pi(y) \le \pi(xy)$$

and hence $xy \in \Uparrow_\pi e$ and $\Uparrow_\pi e$ is a semigroup. Next, we show that $I \overset{\text{def}}{=} X \setminus \Uparrow_\pi e$ is an ideal in X. Assuming that I is not an ideal, we can find elements $x \in I$ and $y \in X$ such that xy or yx belongs to $X \setminus I = \Uparrow_\pi e$. First we consider the case $xy \in \Uparrow_\pi e$. By Theorem 6(2), $exy \in H_e$ and hence there exists an element $g \in H_e$ such that $exyg = e$. Assuming that $ex \in \Uparrow_\pi e$ and applying Proposition 9(6), we conclude that

$$e \le \pi(ex) = \pi(eex) = e\pi(ex) \le e$$

and hence $e = \pi(ex)$. Applying Proposition 9(6) once more, we conclude that

$$e = \pi(ex) = e\pi(x) \le \pi(x)$$

and $x \in \Uparrow_\pi e$, which contradicts the choice of x. Therefore, $ex \notin \Uparrow_\pi e$. Replacing the elements x, y by ex and yg, we can assume that $ex = x$, $ye = y$ and $xy = e$. Consider the product $f = yx$ and observe that

$$ff = yxyx = yex = yx = f,$$

which means that f is an idempotent. By the E-hypocentrality of X, the equality $xy = e$ implies $xe = ex$ or $ye = ey$. If $xe = ex$, then

$$f = yx = yex = yxe = fe.$$

If $ye = ey$, then

$$f = yx = yex = eyx = ef = fe.$$

In both cases we conclude that $f = fe$. By the E-hypocentrality of X, the equality $f = yx$ implies $fy = yf$ or $fx = xf$. If $fy = yf$, then

$$f = ef = xyf = xfy = xyxy = ee = e.$$

If $fx = xf$, then

$$f = fe = fxy = xfy = xyxy = e.$$

In both cases we obtain $e = f$.

Now observe that $xX^1 = exX^1 \subseteq eX^1$ and $eX^1 = xyX^{-1} \subseteq xX^1$, which implies $xX^1 = eX^1$. On the other hand,

$$X^1x = X^1ex = X^1xyx \subseteq X^1yx = X^1f = X^1e$$

and $X^1e = X^1f = X^1yx \subseteq X^1x$, which implies $X^1x = X^1e$. Therefore, $x \in H_e \subseteq {\Uparrow_\pi} e$, which contradicts the choice of x. By analogy we can derive a contradiction from the assumption $yx \in {\Uparrow_\pi} e$. Those contradictions show that ${\Uparrow_\pi} e$ is a prime coideal, equal to ${\Uparrow} e$. □

Now we can prove that for every $e \in E(X)$ its $\mathfrak{Q}$-class ${\Updownarrow} e$ equals $\sqrt[\mathbb{N}]{H_e}$. By Proposition 2, $\sqrt[\mathbb{N}]{H_e} \subseteq {\Updownarrow} e$. To prove that $\sqrt[\mathbb{N}]{H_e} = {\Updownarrow} e$, choose any element $x \in {\Updownarrow} e$. Since X is eventually Clifford, there exists an idempotent $f \in E(X)$ such that $x \in \sqrt[\mathbb{N}]{H_f}$. Then there exists $n \in \mathbb{N}$ such that $x^n \in H_f$ and hence $f \Updownarrow x^n \Updownarrow x \Updownarrow e$, see Proposition 1. By Claim 2,

$$f \in {\Updownarrow} e \subseteq {\Uparrow} e = \{y \in X : e \le \pi(y)\}$$

and hence $e \le f$. By analogy,

$$e \in {\Updownarrow} e = {\Updownarrow} f \subseteq {\Uparrow} f = \{y \in X : f \le \pi(y)\}$$

implies $f \le \pi(e) = e$. The inequalities $e \le f$ and $f \le e$ imply $e = f$ and hence $x \in \sqrt[\mathbb{N}]{H_f} = \sqrt[\mathbb{N}]{H_e}$, and finally, ${\Updownarrow} e = \sqrt[\mathbb{N}]{H_e}$. □

Theorems 6 and 7 imply the following theorem describing properties of E-hypercentral π-regular E-semigroups.

Theorem 8. *Every E-separated π-regular E-semigroup X has the following properties.*

(1) *X is eventually Clifford and $E(X)$ is a semilattice.*
(2) *For every idempotent $e \in E(X)$ we have ${\Updownarrow} e = \sqrt[\mathbb{N}]{H_e}$ and ${\Uparrow} e = \{x \in X : e \le \pi(x)\}$.*
(3) *For any idempotent $e \in E(X)$ the maximal subgroup $H_e \subseteq X$ is an ideal in the semigroup ${\Uparrow} e$.*
(4) *For any $e \in E(X)$ and $x \in {\Uparrow} e$ we have $ex = xe$;*
(5) *The map $\pi : X \to E(X)$ is a homomorphic retraction of X onto $E(X)$.*
(6) *The map $\hbar : X \to H(X)$, $\hbar : x \mapsto x\pi(x)$, is a homomorphic retraction of X onto its Clifford part $H(X)$.*
(7) *The Clifford part $H(X)$ is a subsemigroup of X.*

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E-ВІДОКРЕМЛЮВАНІ НАПІВГРУПИ

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Напівгрупа називаться *E-відокремлюваною*, якщо гомоморфізми в напівґратки розділяють ідемпотенти напівгрупи. Охарактеризовано E-відокремлювані напівгрупи на мові комутативних властивостей ідемпотентів. Також охарактеризовано E-відокремлювані напівгрупи в класі π-регулярних E-напівгруп.

Ключові слова: E-центральна напівгрупа, найменша напівґраткова конгруенція, бінарний квазіпорядок.