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PROPERTIES OF POLYNOMIAL SOLUTIONS OF A DIFFERENTIAL EQUATION OF THE SECOND ORDER WITH POLYNOMIAL COEFFICIENTS OF THE SECOND DEGREE

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An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function f is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known that the condition

$$\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0 \qquad (z \in \mathbb{D})$$

is necessary and sufficient for the convexity of f. Function f is said to be close-to-convex if there exists a convex in $\mathbb{D}$ function Φ such that $\operatorname{Re}(f'(z)/\Phi'(z))>0$ $(z \in \mathbb{D})$. Close-to-convex function f has a characteristic property that the complement G of the domain $f(\mathbb{D})$ can be filled with rays which start from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0) \neq 0$.

We indicate conditions on parameters β_0, β_1, γ_0, γ_1, γ_2 and α_0, α_1, α_2 of the differential equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = \alpha_0 z^2 + \alpha_1 z + \alpha_2,$$

under which this equation has a polynomial solution

$$f(z) = \sum_{n=0}^{p} f_n z^n \quad (\deg f = p \geq 2)$$

close-to-convex or convex in $\mathbb{D}$ together with all its derivatives $f^{(j)}$ $(1 \leq j \leq p-1)$. The results depend on equality or inequality to zero of the parameter γ_2.

For example, it is proved that if $p \geq 3$, $\gamma_2 \neq 0$,

$$\gamma_0 = p\beta_0 + \gamma_1 = \beta_1 + \gamma_2 = \alpha_1\gamma_2 + p\beta_0\alpha_2 = 0$$

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holds, this equation has a polynomial solution

$$f(z)=\alpha_2/\gamma_2+z+\frac{\alpha_0+(p-1)\beta_0}{2+\beta_1}z^2+\sum_{n=3}^{p}f_nz^n,$$

where the coefficients f_n are defined by the equality

$$f_n=\frac{(p-n+1)\beta_0}{(n-1)(n+\beta_1)}f_{n-1}\qquad(3\le n\le p),$$

such that:
1) if $(11p-14)|\beta_0|/4+2|\alpha_0|\le 2-|\beta_1|$ and $11(p-2)|\beta_0|/4\le 3-|\beta_1|$ then f is close-to-convex in $\mathbb{D}$ together with all its derivatives $f^{(j)}$ $(1\le j\le p-1)$;
2) if $(41p-50)|\beta_0|/8+4|\alpha_0|\le 2-|\beta_1|$ and $33(p-2)|\beta_0|/8\le 3-|\beta_1|$ then f is convex in $\mathbb{D}$ together with all its derivatives $f^{(j)}$ $(1\le j\le p-1)$.
A similar result is obtained in the case $\gamma_2=0$.

Key words: linear non-homogeneous differential equation of the second order, polynomial coefficient, polynomial solution, close-to-convex function, convex function.

1. Introduction and auxiliary results

An analytic univalent in $\mathbb{D}=\{z:|z|<1\}$ function

$$f(z)=\sum_{n=0}^{\infty}f_nz^n \tag{1}$$

is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known [1, p. 203] (see also [2, p. 8]) that the condition

$$\operatorname{Re}\{1+zf''(z)/f'(z)\}>0\qquad(z\in\mathbb{D})$$

is necessary and sufficient for the convexity of f. By W. Kaplan [3] the function f is said to be close-to-convex in $\mathbb{D}$ (see also [1, p. 583], [2, p. 11]) if there exists a convex in $\mathbb{D}$ function Φ such that

$$\operatorname{Re}(f'(z)/\Phi'(z))>0\,(z\in\mathbb{D}).$$

The close-to-convex function f has a characteristic property that the complement G of the domain $f(\mathbb{D})$ can be filled with rays which start from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0)\ne 0$. Hence, it follows that the function f is close-to-convex in $\mathbb{D}$ if and only if the function

$$g(z)=z+\sum_{n=2}^{\infty}g_nz^n,\quad g_n=f_n/f_1, \tag{2}$$

is close-to-convex in $\mathbb{D}$. We also remark that function (2) is said to be starlike if $f(\mathbb{D})$ is a starlike domain regarding the origin and the condition

$$\operatorname{Re}\{zg'(z)/g(z)\}>0\ (z\in\mathbb{D})$$

is necessary and sufficient for the starlikeness of g [2, p. 9]. Clearly, every starlike function is close-to-convex.

S. M. Shah [4] indicated conditions on real parameters β_0, β_1, γ_0, γ_1, γ_2 of the differential equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0,$$

under which there exists an entire transcendental solution (1) such that f and all its derivatives are close-to-convex in $\mathbb{D}$. The investigations are continued in the papers [5–10], but in all of this papers the case of polynomial solutions was not investigated. In the papers [11–14] properties of entire solutions of a linear differential equation of n-th order with polynomial coefficients of n-th degree are investigated. Some results from these papers are published also in monograph [2].

In [15], the equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = \alpha_0 z^2 + \alpha_1 z + \alpha_2 \tag{3}$$

is considered with real parameters and the existence and close-to-convexity of its polynomial solutions are studied. In particular, it is proved that in order that the polynomial

$$f(z) = \sum_{n=0}^{p} f_n z^n, \quad \deg f = p \geq 2, \tag{4}$$

be a solution of the differential equation (3), it is necessary that $\gamma_0 = p\beta_0 + \gamma_1 = 0$. Substituting (4) into (3), we get [15]

$$\gamma_2 f_0 = \alpha_2, \quad (\beta_1 + \gamma_2)f_1 = \alpha_1 + p\beta_0 f_0, \quad (2 + 2\beta_1 + \gamma_2)f_2 = \alpha_0 + (p-1)\beta_0 f_1 \tag{5}$$

and for $3 \leq n \leq p$

$$(n(n + \beta_1 - 1) + \gamma_2)f_n = (p - n + 1)\beta_0 f_{n-1}. \tag{6}$$

If we assume that $n(n + \beta_1 - 1) + \gamma_2 \neq 0$ for all $3 \leq n \leq p$, it allows us to rewrite the equality (6) in the form

$$f_n = \frac{(p - n + 1)\beta_0}{n(n + \beta_1 - 1) + \gamma_2} f_{n-1}, \quad 3 \leq n \leq p, \tag{7}$$

whence it follows that $f_p = 0$, if $\beta_0 = 0$. Therefore, further we assume also that $\beta_0 \neq 0$.

In the case of real parameters for the study of the close-to convexity of the polynomial

$$g(z) = z + \sum_{n=2}^{p} g_n z^n, \tag{8}$$

Alexander's criterion [16, 17] was used. Here we are going to consider a case of complex parameters β_0, β_1, γ_0, γ_1, γ_2, α_0, α_1, α_2 and we will use the following lemma [16, 17].

Lemma 1. *If* $\sum_{n=2}^{p} n|g_n| \leq 1$ *then polynomial* (8) *is a starlike function and if* $\sum_{n=2}^{p} n^2|g_n| \leq 1$ *then polynomial* (8) *is a convex function.*

Using Lemma 1 we prove the following statement.

Lemma 2. *Let $\xi_n \neq 0$, $\xi_n = g_n/g_{n-1}$ for $2 \leq n \leq p$ and*

$$\xi = \max\left\{\frac{n}{n-1}|\xi_n| : 3 \leq n \leq p\right\}.$$

If $2|g_2| \leq 1 - \xi$ then polynomial (8) is a starlike function and if $4|g_2| \leq 1 - 3\xi/2$ then polynomial (8) is a convex function.

Proof. Since $g_n = \xi_n g_{n-1}$ for $2 \leq n \leq p$, we have

$$\sum_{n=2}^{p} n|g_n| = \sum_{n=2}^{p} n|\xi_n||g_{n-1}| = $$
$$= \sum_{n=1}^{p-1}(n+1)|\xi_{n+1}||g_n| = $$
$$= 2|g_2| + \sum_{n=2}^{p} \frac{n+1}{n}|\xi_{n+1}|n|g_n|, \qquad \xi_{p+1} = 0,$$

i.e., $\sum_{n=2}^{p}\left(1 - \frac{n+1}{n}|\xi_{n+1}|\right) n|g_n| = 2|g_2|$. Since $\xi_{p+1} = 0$ and $\frac{n+1}{n}|\xi_{n+1}| \leq \xi < 1$ for $2 \leq n \leq p-1$, hence it follows that $(1-\xi)\sum_{n=2}^{p} n|g_n| \leq 2|g_2|$. Therefore, if $2|g_2| \leq 1 - \xi$ then $\sum_{n=2}^{p} n|g_n| \leq 1$ and by Lemma 1 polynomial (8) is a starlike function.

If we put

$$\xi^* = \max\left\{\left(\frac{n}{n-1}\right)^2 |\xi_n| : 3 \leq n \leq p\right\}$$

and suppose that $4|g_2| \leq 1 - \xi^*$ then as above we get $(1-\xi^*)\sum_{n=2}^{p} n^2|g_n| \leq 4|g_2|$, i.e., $\sum_{n=2}^{p} n^2|g_n| \leq 1$ and by Lemma 1 polynomial (8) is a convex function. Since $\xi^* \leq 3\xi/2$, the proof of Lemma 2 is complete. □

In view of (5) and (6) it is clear that the existence of convex or close-to-convex solution (4) of differential equation (3) depends on the equality to zero of the parameter γ_2. Therefore, we will consider two cases: $\gamma_2 \neq 0$ and $\gamma_2 = 0$.

2. The case $\gamma_2 \neq 0$

From the first equality (5) it follows that $f_0 = \alpha_2/\gamma_2$, and the second equality (5) implies

$$(\beta_1 + \gamma_2)f_1 = \alpha_1 + p\beta_0\alpha_2/\gamma_2.$$

For the close-to-convexity of f the condition $f_1 \neq 0$ is necessary. This condition is not necessary for the convexity of the function f, but since we are going to use Lemma 2, then we will assume that $f_1 \neq 0$. Therefore, from the last equality it follows that either $\beta_1 + \gamma_2 \neq 0$ and $\alpha_1 + p\beta_0\alpha_2/\gamma_2 \neq 0$ or $\beta_1 + \gamma_2 = \alpha_1 + p\beta_0\alpha_2/\gamma_2 = 0$.

If $\beta_1 + \gamma_2 \neq 0$ and $\alpha_1 + p\beta_0\alpha_2/\gamma_2 \neq 0$ then $f_1 = \dfrac{\alpha_1\gamma_2 + p\beta_0\alpha_2}{\gamma_2(\beta_1 + \gamma_2)}$, and if $2 + 2\beta_1 + \gamma_2 \neq 0$ then from the third equality (5) we obtain

$$f_2 = \frac{(p-1)\beta_0(\alpha_1\gamma_2 + p\beta_0\alpha_2) + \alpha_0\gamma_2(\beta_1 + \gamma_2)}{\gamma_2(\beta_1 + \gamma_2)(2 + 2\beta_1 + \gamma_2)}.$$

Thus, the desired solution should be

$$f(z) = \frac{\alpha_2}{\gamma_2} + \frac{\alpha_1\gamma_2 + p\beta_0\alpha_2}{\gamma_2(\beta_1 + \gamma_2)} z + \frac{(p-1)\beta_0(\alpha_1\gamma_2 + p\beta_0\alpha_2) + \alpha_0\gamma_2(\beta_1 + \gamma_2)}{\gamma_2(\beta_1 + \gamma_2)(2 + 2\beta_1 + \gamma_2)} z^2 + \sum_{n=3}^{p} f_n z^n \tag{9}$$

where the coefficients f_n satisfy (7). The following theorem is true.

Theorem 1. *Let $p \geq 3$, $\gamma_2 \neq 0$, $\gamma_0 = p\beta_0 + \gamma_1 = 0$, $\beta_1 + \gamma_2 \neq 0$, $\alpha_1\gamma_2 + p\beta_0\alpha_2 \neq 0$. Then:*

1) *if*

$$\frac{5p-6}{2}|\beta_0| + 2\left|\frac{\alpha_0\gamma_2(\beta_1 + \gamma_2)}{\alpha_1\gamma_2 + p\beta_0\alpha_2}\right| \leq 2 - 2|\beta_1| - |\gamma_2| \tag{10}$$

then differential equation (3) *has polynomial solution* (9) *close-to-convex in* $\mathbb{D}$ *and if* $3(p-2)|\beta_0|/2 \leq 2 - |\beta_1| - |\gamma_2|/3$ *all its derivatives* $f^{(j)}$ $(1 \leq j \leq p-1)$ *are close-to-convex;*

2) *if*

$$\frac{19p-22}{4}|\beta_0| + 4\left|\frac{\alpha_0\gamma_2(\beta_1 + \gamma_2)}{\alpha_1\gamma_2 + p\beta_0\alpha_2}\right| \leq 2 - 2|\beta_1| - |\gamma_2| \tag{11}$$

then differential equation (3) *has polynomial solution* (9) *convex in* $\mathbb{D}$ *and if* $11(p-2)|\beta_0|/4 \leq 2 - |\beta_1| - |\gamma_2|/3$ *all its derivatives* $f^{(j)}$ $(1 \leq j \leq p-1)$ *are convex.*

Proof. For polynomial (8) with $g_n = f_n/f_1$ we have

$$g_2 = \frac{(p-1)\beta_0}{2 + 2\beta_1 + \gamma_2} + \frac{\alpha_0\gamma_2(\beta_1 + \gamma_2)}{(2 + 2\beta_1 + \gamma_2)(\alpha_1\gamma_2 + p\beta_0\alpha_2)} = \xi_2 = \xi_2 g_1,$$

and since (10) implies $|2 + 2\beta_1 + \gamma_2| \geq 2 - 2|\beta_1| - |\gamma_2| > 0$, we get

$$|g_2| = |\xi_2| \leq \frac{1}{2 - 2|\beta_1| - |\gamma_2|}\left((p-1)|\beta_0| + \left|\frac{\alpha_0\gamma_2(\beta_1 + \gamma_2)}{\alpha_1\gamma_2 + p\beta_0\alpha_2}\right|\right). \tag{12}$$

For $3 \leq n \leq p$ from (7) we obtain

$$g_n = \frac{f_n}{f_1} = \frac{\xi_n f_{n-1}}{f_1} = \xi_n g_{n-1},$$

where $\xi_n = \dfrac{(p-n+1)\beta_0}{n(n+\beta_1-1)+\gamma_2}$ and

$$\frac{n}{n-1}|\xi_n| \leq \frac{(p-n+1)|\beta_0|}{(n-1)(n-|\beta_1|-1-|\gamma_2|/n)} \leq \frac{(p-n+1)|\beta_0|}{(n-1)(2-2|\beta_1|-|\gamma_2|)},$$

i.e.,

$$\xi = \max\left\{\frac{n}{n-1}|\xi_n| : 3 \le n \le p\right\} \le \frac{(p-2)|\beta_0|}{2(2-2|\beta_1|-|\gamma_2|)}. \tag{13}$$

It is easy to check that (10), (12) and (13) imply the inequality $2|g_2| \le 1-\xi$. Therefore, by Lemma 2 the polynomial g is a starlike function and, thus, function (9) is close-to-convex in $\mathbb{D}$.

For $1 \le j \le p-2$ the derivative

$$f^{(j)}(z) = j!f_j + (j+1)!f_{j+1}z + \sum_{n=2}^{p-j}(n+1)(n+2)\dots(n+j)f_{n+j}z^n. \tag{14}$$

is close-to-convex in $\mathbb{D}$ if and only if the function

$$g_j(z) = z + \sum_{n=2}^{p-j} g_{n,j}z^n, \qquad g_{n,j} = \frac{(n+1)(n+2)\dots(n+j)f_{n+j}}{(j+1)!f_{j+1}}, \tag{15}$$

is close-to-convex in $\mathbb{D}$. For $1 \le j \le p-2$ and $2 \le n \le p-j$ we have $3 \le n+j \le p$, and in view of (7) and (15) we get

$$\begin{aligned} g_{n,j} =& \frac{(n+1)(n+2)\dots(n+j)}{(j+1)!f_{j+1}}\frac{(p-n-j+1)\beta_0}{(n+j)(n+j+\beta_1-1)+\gamma_2}f_{n+j-1} = \\ =& \frac{(n+1)(n+2)\dots(n+j)}{(j+1)!f_{j+1}}\xi_{n+j}\frac{(j+1)!f_{j+1}}{n(n+1)\dots(n+j-1)}g_{n-1,j} = \\ =& \frac{n+j}{n}\xi_{n+j}g_{n-1,j}, \end{aligned}$$

where as above $\xi_n = \dfrac{(p-n+1)\beta_0}{n(n+\beta_1-1)+\gamma_2}$. Therefore, to apply Lemma 2, we need to find a condition under which

$$2|g_{2,j}| \le 1 - \max_{3 \le n \le p-j}\frac{n}{n-1}\frac{n+j}{n}|\xi_{n+j}| = 1 - \max_{3 \le n \le p-j}\frac{n+j}{n-1}|\xi_{n+j}|. \tag{16}$$

From (7) and (15) we have

$$\begin{aligned} 2|g_{2,j}| &= 2\frac{3\dots(j+2)|f_{j+2}|}{(j+1)!|f_{j+1}|} = \\ &= 2\frac{j+2}{2}\frac{(p-(j+2)+1)|\beta_0|}{|(j+2)(j+2+\beta_1-1)+\gamma_2|} \le \\ &\le \frac{(p-j-1)|\beta_0|}{j+1-|\beta_1|-|\gamma_2|/(j+2)} \end{aligned} \tag{17}$$

and

$$\begin{aligned} \max_{3 \le n \le p-j}\frac{n+j}{n-1}|\xi_{n+j}| &\le \max_{3 \le n \le p-j}\frac{n+j}{n-1}\frac{(p-n-j+1)|\beta_0|}{(n+j)(n+j-|\beta_1|-1)-|\gamma_2|)} \le \\ &\le \max_{3 \le n \le p-j}\frac{1}{n-1}\frac{(p-n-j+1)|\beta_0|}{n+j-|\beta_1|-1-|\gamma_2|/(n+j)} \le \\ &\le \frac{1}{2}\frac{(p-j-1)|\beta_0|}{j+1-|\beta_1|-|\gamma_2|/(j+2)}. \end{aligned} \tag{18}$$

From (17) and (18) it follows that if

$$3(p-j-1)|\beta_0|/2 \le j+1-|\beta_1|-|\gamma_2|/(2+j) \tag{19}$$

for $1 \le j \le p-2$ then (16) holds and by Lemma 2 the derivative $f^{(j)}$ for $1 \le j \le p-2$ is close-to-convex in $\mathbb{D}$.

Finally, we remark that (19) holds for all $1 \le j \le p-2$ if

$$3(p-2)|\beta_0|/2 \le 2-|\beta_1|-|\gamma_2|/3.$$

Since $f^{(p-1)}$ is a linear function and, thus, it is close-to-convex, the first part of Theorem 1 is proved.

If condition (11) holds then from (12) and (13) we obtain the inequality $4|g_2| \le \le 1-3\xi/2$, and by Lemma 2 polynomial (9) is a convex function.

If

$$\frac{11(p-j-1)|\beta_0|}{4} \le j+1-|\beta_1|-|\gamma_2|/(2+j) \tag{20}$$

for some $1 \le j \le p-2$ then (17) and (18) imply

$$4|g_{2,j}| \le 1-\frac{3}{2}\max_{3\le n\le p-j}\frac{n+j}{n-1}|\xi_{n+j}|.$$

Therefore, by Lemma 2 function (15) is convex and, thus, function (14) is convex. Finally, we remark that (20) holds for all $1 \le j \le p-2$ if $11(p-2)|\beta_0|/4 \le 2-|\beta_1|-|\gamma_2|/3$. The proof of Theorem 1 is complete. □

Now suppose that $\beta_1+\gamma_2 = \alpha_1+p\beta_0\alpha_2/\gamma_2 = 0$. Then from the second equality (5) it follows that f_1 may be arbitrary. If we choose $f_1 = 1$ then under the condition $2+\beta_1 \ne 0$ in view of the third equality (5) we get $f_2 = \dfrac{\alpha_0+(p-1)\beta_0}{2+\beta_1}$. From (7) under the condition $n+\beta_1 \ne 0$ we obtain

$$f_n = \frac{(p-n+1)\beta_0}{(n-1)(n+\beta_1)}f_{n-1}, \quad 3 \le n \le p. \tag{21}$$

Thus, the desired solution has the form

$$f(z) = \frac{\alpha_2}{\gamma_2}+z+\frac{\alpha_0+(p-1)\beta_0}{2+\beta_1}z^2+\sum_{n=3}^{p} f_n z^n, \tag{22}$$

where the coefficients f_n satisfy (21), and we will come to such a theorem.

Theorem 2. *Let $p \ge 3$, $\gamma_2 \ne 0$, $\gamma_0 = p\beta_0+\gamma_1 = \beta_1+\gamma_2 = \alpha_1\gamma_2+p\beta_0\alpha_2 = 0$. Then:*

1) *if*

$$\frac{11p-14}{4}|\beta_0|+2|\alpha_0| \le 2-|\beta_1| \tag{23}$$

then differential equation (3) *has polynomial solution* (22) *close-to-convex in* $\mathbb{D}$ *and if* $9(p-2)|\beta_0|/4 \le 3-|\beta_1|$ *all its derivatives* $f^{(j)}$ $(1 \le j \le p-1)$ *are close-to-convex;*

2) *if*

$$\frac{41p-50}{8}|\beta_0|+4|\alpha_0|\le 2-|\beta_1| \tag{24}$$

then differential equation (3) *has polynomial solution* (22) *convex in* $\mathbb{D}$ *and if* $33(p-2)|\beta_0|/8\le 3-|\beta_1|$ *all its derivatives* $f^{(j)}$ $(1\le j\le p-1)$ *are convex.*

Proof. For polynomial (8) with $g_n=f_n$ for $1\le n\le p$ now we have

$$|g_2|=\left|\frac{\alpha_0+(p-1)\beta_0}{2+\beta_1}\right|\le\frac{|\alpha_0|+(p-1)|\beta_0|}{2-|\beta_1|} \tag{25}$$

and in view of (21)

$$\begin{aligned}\xi &\le \max_{3\le n\le p}\frac{n}{n-1}\frac{(p-n+1)|\beta_0|}{(n-1)(n-|\beta_1|)}\le\\ &\le\frac{3(p-2)|\beta_0|}{4(3-|\beta_1|)}<\\ &<\frac{3(p-2)|\beta_0|}{4(2-|\beta_1|)}.\end{aligned} \tag{26}$$

From (23), (25) and (26) it follows that $2|g_2|\le 1-\xi$. Then by Lemma 2 the function g is starlike and, thus, function (22) is close-to-convex.

If (24) holds then using (25), (26) and Lemma 2 similarly we prove the convexity of polynomial (22).

Let us turn to the derivative $f^{(j)}$, $1\le j\le p-2$. For the coefficients $g_{n,j}$ of function (15) now in view of (21) we have

$$\begin{aligned}g_{n,j}&=\frac{n+j}{n}\xi_{n+j}g_{n-1,j}=\\ &=\frac{n+j}{n}\frac{(p-n-j+1)\beta_0}{(n+j-1)(n+j+\beta_1)}g_{n-1,j}.\end{aligned}$$

Therefore,

$$|g_{2,j}|\le\frac{2+j}{2}\frac{(p-j-1)|\beta_0|}{(j+1)(j+2-|\beta_1|)} \tag{27}$$

and

$$\begin{aligned}\max_{3\le n\le p}\frac{n+j}{n-1}|\xi_{n+j}|&\le\max_{3\le n\le p}\frac{n+j}{n-1}\frac{(p-n-j+1)|\beta_0|}{(n+j-1)(n+j-|\beta_1|)}\le\\ &\le\frac{1}{2}\frac{(2+j)(p-j-1)|\beta_0|}{(j+1)(2+j-|\beta_1|)}.\end{aligned} \tag{28}$$

If for some $1\le j\le p-2$

$$3(2+j)(p-j-1)|\beta_0|/2\le(j+1)(j+2-|\beta_1|) \tag{29}$$

then (27) and (28) imply

$$2|g_{2,j}|\le 1-\max_{3\le n\le p}\frac{n+j}{n-1}|\xi_{n+j}|$$

and by Lemma 2 $f^{(j)}$ is close-to-convex in $\mathbb{D}$.

If for some $1 \le j \le p-2$

$$11(2+j)(p-j-1)|\beta_0|/4 \le (j+1)(j+2-|\beta_1|) \tag{30}$$

then (27) and (28) imply

$$4|g_{2,j}| \le 1 - \frac{3}{2}\max_{3\le n\le p}\frac{n+j}{n-1}|\xi_{n+j}|$$

and by Lemma 2 $f^{(j)}$ is convex in $\mathbb{D}$.

Finally, we remark that (29) holds for all $1 \le j \le p-2$ if $9(p-2)|\beta_0|/4 \le 3-|\beta_1|$, and (30) holds for all $1 \le j \le p-2$ if $33(p-2)|\beta_0|/8 \le 3-|\beta_1|$. Theorem 2 is proved. $\square$

3. The case $\gamma_2 = 0$

From first equality (5) it follows that $\alpha_2 = 0$ and f_0 may be arbitrary. If we choose $f_0 = 0$ then from (5) and (7) we get

$$\beta_1 f_1 = \alpha_1, \qquad 2(1+\beta_1)f_2 = \alpha_0 + (p-1)\beta_0 f_1 \tag{31}$$

and

$$f_n = \frac{(p-n+1)\beta_0}{n(n+\beta_1-1)}f_{n-1}, \qquad 3 \le n \le p. \tag{32}$$

Since we consider $f_1 \ne 0$, from the first equality (31) it follows that either $\beta_1 \ne 0$ and $\alpha_1 \ne 0$ or $\beta_1 = \alpha_1 = 0$. If $\beta_1 \ne 0$ and $\alpha_1 \ne 0$ then $f_1 = \dfrac{\alpha_1}{\beta_1}$ and

$$f_2 = \frac{\alpha_0\beta_1 + (p-1)\alpha_1\beta_0}{2\beta_1(1+\beta_1)}.$$

Thus, the desired solution has the form

$$f(z) = \frac{\alpha_1}{\beta_1}z + \frac{\alpha_0\beta_1 + (p-1)\alpha_1\beta_0}{2\beta_1(1+\beta_1)}z^2 + \sum_{n=3}^{p} f_n z^n, \tag{33}$$

where the coefficients f_n satisfy (32), and we will come to the following theorem.

Theorem 3. *Let $p \ge 3$, $\gamma_2 = \alpha_2 = \gamma_0 = \gamma_1 + p\beta_0 = 0$, $\beta_1 \ne 0$ and $\alpha_1 \ne 0$. Then:*

1) *if*

$$\frac{3p-4}{2}|\beta_0| + \left|\frac{\alpha_0\beta_1}{\alpha_1}\right| \le 1 - |\beta_1| \tag{34}$$

then differential equation (3) has polynomial solution (33) close-to-convex in $\mathbb{D}$ and if $3(p-2)|\beta_0|/2 \le 2-|\beta_1|$ all its derivatives $f^{(j)}$ $(1 \le j \le p-1)$ are close-to-convex;

2) *if*

$$\frac{11p-14}{4}|\beta_0| + 2\left|\frac{\alpha_0\beta_1}{\alpha_1}\right| \le 1 - |\beta_1| \tag{35}$$

then differential equation (3) has polynomial solution (33) convex in $\mathbb{D}$ and if $11(p-2)|\beta_0|/4 \le 2-|\beta_1|$ all its derivatives $f^{(j)}$ $(1 \le j \le p-1)$ are convex.

The proof of this theorem is the same as proofs those of the previous theorems. We remark only that now $|g_2| \leq \frac{|\alpha_0||\beta_1|+(p-1)|\alpha_1||\beta_0|}{2|\alpha_1|(1-|\beta_1|)}$, $\xi_n = \frac{(p-n+1)\beta_0}{n(n+\beta_1-1)}$ and $\xi \leq \frac{(p-2)|\beta_0|}{2(1-|\beta_1|)}$, whence it follows that $2|g_2| \leq 1-\xi$ if (34) holds and $4|g_2| \leq 1-3\xi/2$ if (35) holds. For some $1 \leq j \leq p-1$ as above we have $g_{n,j} = \frac{n+j}{n}\xi_{n+j}g_{n-1,j}$, where now

$$\xi_{n+j} = \frac{(p-n-j+1)\beta_0}{(n+j)(n+j-\beta_1-1)},$$

whence $|g_{2,j}| \leq \frac{(p-j-1)|\beta_0|}{2(j+1-|\beta_1|)}$ and

$$\xi := \max_{3\leq n\leq p} \frac{n+j}{n-1}|\xi_{n+j}| \leq \frac{(p-j-1)|\beta_0|}{2(j+1-|\beta_1|)}.$$

Therefore, $2|g_{2,j}| \leq 1-\xi$ if $3(p-j-1)|\beta_0|/2 \leq j+1-|\beta_1|$ and $4|g_{2,j}| \leq 1-3\xi/2$ if $11(p-j-1)|\beta_0|/4 \leq j+1-|\beta_1|$. It remains to notice that the last conditions hold for all $1 \leq j \leq p-1$ provided $3(p-2)|\beta_0|/2 \leq 2-|\beta_1|$ and $11(p-2)|\beta_0|/4 \leq$ $\leq 2-|\beta_1|$ respectively and use Lemma 2.

If $\beta_1 = \alpha_1 = 0$ from (31) it follows that f_1 may be arbitrary. If we choose $f_1 = 1$ then $f_2 = \frac{\alpha_0+(p-1)\beta_0}{2}$ and

$$f_n = \frac{(p-n+1)\beta_0}{n(n-1)}f_{n-1}, \quad 3 \leq n \leq p. \tag{36}$$

Therefore, the desired solution has the form

$$f(z) = z + \frac{\alpha_0+(p-1)\beta_0}{2}z^2 + \sum_{n=3}^{p} f_n z^n, \tag{37}$$

where the coefficients f_n satisfy (36), and we will come to the following theorem.

Theorem 4. *Let $p \geq 3$, $\gamma_2 = \alpha_2 = \gamma_0 = \gamma_1 + p\beta_0 = \beta_1 = \alpha_1 = 0$. Then:*

1) *if $(5p-6)|\beta_0|/4 + |\alpha_0| \leq 1$ then differential equation (3) has polynomial solution (37) close-to-convex in $\mathbb{D}$ and if $3(p-2)|\beta_0| \leq 4$ all its derivatives $f^{(j)}$ $(1 \leq j \leq p-1)$ are close-to-convex in $\mathbb{D}$;*
2) *if $(19p-22)|\beta_0|/8 + 2|\alpha_0| \leq 1$ then differential equation (3) has polynomial solution (37) convex in $\mathbb{D}$ and if $11(p-2)|\beta_0| \leq 8$ all its derivatives $f^{(j)}$ $(1 \leq j \leq p-1)$ are convex in $\mathbb{D}$.*

Proof. Choosing $g_n = f_n$, we have $2|g_2| \leq (p-1)|\beta_0| + |\alpha_0|$ and $\xi \leq (p-2)|\beta_0|/4$. Therefore, in view of the condition $(5p-6)|\beta_0|/4 + |\alpha_0| \leq 1$ we get $2|g_2| \leq 1-\xi$ and, thus, polynomial (37) is starlike. Also in view on the condition $(19p-22)|\beta_0|/8+2|\alpha_0| \leq 1$ we get $4|g_2| \leq 1-3\xi/2$, i. e. polynomial (37) is convex.

Similarly, for some $1 \leq j \leq p-1$ we have $2|g_{2,j}| \leq \frac{(p-j-1)|\beta_0|}{j+1}$ and

$$\xi = \max_{3\leq n\leq p} \frac{n+j}{n-1}|\xi_{n+j}| \leq \frac{(p-j-1)|\beta_0|}{2(j+1)},$$

whence $2|g_{2,j}| \le 1-\xi$ if $3(p-j-1)|\beta_0| \le 2(j+1)$ and $4|g_{2,j}| \le 1-3\xi/2$ if $11(p-j-1)|\beta_0| \le 4(j+1)$. Since the last conditions hold if $3(p-2)|\beta_0| \le 4$ and $11(p-2)|\beta_0| \le 8$ respectively, Theorem 4 is proved. □

4. Additions

First of all, we note that the condition $p \ge 3$ is not essential in Theorems 1 - 4. Repeating their proofs, one can prove for $p = 2$ the following statements.

Proposition 1. *Let $\gamma_2 \ne 0$, $\gamma_0 = 2\beta_0 + \gamma_1 = 0$, $\beta_1 + \gamma_2 \ne 0$, $\alpha_1\gamma_2 + 2\beta_0\alpha_2 \ne 0$. Then differential equation* (3) *has a polynomial solution*

$$f(z) = \frac{\alpha_2}{\gamma_2} + \frac{\alpha_1\gamma_2 + 2\beta_0\alpha_2}{\gamma_2(\beta_1+\gamma_2)}z + \frac{\beta_0(\alpha_1\gamma_2 + 2\beta_0\alpha_2) + \alpha_0\gamma_2(\beta_1+\gamma_2)}{\gamma_2(\beta_1+\gamma_2)(2+2\beta_1+\gamma_2)}z^2$$

which is close-to-convex if the condition

$$2|\beta_0| + 2\left|\frac{\alpha_0\gamma_2(\beta_1+\gamma_2)}{\alpha_1\gamma_2 + 2\beta_0\alpha_2}\right| \le 2 - 2|\beta_1| - |\gamma_2|$$

holds, and convex in $\mathbb{D}$ if the condition

$$4\beta_0| + 4\left|\frac{\alpha_0\gamma_2(\beta_1+\gamma_2)}{\alpha_1\gamma_2 + 2\beta_0\alpha_2}\right| \le 2 - 2|\beta_1| - |\gamma_2|$$

holds.

Proposition 2. *Let $\gamma_2 \ne 0$, $\gamma_0 = 2\beta_0 + \gamma_1 = \beta_1 + \gamma_2 = \alpha_1\gamma_2 + 2\beta_0\alpha_2 = 0$. Then differential equation* (3) *has a polynomial solution*

$$f(z) = \frac{\alpha_2}{\gamma_2} + z + \frac{\alpha_0+\beta_0}{2+\beta_1}z^2$$

which is close-to-convex if the condition $2|\beta_0| + 2|\alpha_0| \le 2 - |\beta_1|$ holds, and convex in $\mathbb{D}$ if the condition $4|\beta_0| + 4|\alpha_0| \le 2 - |\beta_1|$ holds.

Proposition 3. *Let $\gamma_2 = \alpha_2 = \gamma_0 = \gamma_1 + 2\beta_0 =$, $\beta_1 \ne 0$ and $\alpha_1 \ne 0$. Then differential equation* (3) *has a polynomial solution*

$$f(z) = \frac{\alpha_1}{\beta_1}z + \frac{\alpha_0\beta_1 + \alpha_1\beta_0}{2\beta_1(1+\beta_1)}z^2$$

which is starlike if the condition $|\beta_0| + |\alpha_0\beta_1/\alpha_1| \le 1 - |\beta_1|$ holds, and convex in $\mathbb{D}$ if the condition $2|\beta_0| + 2|\alpha_0\beta_1/\alpha_1| \le 1 - |\beta_1|$ holds.

Proposition 4. *Let $\gamma_2 = \alpha_2 = \gamma_0 = \gamma_1 + 2\beta_0 = \beta_1 = \alpha_1 = 0$. Then differential equation* (3) *has a polynomial solution*

$$f(z) = z + \frac{\alpha_0+\beta_0}{2}z^2$$

which is starlike if the condition $|\beta_0| + |\alpha_0| \le 1$ holds, and convex in $\mathbb{D}$ if the condition $2|\beta_0| + 2|\alpha_0| \le 1$ holds.

Recall that before obtaining the above results we demanded the fulfillment of conditions $n(n+\beta_1-1)+\gamma_2\neq 0$ for all $3\leq n\leq p$ and $\beta_0\neq 0$. Here we suppose that $\beta_0=0$. Then the equality $\gamma_0=p\beta_0+\gamma_1=0$ implies $\gamma_0=\gamma_1=0$, and thus, from (5) and (7) we get

$$\gamma_2 f_0=\alpha_2, \qquad (\beta_1+\gamma_2)f_1=\alpha_1, \qquad (2+2\beta_1+\gamma_2)f_2=\alpha_0 \tag{38}$$

and for $3\leq n\leq p$

$$(n(n+\beta_1-1)+\gamma_2)f_n=0. \tag{39}$$

From (39) it follows that if $p(p+\beta_1-1)+\gamma_2=0$ then $f_p\neq 0$ may be arbitrary. Two cases are possible:

1) $n(n+\beta_1-1)+\gamma_2\neq 0$ for all $3\leq n<p$ or $p=3$; and
2) there is only one $3\leq p_1<p$ such that $p_1(p_1+\beta_1-1)+\gamma_2=0$.

In the first case we have

$$f(z)=f_0+f_1z+f_2z^2+f_pz^p$$

for $p\geq 3$. If $\gamma_2\neq 0$ from (38) we obtain

$$f_0=\frac{\alpha_2}{\gamma_2}, \qquad f_1=\frac{\alpha_1}{\beta_1+\gamma_2}, \qquad f_2=\frac{\alpha_0}{2+2\beta_1+\gamma_2}. \tag{40}$$

To use Lemma 1, we need to choose $f_p\neq 0$ so that $2|f_2/f_1|+p|f_p/f_1|\leq 1$, i. e.

$$2|\alpha_0/(2+2\beta_1+\gamma_2)|+p|f_p|\leq|\alpha_1/(\beta_1+\gamma_2)| \tag{41}$$

(clearly, this is possible if $2|\alpha_0/(2+2\beta_1+\gamma_2)|<|\alpha_1/(\beta_1+\gamma_2)|$). If $\gamma_2=0$ then $\alpha_2=0$ and coefficient f_0 can be chosen equal to zero. Then

$$f_0=0, \qquad f_1=\alpha_1/\beta_1, \qquad f_2=\alpha_0/(2+2\beta_1) \tag{42}$$

and we need to choose $f_p\neq 0$ so that

$$|\alpha_0/(1+\beta_1)|+p|f_p|\leq|\alpha_1/\beta_1| \tag{43}$$

(this is possible if $|\alpha_0/(1+\beta_1)|<|\alpha_1/\beta_1|$).

Thus, the following statement is valid.

Proposition 5. *Let* $\beta_0=\gamma_0=\gamma_1=0$, *and* (39) *holds only for* $n=p\geq 3$. *Then differential equation* (3) *has a polynomial solution*

$$f(z)=f_0+f_1z+f_2z^2+f_pz^p$$

close-to-convex in $\mathbb{D}$ *provided either* $\gamma_2\neq 0$, $\alpha_1\neq 0$ *and the coefficients are defined by* (40) *and* (41) *or* $\gamma_2=0$, $\alpha_1\neq 0$ *and the coefficients are defined by* (42) *and* (43).

Remark 1. If in Proposition 5 conditions (41) and (43) are replaced by the conditions

$$4|\alpha_0/(2+2\beta_1+\gamma_2)|+p^2|f_p|\leq|\alpha_1/(\beta_1+\gamma_2)|$$

and

$$2|\alpha_0/(1+\beta_1)|+p^2|f_p|\leq|\alpha_1/\beta_1|$$

respectively then close-to-convexity should be replaced by convexity.

If $p > 3$, $p(p+\beta_1-1)+\gamma_2=0$ and $p_1(p_1+\beta_1-1)+\gamma_2=0$ for some $3 \le p_1 < p$ then if $\gamma_2 \neq 0$ from (38) we obtain (40) and we choose $f_{p_1} \neq 0$, $f_p \neq 0$ so that

$$2|\alpha_0/(2+2\beta_1+\gamma_2)| + p_1|f_{p_1}| + p|f_p| \le |\alpha_1/(\beta_1+\gamma_2)|. \tag{44}$$

If $\gamma_2 = 0$ then from (38) we obtain (42) and we choose $f_{p_1} \neq 0$, $f_p \neq 0$ so that

$$|\alpha_0/(1+\beta_1)| + p_1|f_{p_1}| + p|f_p| \le |\alpha_1/\beta_1|. \tag{45}$$

Proposition 6. *Let $\beta_0=\gamma_0=\gamma_1=0$, and (39) holds for $n=p_1$ and $n=p\ge 4$, $3\le p_1<p$. Then differential equation (3) has a polynomial solution*

$$f(z) = f_0 + f_1 z + f_2 z^2 + f_{p_1} z^{p_1} + f_p z^p$$

close-to-convex in $\mathbb{D}$ provided either $\gamma_2 \neq 0$, $\alpha_1 \neq 0$ and the coefficients are defined by (40) and (44) or $\gamma_2=\alpha_2=0$, $\alpha_1\neq 0$ and the coefficients are defined by (42) and (45).

Remark 2. If in Proposition 6 conditions (44) and (45) replaced by the conditions

$$4|\alpha_0/(2+2\beta_1+\gamma_2)| + p_1^2|f_{p_1}| + p^2|f_p| \le |\alpha_1/(\beta_1+\gamma_2)|$$

and

$$2|\alpha_0/(1+\beta_1)| + p_1^2|f_{p_1}| + p^2|f_p| \le |\alpha_1/\beta_1|$$

respectively then close-to-convexity should be replaced by convexity.

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ВЛАСТИВОСТІ ПОЛІНОМІАЛЬНИХ РОЗВ'ЯЗКІВ ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ ДРУГОГО ПОРЯДКУ З ПОЛІНОМІАЛЬНИМИ КОЕФІЦІЄНТАМИ ДРУГОГО СТЕПЕНЯ

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Аналітична однолиста в $\mathbb{D} = \{z : |z| < 1\}$ функція f називається опуклою, якщо $f(\mathbb{D})$ - опукла область. Добре відомо, що умова

$$\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0 \qquad (z \in \mathbb{D})$$

є необхідною і достатньою для опуклості f. Функція f називається близькою до опуклої, якщо існує така опукла в $\mathbb{D}$ функція Φ, що

$$\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0 \qquad (z \in \mathbb{D}).$$

Близька до опуклої функція f характеризується тим, що доповнення G до області $f(\mathbb{D})$ можна покрити променями, які виходять з ∂G і лежать в G. Кожна близька до опуклої в $\mathbb{D}$ функція f є однолистою в $\mathbb{D}$ і тому

$f'(0) \neq 0$.
Знайдено умови на параметри β_0, β_1, γ_0, γ_1, γ_2 і α_0, α_1, α_2 диференціального рівняння

$$z^2 w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = \alpha_0 z^2 + \alpha_1 z + \alpha_2,$$

за яких це рівняння має поліноміальний розв'язок

$$f(z) = \sum_{n=0}^{p} f_n z^n \qquad (\deg f = p \geq 2),$$

близький до опуклого або опуклий в $\mathbb{D}$ разом з усіма його похідними $f^{(j)}$ $(1 \leq j \leq p-1)$. Результати залежать від рівності чи нерівності нулеві параметра γ_2.
Наприклад, доведено, що за умов $p \geq 3$, $\gamma_2 \neq 0$,

$$\gamma_0 = p\beta_0 + \gamma_1 = \beta_1 + \gamma_2 = \alpha_1\gamma_2 + p\beta_0\alpha_2 = 0.$$

це рівняння має поліноміальний розв'язок

$$f(z) = \alpha_2/\gamma_2 + z + \frac{\alpha_0 + (p-1)\beta_0}{2+\beta_1} z^2 + \sum_{n=3}^{p} f_n z^n,$$

де коефіцієнти f_n визначаються рівністю

$$f_n = \frac{(p-n+1)\beta_0}{(n-1)(n+\beta_1)} f_{n-1} \qquad (3 \leq n \leq p),$$

такий що:
1) якщо $(11p-14)|\beta_0|/4 + 2|\alpha_0| \leq 2 - |\beta_1|$ і $11(p-2)|\beta_0|/4 \leq 3 - |\beta_1|$, то f є близьким до опуклого в $\mathbb{D}$ разом з усіма його похідними $f^{(j)}$ $(1 \leq j \leq p-1)$;
2) якщо $(73p-82)|\beta_0|/16 + 4|\alpha_0| \leq 2 - |\beta_1|$ і $33(p-2)|\beta_0|/8 \leq 3 - |\beta_1|$, то f є опуклим в $\mathbb{D}$ разом з усіма його похідними $f^{(j)}$ $(1 \leq j \leq p-1)$.
Подібний результат отримано й у випадку $\gamma_2 = 0$.

Ключові слова: лінійне неоднорідне диференціальне рівняння другого порядку, поліноміальні коефіцієнти, поліноміальний розв'язок, близька до опуклої функція, опукла функція.