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NEIGHBORHOODS OF DIRICHLET SERIES ABSOLUTELY CONVERGENT IN A HALF-PLANE

Myroslav SHEREMETA

Ivan Franko National University of Lviv,
Universytetska Str., 1, 79000, Lviv, Ukraine
e-mail: m.m.sheremeta@gmail.com

For an absolutely convergent in the half-plane $\Pi_0 = \{s : \mathrm{Re}\, s < 0\}$ Dirichlet series $F(s) = e^s + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\}$ the set $O_{j,\delta}(F)$ of Dirichlet series $G(s) = e^s + \sum_{k=1}^{\infty} g_k \exp\{s\lambda_k\}$ such that $\sum_{k=1}^{\infty} \lambda_k^j |g_k - f_k| \le \delta$ is called a neighborhood of the function F. It is proved that each function G from the neighborhood $O_{1,1}(E)$ of the function $E(s) = e^s$ is a pseudostarlike function in Π_0, and if G is pseudostarlike in Π_0 and $g_k \le 0$ for all $k \ge 1$ then $G \in O_{1,1}(E)$. A similar connection exists between $O_{2,1}(E)$ and pseudoconvex functions in Π_0. The neighborhoods $O_{1,\delta}(F)$ and $O_{2,\delta}(F)$ are investigated also in the cases when F is either pseudostarlike or pseudoconvex of the order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1)$. Assuming that $0 \le \alpha < 1$, $0 < \beta < \beta_1 \le 1$, the coefficients f_k and g_k are negative and the function F is pseudostarlike of the order α and the type β it is proved, in particular, that if $G \in O_{1,\delta}(F)$ with $\delta = \dfrac{2(1-\alpha)(\beta_1 - A\beta)}{1+\beta_1}$, where $A = \dfrac{(1+\beta_1)\lambda_1 - 2\alpha\beta_1 - (1-\beta_1)}{(1+\beta)\lambda_1 - 2\alpha\beta - (1-\beta)}$, then G is pseudostarlike of the order α and the type β_1. On the contrary, if G is pseudostarlike of the order α and the type β_1 then $G \in O_{1,\delta}(F)$ with

$$\delta = 2\lambda_1(1-\alpha)\left(\frac{\beta_1}{\lambda_1(1+\beta_1) - 2\alpha\beta_1 - (1-\beta_1)} + \frac{\beta}{\lambda_1(1+\beta) - 2\alpha\beta - (1-\beta)}\right).$$

Key words: Dirichlet series, pseudostarlikeness, pseudoconvexity, neighborhood of function.

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1. Introduction

Denote by A the class of analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ functions

$$f(z) = z + \sum_{k=2}^{\infty} f_k z^k. \tag{1}$$

For $f \in A$, following A.W. Goodman [1] and S. Ruscheweyh [2], its neighborhood is a set of the form

$$N_\delta(f) = \Big\{ g(z) = z + \sum_{k=2}^{\infty} g_k z^k \in A : \sum_{k=2}^{\infty} k|g_k - f_k| \le \delta \Big\}.$$

The neighborhoods of various classes of analytical in $\mathbb{D}$ functions were studied by many authors (we concentrate here only on articles [3, 4, 5, 6, 7, 8, 9]). The Dirichlet series of the form

$$F(s) = e^s + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\}, \quad s = \sigma + it, \tag{2}$$

absolutely convergent in the half-plane $\Pi_0 = \{s : \operatorname{Re} s < 0\}$ are direct generalizations of functions (1) analytic in $\mathbb{D}$, where the exponents satisfy the condition $1 < \lambda_k \uparrow +\infty$ $(1 \leqslant k \uparrow +\infty)$. The class of such Dirichlet series will be denoted by $SD(0)$. We say that $F \in SD^*(0)$ if $F \in SD(0)$ and $f_k \le 0$ for all $k \ge 1$.

Geometric theory of the class $SD(0)$ was initiated in [10] and [11, p. 133–154]. It is known [10] that each function $F \in SD(0)$ is non-univalent in Π_0, but there exist conformal in Π_0 functions (2), and if $\sum_{k=2}^{\infty} \lambda_k |f_k| \le 1$ then function (2) is conformal in Π_0. A conformal function (2) in Π_0 is said to be pseudostarlike if $\operatorname{Re}\{F'(s)/F(s)\} > 0$ and is said to be pseudoconvex if $\operatorname{Re}\{F''(s)/F'(s)\} > 0$ for $s \in \Pi_0$. In [10] (see also [11, p. 139]) it is proved that if $\sum_{k=2}^{\infty} \lambda_k |f_k| \le 1$ then function (2) is pseudostarlike and if $\sum_{k=2}^{\infty} \lambda_k^2 |f_k| \le 1$ then function (2) is pseudoconvex. A conformal function (2) in Π_0 is said [12] to be pseudostarlike of the order $\alpha \in [0, 1)$ if $\operatorname{Re}\{F'(s)/F(s)\} > \alpha$ for $s \in \Pi_0$, i.e., $\left|\frac{F'(s)}{F(s)} - 1\right| < \left|\frac{F'(s)}{F(s)} - (2\alpha - 1)\right|$ for $s \in \Pi_0$. Finally, a conformal function (2) in Π_0 is said [12] to be pseudostarlike of the order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1]$ if

$$\left|\frac{F'(s)}{F(s)} - 1\right| < \beta \left|\frac{F'(s)}{F(s)} - (2\alpha - 1)\right|, \qquad s \in \Pi_0.$$

Similarly, a conformal function (2) in Π_0 is said to be pseudoconvex of the order $\alpha \in [0, 1)$ if $\operatorname{Re}\{F''(s)/F'(s)\} > \alpha$, and pseudoconvex of the order α and the type $\beta \in (0, 1]$ if

$$\left|\frac{F''(s)}{F'(s)} - 1\right| < \beta \left|\frac{F''(s)}{F'(s)} - (2\alpha - 1)\right|, \quad s \in \Pi_0.$$

By $PS_{\alpha,\beta}$ we denote the class of pseudostarlike functions of the order $\alpha \in [0,1)$ and the type $\beta \in (0,1]$ and by $PC_{\alpha,\beta}$ we denote the class of pseudoconvex functions of the order $\alpha \in [0,1)$ and the type $\beta \in (0,1]$.

For $j > 0$ and $\delta > 0$ we define the neighborhood of $F \in SD_0$ as follows

$$O_{j,\delta}(F) = \Big\{G(s) = e^s + \sum_{k=1}^{\infty} g_k \exp\{s\lambda_k\} \in SD_0 : \sum_{k=1}^{\infty} \lambda_k^j |g_k - f_k| \le \delta\Big\}.$$

Similarly, for $F \in SD_0^*$

$$O^*_{j,\delta}(F) = \Big\{G(s) = e^s + \sum_{k=1}^{\infty} g_k \exp\{s\lambda_k\} \in SD_0^* : \sum_{k=1}^{\infty} \lambda_k^j |g_k - f_k| \le \delta\Big\}.$$

Here we will establish a connection between the classes $PS_{\alpha,\beta}$, $PC_{\alpha,\beta}$ and $O_{j,\delta}(F)$, $O^*_{j,\delta}(F)$.

2. Neighborhoods of the function $E(s) = e^s$

We need the following lemmas.

Lemma 1 ([12])**.** *In order for function $F \in SD_0$ to be pseudostarlike of the order $\alpha \in [0,1)$ and the type $\beta \in (0,1]$ it is sufficient and in the case when $f_k \le 0$ for all $k \ge 1$, it is necessary that*

$$\sum_{k=1}^{\infty} \{(1+\beta)\lambda_k - 2\beta\alpha - (1-\beta)\}|f_k| \le 2\beta(1-\alpha). \tag{3}$$

Also in order for function $F \in SD_0$ to be pseudoconvex of the order $\alpha \in [0,1)$ and the type $\beta \in (0,1]$ it is sufficient and in the case when $f_k \le 0$ for all $k \ge 1$, it is necessary that

$$\sum_{k=1}^{\infty} \lambda_k \{(1+\beta)\lambda_k - 2\beta\alpha - (1-\beta)\}|f_k| \le 2\beta(1-\alpha).$$

Lemma 2. *Let $F \in SD_0$. Then $G \in O_{2,\delta}(F)$ if and only if $G' \in O_{1,\delta}(F')$.*

Proof. Clearly

$$F'(s) = e^s + \sum_{k=1}^{\infty} \lambda_k f_k \exp\{s\lambda_k\} \in SD_0.$$

Therefore, $G' \in O_{1,\delta}(F')$ if and only if $\sum_{k=1}^{\infty} \lambda_k |\lambda_k g_k - \lambda_k f_k| \le \delta$, i. e. if and only if $\sum_{k=1}^{\infty} \lambda_k^2 |g_k - f_k| \le \delta$. The last condition holds if and only if $G \in O_{2,\delta}(F)$. □

Now we prove such theorem.

Theorem 1. *For the function $E(s) = e^s$ the following correlations hold: $O_{1,1}(E) \subset PS_{0,1}$, $O^*_{1,1}(E) = PS_{0,1} \bigcap SD_0^*$, $O_{2,1}(E) \subset PC_{0,1}$, and $O^*_{2,1}(E) = PC_{0,1} \bigcap SD_0^*$.*

Proof. If $G \in O_1(1,E)$ then $G \in SD_0$ and $\sum_{k=1}^{\infty} \lambda_k |g_k| \le 1$. Therefore,

$$
\begin{aligned}
|G'(s) - G(s)| =& \Big| \sum_{k=1}^{\infty} (\lambda_k - 1) g_k \exp\{s\lambda_k\} \Big| \le \\
& \le \sum_{k=1}^{\infty} \lambda_k |g_k| \exp\{\sigma\lambda_k\} - \sum_{k=1}^{\infty} |g_k| \exp\{\sigma\lambda_k\} \le \\
& \le \sum_{k=1}^{\infty} (\lambda_k - 1) |g_k| \exp\{\sigma\lambda_k\} \le \\
& \le e^{\sigma} \sum_{k=1}^{\infty} \lambda_k |g_k| - \sum_{k=1}^{\infty} |g_k| \exp\{\sigma\lambda_k\} \le \\
& \le e^{\sigma} - \sum_{k=1}^{\infty} |g_k| \exp\{\sigma\lambda_k\}.
\end{aligned}
$$

On the other hand,

$$
|G(s)| = \Big| e^s + \sum_{k=1}^{\infty} g_k \exp\{s\lambda_k\} \Big| \ge e^{\sigma} - \sum_{k=1}^{\infty} |g_k| \exp\{\sigma\lambda_k\},
$$

and, thus, $|G'(s) - G(s)| \le |G(s)|$, i.e., $\left| \frac{G'(s)}{G(s)} - 1 \right| \le 1$ for all $s \in \Pi_0$. Hence it follows that $\mathrm{Re}\{G'(s)/G(s)\} > 0$, i.e., $G \in PS_{0,1}$ and $O_{1,1}(E) \subset PS_{0,1}$.

From above it follows that $O^*_{1,1}(E) \subset PS_{0,1}$. On the other hand, if $G \in SD^*_0$ and $G \in PS_{0,1}$ then by Lemma 1 $\sum_{k=1}^{\infty} \lambda_k |f_k| \le 1$, i. e. $G \in O^*_{1,1}(E)$. Thus, $PS_{0,1} \bigcap SD^*_0 \subset O^*_{1,1}(E)$ and $PS_{0,1} \bigcap SD^*_0 = O^*_{1,1}(E)$.

We remark that since $G''(s)/G'(s) = H'(s)/H(s)$, where $H(s) = e^s + \sum_{k=1}^{\infty} h_k \exp\{s\lambda_k\}$ and $h_k = \lambda_k g_k$, the function G is pseudoconvex of the order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1]$ if and only if the function H is pseudostarlike of the order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1]$. Therefore, from the results proved above for pseudostarlike functions, one can easily obtain the corresponding results for pseudoconvex functions. In particular, in view of Lemma 2 from above we get $O_{2,1}(E) \subset PS_{0,1}$ and $PS_{0,1} \bigcap SD^*_0 = O^*_{2,1}(E)$. The proof of Theorem 1 is complete. □

3. Neighborhoods of pseudostarlike and pseudoconvex functions of the order α.

The following theorem is true.

Theorem 2. *Let* $0 \le \alpha_1 < \alpha < 1$ *and* $F \in SD^*_0 \bigcap PS_{\alpha,1}$.

1. *If* $G \in O^*_{1,\delta}(F)$ *with* $\delta = (\alpha - \alpha_1)\frac{\lambda_1 - 1}{\lambda_1 - \alpha}$ *then* $G \in PS_{\alpha_1,1}$.

2. *On the contrary, if* $G \in SD_0^* \bigcap PS_{\alpha_1,1}$ *then* $G \in O_{1,\delta}^*(F)$ *with*

$$\delta = \frac{\lambda_1(1-\alpha)}{\lambda_1-\alpha} + \frac{\lambda_1(1-\alpha_1)}{\lambda_1-\alpha_1}.$$

Proof. Since $F \in PS_{\alpha,1}$, by Lemma 1 with $\beta = 1$ we have

$$\sum_{k=1}^{\infty}\{\lambda_k-\alpha\}|f_k| \le 1-\alpha. \tag{4}$$

Therefore, for $\alpha_1 < \alpha$ in view of (4) we get

$$\begin{aligned}\sum_{k=1}^{\infty}\{\lambda_k-\alpha_1\}|g_k| &\le \sum_{k=1}^{\infty}\{\lambda_k-\alpha_1\}(|g_k-f_k|+|f_k|) \le\\ &\le \sum_{k=1}^{\infty}\{\lambda_k-\alpha+\alpha-\alpha_1\}|f_k| + \sum_{k=1}^{\infty}\{\lambda_k-\alpha_1\}|f_k-g_k| \le\\ &\le \sum_{k=1}^{\infty}\{\lambda_k-\alpha\}|f_k| + (\alpha-\alpha_1)\sum_{k=1}^{\infty}|f_k| + \sum_{k=1}^{\infty}\lambda_k|f_k-g_k| \le\\ &\le 1-\alpha+\delta+(\alpha-\alpha_1)\sum_{k=1}^{\infty}|f_k|.\end{aligned}$$

But in view of (4)

$$(\lambda_1-\alpha)\sum_{k=1}^{\infty}|f_k| \le \sum_{k=1}^{\infty}\{\lambda_k-\alpha\}|f_k| \le 1-\alpha,$$

that is $\sum\limits_{k=1}^{\infty}|f_k| \le (1-\alpha)/(\lambda_1-\alpha)$ and, thus,

$$\sum_{k=1}^{\infty}\{\lambda_k-\alpha_1\}|g_k| \le 1-\alpha+\delta+(\alpha-\alpha_1)\frac{1-\alpha}{\lambda_1-\alpha} = 1-\alpha_1,$$

i.e., by Lemma 1 with $\beta = 1$ the function G is pseudostarlike of the order α_1.

Now suppose that $G \in SD_0^* \bigcap PS_{\alpha_1,1}$. Then in view of (4) we have

$$\begin{aligned}\sum_{k=1}^{\infty}\lambda_k|g_k-f_k| &= \sum_{k=1}^{\infty}\frac{\lambda_k}{\lambda_k-\alpha_1}\{\lambda_k-\alpha_1\}|g_k-f_k| \le\\ &\le \frac{\lambda_1}{\lambda_1-\alpha_1}\sum_{k=1}^{\infty}\{\lambda_k-\alpha_1\}|g_k-f_k| \le\\ &\le \frac{\lambda_1}{\lambda_1-\alpha_1}\left(\sum_{k=1}^{\infty}\frac{\lambda_k-\alpha_1}{\lambda_k-\alpha}\{\lambda_k-\alpha\}|f_k| + \sum_{k=1}^{\infty}\{\lambda_k-\alpha_1\}|g_k|\right) \le\\ &\le \frac{\lambda_1}{\lambda_1-\alpha_1}\left(\frac{\lambda_1-\alpha_1}{\lambda_1-\alpha}(1-\alpha)+1-\alpha_1\right) =\end{aligned}$$

$$= \frac{\lambda_1(1-\alpha)}{\lambda_1-\alpha} + \frac{\lambda_1(1-\alpha_1)}{\lambda_1-\alpha_1} = \\ = \delta,$$

i.e., $G \in O^*_{1,\delta}(F)$ with such δ. Theorem 2 is proved. □

Using Lemma 2 and the remark in the proof of Theorem 1, we can prove the following statement.

Proposition 1. *Let $0 \le \alpha_1 < \alpha < 1$ and $F \in SD^*_0 \bigcap PC_{\alpha,1}$. If $G \in O^*_{2,\delta}(F)$ with $\delta = (\alpha-\alpha_1)\dfrac{\lambda_1-1}{\lambda_1-\alpha}$ then $G \in PC_{\alpha_1,1}$. On the contrary, if $G \in SD^*_0 \bigcap PC_{\alpha_1,1}$ then $G \in O^*_{2,\delta}(F)$ with*

$$\delta = \frac{\lambda_1(1-\alpha)}{\lambda_1-\alpha} + \frac{\lambda_1(1-\alpha_1)}{\lambda_1-\alpha_1}.$$

4. Neighborhoods of pseudostarlike and pseudoconvex functions of the order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1)$

In the general case the following theorem is true.

Theorem 3. *Let $0 \le \alpha < 1$, $0 < \beta < \beta_1 \le 1$ and $F \in SD^*_0 \bigcap PS_{\alpha,\beta}$.*

1. *If $G \in O^*_{1,\delta}(F)$ with $\delta = \dfrac{2(1-\alpha)(\beta_1 - A\beta)}{1+\beta_1}$, where*
$$A = \frac{(1+\beta_1)\lambda_1 - 2\alpha\beta_1 - (1-\beta_1)}{(1+\beta)\lambda_1 - 2\alpha\beta - (1-\beta)},$$
then $G \in PS_{\alpha,\beta_1}$.
2. *On the contrary, if $G \in SD^*_0 \bigcap PS_{\alpha,\beta_1}$ then $G \in O^*_{1,\delta}(F)$ with*

$$\delta = 2\lambda_1(1-\alpha)\left(\frac{\beta_1}{\lambda_1(1+\beta_1)-2\alpha\beta_1-(1-\beta_1)} + \frac{\beta}{\lambda_1(1+\beta)-2\alpha\beta-(1-\beta)}\right). \tag{5}$$

Proof. At first we remark that $\beta_1 - A\beta > 0$ and

$$\max_{k\ge1} \frac{(1+\beta_1)\lambda_k - 2\alpha\beta_1 - (1-\beta_1)}{(1+\beta)\lambda_k - 2\alpha\beta - (1-\beta)} = A.$$

Since $F \in PS_{\alpha,\beta}$, by Lemma 1 for $0 < \beta < \beta_1 \le 1$ we have

$$\sum_{k=1}^{\infty}\{(1+\beta_1)\lambda_k - 2\alpha\beta_1 - (1-\beta_1)\}|g_k| \le$$
$$\le \sum_{k=1}^{\infty}\{(1+\beta_1)\lambda_k-2\alpha\beta_1-(1-\beta_1)\}|g_k-f_k| + \sum_{k=1}^{\infty}\{(1+\beta_1)\lambda_k-2\alpha\beta_1-(1-\beta_1)\}|f_k| \le$$
$$\le (1+\beta_1)\sum_{k=1}^{\infty}\lambda_k|g_k-f_k| + \sum_{k=1}^{\infty}\frac{(1+\beta_1)\lambda_k-2\alpha\beta_1-(1-\beta_1)}{(1+\beta)\lambda_k-2\alpha\beta-(1-\beta)}\{(1+\beta)\lambda_k-2\alpha\beta-(1-\beta)\}|f_k| \le$$
$$\le (1+\beta_1)\delta + \max_{k\ge1}\left\{\frac{(1+\beta_1)\lambda_k-2\alpha\beta_1-(1-\beta_1)}{(1+\beta)\lambda_k-2\alpha\beta-(1-\beta)}\right\}\sum_{k=1}^{\infty}\{(1+\beta)\lambda_k-2\alpha\beta-(1-\beta)\}|f_k| \le$$

$$\le (1+\beta_1)\delta + 2A\beta(1-\alpha) \le$$
$$\le 2\beta_1(1-\alpha),$$

i.e., by Lemma 1 $G \in PS_{\alpha,\beta_1}$.

Now suppose that $G \in SD_0^* \bigcap PS_{\alpha,\beta_1}$. Then in view of inequality (3) we have

$$\sum_{k=1}^{\infty}\lambda_k|g_k - f_k| =$$
$$= \sum_{k=1}^{\infty}\frac{\lambda_k}{(1+\beta_1)\lambda_k - 2\alpha\beta_1 - (1-\beta_1)}\{(1+\beta_1)\lambda_k - 2\alpha\beta_1 - (1-\beta_1)\}|g_k - f_k| \le$$
$$\le \frac{\lambda_1}{(1+\beta_1)\lambda_1 - 2\alpha\beta_1 - (1-\beta_1)}\sum_{k=1}^{\infty}(1+\beta_1)\lambda_k - 2\alpha\beta_1 - (1-\beta_1)\}|g_k - f_k| \le$$
$$\le \frac{\lambda_1}{(1+\beta_1)\lambda_1 - 2\alpha\beta_1 - (1-\beta_1)}\times$$
$$\times\left(\sum_{k=1}^{\infty}\{(1+\beta_1)\lambda_k - 2\alpha\beta_1 - (1-\beta_1)\}|g_k| + \sum_{k=1}^{\infty}\{(1+\beta_1)\lambda_k - 2\alpha\beta_1 - (1-\beta_1)\}|f_k|\right) \le$$
$$\le \frac{\lambda_1}{(1+\beta_1)\lambda_1 - 2\alpha\beta_1 - (1-\beta_1)}\times$$
$$\times\left(2\beta_1(1-\alpha) + \sum_{k=1}^{\infty}\frac{(1+\beta_1)\lambda_k - 2\alpha\beta_1 - (1-\beta_1)}{(1+\beta)\lambda_k - 2\alpha\beta - (1-\beta)}\{(1+\beta)\lambda_k - 2\alpha\beta - (1-\beta)\}|f_k|\right) \le$$
$$\le \frac{\lambda_1}{(1+\beta_1)\lambda_1 - 2\alpha\beta_1 - (1-\beta_1)}\left(2\beta_1(1-\alpha) + 2\beta(1-\alpha)\frac{(1+\beta_1)\lambda_1 - 2\alpha\beta_1 - (1-\beta_1)}{(1+\beta)\lambda_1 - 2\alpha\beta - (1-\beta)}\right) =$$
$$= \delta,$$

i.e., $G \in O_{1,\delta}^*(F)$ with such δ. The proof of Theorem 3 is complete. □

Using Lemma 2 and the remark in the proof of Theorem 1, we can prove also the following statement.

Proposition 2. *Let* $0 \le \alpha < 1$, $0 < \beta < \beta_1 \le 1$ *and* $F \in SD_0^* \bigcap PC_{\alpha,\beta}$. *If* $G \in O_{2,\delta}^*(F)$ *with* $\delta = \dfrac{2(1-\alpha)(\beta_1 - A\beta)}{1+\beta_1}$, *where*

$$A = \frac{(1+\beta_1)\lambda_1 - 2\alpha\beta_1 - (1-\beta_1)}{(1+\beta)\lambda_1 - 2\alpha\beta - (1-\beta)},$$

then $G \in PC_{\alpha,\beta_1}$. *On the contrary, if* $G \in SD_0^* \bigcap PC_{\alpha,\beta_1}$ *then* $G \in O_{2,\delta}^*(F)$ *with* δ *specified in* (5).

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ОКОЛИ АБСОЛЮТНО ЗБІЖНИХ У ПІВПЛОЩИНІ РЯДІВ ДІРІХЛЕ

Мирослав ШЕРЕМЕТА

Львівський національний університет імені Івана Франка,
вул. Університетська 1, 79000, м. Львів
e-mail: m.m.sheremeta@gmail.com

Для абсолютно збіжного у півплощині $\Pi_0 = \{s : \operatorname{Re} s < 0\}$ ряду Діріхле $F(s) = e^s + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\}$ множина $O_{j,\delta}(F)$ таких рядів Діріхле $G(s) = e^s + \sum_{k=1}^{\infty} g_k \exp\{s\lambda_k\}$, що $\sum_{k=1}^{\infty} \lambda_k^j |g_k - f_k| \le \delta$, називається околом функції F. Доведено, що кожна функція G з околу $O_{1,1}(E)$ функції $E(s) = e^s$ є псевдозірковою в Π_0, а якщо G є псевдозірковою в Π_0 і $g_k \le 0$ для всіх $k \ge 1$, то $G \in O_{1,1}(E)$. Подібний зв'язок існує між околом $O_{2,1}(E)$ і псевдоопуклими в Π_0 функціями. Досліджено також околи $O_{1,\delta}(F)$ і $O_{2,\delta}(F)$ у випадках, коли F є або псевдозірковою, або псевдоопуклою порядку $\alpha \in [0, 1)$ і типу $\beta \in (0, 1)$. За умов, що $0 \le \alpha < 1$, $0 < \beta < \beta_1 \le 1$, коефіцієнти f_k і g_k від'ємні, а F є псевдозірковою порядку α і типу β, зокрема, доведено таке: якщо $G \in O_{1,\delta}(F)$ з $\delta = \dfrac{2(1-\alpha)(\beta_1 - A\beta)}{1+\beta_1}$, де $A = \dfrac{(1+\beta_1)\lambda_1 - 2\alpha\beta_1 - (1-\beta_1)}{(1+\beta)\lambda_1 - 2\alpha\beta - (1-\beta)}$, то G псевдозіркова функція порядку α і типу β_1. Навпаки, якщо G є псевдозірковою порядку α і типу β_1, то $G \in O_{1,\delta}(F)$ з
$$\delta = 2\lambda_1(1-\alpha)\left(\frac{\beta_1}{\lambda_1(1+\beta_1) - 2\alpha\beta_1 - (1-\beta_1)} + \frac{\beta}{\lambda_1(1+\beta) - 2\alpha\beta - (1-\beta)}\right).$$

Ключові слова: ряд Діріхле, псевдозірковість, псевдоопуклість, окіл функції.