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LOGARITHMIC DERIVATIVE AND ANGULAR DENSITY OF ZEROS FOR THE BLASCHKE PRODUCT

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Let $z_0 = 1$ be the only boundary point of zeros (a_n) of the Blaschke product $B(z)$,

$$\Gamma_m = \bigcup_{j=1}^{m} \{z : |z| < 1, \arg(1-z) = -\theta_j\} = \bigcup_{j=1}^{m} l_{\theta_j},$$

$$-\pi/2 + \eta < \theta_1 < \theta_2 < \ldots < \theta_m < \pi/2 - \eta,$$

be a finite system of rays, $0 < \eta < 1$. We found asymptotics of the logarithmic derivative of $B(z)$ as $z = 1 - re^{-i\varphi} \to 1$, $-\pi/2 < \varphi < \pi/2$, under the condition of existing the angular density of its zeros related to the comparison function $(1-r)^{-\rho}$, $0 < \rho < 1$.. We also considered the inverse problem for $B(z)$, whose zeros lie on Γ_m.

Key words: logarithmic derivative, Blaschke product, angular density of zeros

1. Introduction

Let (a_n) be a sequence of numbers from $\mathbb{C}$ such that

$$0 < |a_1| \leqslant |a_2| \leqslant \ldots \leqslant |a_n| \leqslant \ldots < 1$$

2020 Mathematics Subject Classification: 30D15, 30D35,30J10 and $\sum\limits_{n=1}^{+\infty}(1-|a_n|)<+\infty$. Then any function of the form

$$B(z)=\prod_{n=1}^{+\infty}\frac{\overline{a}_n}{|a_n|}\cdot\frac{a_n-z}{1-\overline{a}_n z}$$

is called a Blaschke product and is an analytic function in the unit disc $\mathbb{D}=\{z\colon |z|<1\}$. The Blaschke products form an important subclass of the space H^p, $p>0$ (see, for example, [1, p. 89]), that is analytic functions in $\mathbb{D}$, which satisfy the condition

$$\sup_{0\leqslant r<1}\int_0^{2\pi}\left|f(re^{i\theta})\right|^p d\theta<+\infty.$$

In particular, each function $f\in H^p$ can be represented in the form $f(z)=B(z)g(z)$, where $g\in H^p$ and $g(z)\neq 0$ in $\mathbb{D}$, B is the Blaschke product constructed by zeros f. We notice ([2, p. 81]), that the product B is a meromorphic function on $\mathbb{C}$ except for accumulation points of its zeros. In [3] R. Haloyan showed the connection between an angular density of zeros of B related to the comparison function $(1-r)^{-\rho}$, $0<\rho<1$, and the asymptotics of its logarithm as $z\to 1$.

The asymptotics and estimates of the logarithmic derivative of meromorphic functions outside exceptional sets play an important role in various fields of mathematics, in particular, in Nevanlinna's theory of value distribution [4, 5] and in the analytic theory of differential equations [6, 7].

We research the connection between the asymptotics of logarithmic derivative of the Blaschke product B with only one accumulation point of zeros on $\partial\mathbb{D}$ and the existence of the angular density of its zeros in this paper. Without loss of generality, we assume further that such accumulation point is $z_0=1$. Indeed, otherwise we would consider the Blaschke product

$$B^*(z)=B\left(z\cdot e^{i\varphi_0}\right)=\prod_{n=1}^{+\infty}\frac{\overline{a}_n}{|a_n|}\frac{a_n-ze^{i\varphi_0}}{1-\overline{a}_n ze^{i\varphi_0}}=\prod_{n=1}^{+\infty}\frac{\overline{a}_n^*}{|a_n^*|}\frac{a_n^*-z}{1-\overline{a}_n^* z},$$

with $a_n^*=a_n e^{-i\varphi_0}$, here $e^{i\varphi_0}$ is the accumulation point of zeros (a_n) of the product B.

For logarithmic derivative of B from the formula

$$\frac{B'(z)}{B(z)}=\int_0^1\frac{(1-t^2)dn(t)}{(z-te^{i\varphi_n})\,(1-zte^{-i\varphi_n})},\quad |\varphi_n|=|\arg a_n|\to 0,\quad n\to+\infty,$$

it follows that $B'(z)/B(z)=O(1)$ as $z\to e^{i\alpha}$, $|\alpha|>\delta$, $0<\delta<1$, since

$$\frac{B'(z)}{B(z)}=(1+o(1))\int_0^1\frac{(1-t^2)dn(t)}{(e^{i\alpha}-te^{i\varphi_n})\left(1-te^{-i(\varphi_n-\alpha)}\right)}=$$

$$=(1+o(1))\,e^{-i\alpha}\int_0^1\frac{(1-t^2)dn(t)}{1-2t\cos(\alpha-\varphi_n)+t^2},$$

$z \to e^{i\alpha}$, and the integral above converges.

2. Definitions and main results

Let $\eta > 0$ and $a_n = 1 - r_n e^{-i\theta_n}$, $-\pi/2 + \eta < \theta_n < \pi/2 - \eta$ be a sequence of zeros of the Blaschke product B, let $n(t) = n(t, B)$ be a number of (a_n) in the disc $\{z\colon |z| \leqslant t\}$ such that $1 - r_n \leqslant t$, $0 < t < 1$, $r_n \to 0+$ as $n \to +\infty$. It is easy to see that $n(t) = \widetilde{n}(1-t)$, where $\widetilde{n}(t)$ is a counting function of zeros (a_n) of the product B such that $r_n \geqslant t$, $0 < t < 1$.

We denote by $\mathscr{B}(\rho)$ the set of Blaschke products B, zeros of which satisfy the condition

$$\varlimsup_{t\to 1-} \frac{n(t)}{(1-t)^{-\rho}} < +\infty, \; 0 < \rho < 1.$$

A set $E \subset \mathbb{D}$ is said to be a *set of zero μ-density*, $1 < \mu \leqslant 2$, if this set can be covered by a sequence of disks $K(z_j, t_j) = \{z\colon |z - z_j| < t_j\}$ such that

$$\sum_{|1-z_j|\leqslant 1-r} t_j^{\mu} = o\left((1-r)^{\mu}\right), \; r \to 1-.$$

Let $n(t;\psi)$ be the number of zeros $a_n = 1 - r_n e^{-i\psi_n}$ of the product B, for which $1 - r_n \leqslant t$, $-\pi/2 + \eta < \psi_n \leqslant \psi$. We will say that zeros of $B \in \mathscr{B}(\rho)$ have an angular density at the point 1, if for all ψ, $|\psi| < \pi/2 - \eta$, except a countable number of values ψ at most, the finite limit

$$\lim_{t\to 1-} \frac{n(t;\psi)}{(1-t)^{-\rho}} = \Delta(\psi)$$

exists. Let us set

$$h(\varphi;\psi) = e^{i\rho(|\varphi-\psi| - \pi \operatorname{sign}(\varphi-\psi))} - e^{i\rho(\varphi+\psi)} = e^{i\rho\varphi}\left(e^{i\rho(-\psi-\pi\operatorname{sign}(\varphi-\psi))} - e^{i\rho\psi}\right),$$

$$H(\varphi) = \int_{-\pi/2}^{\pi/2} h(\varphi;\psi)\, d\Delta(\psi).$$

Theorem 1. *Let $0 < \rho < 1$, $1 < \mu \leqslant 2$, $B \in \mathscr{B}(\rho)$ and let the zeros of B have the angular density at the point 1. Then there exists a set of zero μ-density $E \subset \mathbb{D}$ such that for $z = 1 - re^{-i\varphi} \notin E$, $\varphi \in (-\pi/2 + \eta; \pi/2 - \eta)$*

$$(1) \qquad (1-z)\frac{B'(z)}{B(z)} = (1 + o(1))\,\frac{\pi\rho}{\sin\pi\rho} H(\varphi) r^{-\rho}, \; r \to 0+.$$

Let

$$-\pi/2 + \eta < \psi_1 < \psi_2 < \ldots < \psi_m < \pi/2 - \eta,$$

$0 < \eta < 1$ and let

$$\Gamma_m = \bigcup_{j=1}^{m} \{z\colon |z| < 1, \arg(1-z) = -\psi_j\} = \bigcup_{j=1}^{m} l_{\psi_j}$$

be a finite system of rays; let $\mathscr{B}(\rho;\Gamma_m)$ be a subclass of products B from the class $\mathscr{B}(\rho)$, whose zeros (a_n) of which lie on Γ_m; let $n(t, \psi_j) = n(t, \psi_j; B)$ be a number of zeros of B lying on the ray l_{ψ_j} such that $1 - r_n \leqslant t$.

Theorem 2. *Let* $0 < \rho < 1$, $\Delta_j \geqslant 0$, $B \in \mathscr{B}(\rho;\Gamma_m)$ *and let for* $z = 1 - re^{-i\varphi}$, $\varphi \in \left(-\frac{\pi}{2};\frac{\pi}{2}\right) \setminus \bigcup_{j=1}^{m} \psi_j$,

$$(1-z)\frac{B'(z)}{B(z)} = (1+o(1))\,\frac{\pi\rho}{\sin\pi\rho}\sum_{j=1}^{m}\Delta_j h(\varphi;\psi_j)r^{-\rho},\ r \to 0+,$$

moreover, for any $\delta > 0$ *the above relation holds uniformly relative to* φ *on the set* $(-\pi/2;\pi/2) \setminus \bigcup_{j=1}^{m}(\psi_j - \delta, \psi_j + \delta)$. *Then for* $j = \overline{1,m}$ *we have that*

$$n(t;\psi_j) = (1+o(1))\,\Delta_j(1-t)^{-\rho}, \quad t \to 1-. \tag{2}$$

3. Additional results

We will use the following propositions, which we formulate in the form of lemmas.

Lemma 1. *Let* $0 < \rho < 1$, $0 < \Delta < +\infty$ *and let the zeros* $a_n = 1 - r_n e^{-i\psi}$ *of the product* B *lie on the ray* l_ψ, $-\pi/2 < \psi < \pi/2$ *and*

$$n(t) = (1+o(1))\,\Delta(1-t)^{-\rho},\ t \to 1-. \tag{3}$$

Then for $z = 1 - re^{-i\varphi}$, $\varphi \in (-\pi/2;\pi/2) \setminus \{\psi\}$, *we have that*

$$(1-z)\frac{B'(z)}{B(z)} = (1+o(1))\,\frac{\pi\rho\Delta}{\sin\pi\rho}h(\varphi;\psi)r^{-\rho},\ r \to 0+, \tag{4}$$

moreover, the equality (4) *holds uniformly for any* $\delta > 0$ *relative to* φ *on the set* $(-\pi/2;\psi-\delta] \cup [\psi+\delta;\pi/2)$.

Proof. After replacing z by $z = (w-1)/w$ we have

$$g(w) = B\left(\frac{w-1}{w}\right) = \left(\prod_{n=1}^{+\infty}\frac{1}{|a_n|}\right)\prod_{n=1}^{+\infty}\left(1-\frac{w}{b_n}\right) / \prod_{n=1}^{+\infty}\left(1-\frac{w}{c_n}\right) = p_0\frac{g_1(w)}{g_2(w)},$$

where $p_0 = \prod_{n=1}^{+\infty}|a_n|^{-1}$, $b_n = 1/(1-a_n) = e^{i\psi}/r_n$ are zeros of the entire function g_1,

$$c_n = -\overline{a}_n/(1-\overline{a}_n) = -e^{-i\psi}/r_n + 1 = e^{-i(\psi+\pi)}/r_n + 1$$

are zeros of g_2 and

$$n(\tau,0,g_1) = n(\tau,0,g_2) = \Delta\tau^\rho + o\left(\tau^\rho\right),\ \tau \to +\infty.$$

Set

$$\widetilde{g}_2(w) = \prod_{n=1}^{+\infty}\left(1-\frac{w}{\widetilde{c}_n}\right),\ \widetilde{c}_n = c_n - 1 = \frac{1}{r_n}e^{-i(\psi+\pi)}.$$

Let $\widehat{l}_\alpha = \{w\colon |w| \geqslant 1, \arg w = \alpha\}$, $l^*_\alpha = \{w\colon |w| \geqslant 1, \arg(w-1) = \alpha\}$ and let

$$\ln g(w) = \ln p_0 + \ln g_1(w) - \ln\widetilde{g}_2(w) + \sum_{n=1}^{+\infty}\ln\left(\frac{1-w/\widetilde{c}_n}{1-w/c_n}\right)$$

be an univalent branch of $\mathrm{Ln}\, g(w)$ on $\mathbb{C} \setminus \left(\widehat{l}_\psi \cup \widehat{l}_{-\psi-\pi} \cup l^*{}_{-\psi-\pi}\right)$ such that $\ln g(0) = \ln g_1(0) = \ln \widetilde{g}_2(0) = 0$, $\ln p_0 > 0$, $\ln \left(1 - \frac{w}{\widetilde{c}_n}\right)\Big|_{w=0} = \ln \left(1 - \frac{w}{c_n}\right)\Big|_{w=0} = 0$. Hence

$$w\frac{g'(w)}{g(w)} = w\left(\frac{g_1'(w)}{g_1(w)} - \frac{\widetilde{g}_2'(w)}{\widetilde{g}_2(w)}\right) + w\sum_{n=1}^{+\infty} \frac{c_n - \widetilde{c}_n}{(w-c_n)(w-\widetilde{c}_n)}. \tag{5}$$

Since ([8, p. 92]) for $\delta \leqslant \varphi \leqslant 2\pi - \delta$, $0 < \delta < 1$, we get that

$$\left|t - re^{i\varphi}\right| \geqslant (t + r)\sin\frac{\delta}{2},\ t > 0,$$

for $\theta \in [-\psi - \pi + \delta, -\psi + \pi - \delta]$ and $\tau \geqslant 2/\sin\frac{\delta}{2}$ we obtain

$$|w - \widetilde{c}_n| = \left|\tau e^{i\theta} - \frac{1}{r_n}e^{i(-\psi-\pi)}\right| = \left|\tau - \frac{1}{r_n}e^{i(-\psi-\theta-\pi)}\right| \geqslant \left(\tau + \frac{1}{r_n}\right)\sin\frac{\delta}{2},$$
$$|w - c_n| \geqslant |w - \widetilde{c}_n| - 1 \geqslant \left(\tau + \frac{1}{r_n}\right)\sin\frac{\delta}{2} - 1 \geqslant \frac{1}{2}\left(\tau + \frac{1}{r_n}\right)\sin\frac{\delta}{2}.$$

Then ($t_n = 1/r_n > 1$),

$$\begin{aligned}\left|w\sum_{n=1}^{+\infty}\frac{1}{(w-c_n)(w-\widetilde{c}_n)}\right| &\leqslant \frac{2\tau}{\sin^2\delta/2}\sum_{n=1}^{+\infty}\frac{1}{(\tau+t_n)} \leqslant \\ &\leqslant \frac{2\tau}{\sin^2\delta/2}\int_1^{+\infty}\frac{dn(t,0,\widetilde{g}_2)}{(t+\tau)^2} = \\ &= \frac{4\tau}{\sin^2\delta/2}\int_1^{+\infty}\frac{n(t,0,\widetilde{g}_2)dt}{(t+\tau)^3} \leqslant \\ &\leqslant \frac{8\Delta\tau}{\sin^2\delta/2}\int_1^{+\infty}\frac{(t+\tau)^\rho}{(t+\tau)^3}d(t+\tau) \leqslant \\ &\leqslant \frac{8\Delta\tau^{\rho-1}}{(2-\rho)\sin^2\delta/2} = \\ &= o(1), \quad \tau \to +\infty.\end{aligned} \tag{6}$$

For $w = \tau e^{i\theta}$, $\psi < \theta < \psi + 2\pi$, taking into account Theorem 1 from [3], we get

$$w\frac{g_1'(w)}{g_1(w)} = (1 + o(1))\,\frac{\pi\rho\Delta}{\sin\pi\rho}e^{i\rho(\theta-\psi-\pi)}\tau^\rho, \quad \tau \to +\infty;$$

and for $-\psi - \pi < \theta < -\psi + \pi$

$$w\frac{\widetilde{g}_2'(w)}{\widetilde{g}_2(w)} = (1 + o(1))\,\frac{\pi\rho\Delta}{\sin\pi\rho}e^{i\rho(\theta+\psi)}\tau^\rho, \quad \tau \to +\infty.$$

By (5), (6) and the two above equalities for $\theta \in (\psi, -\psi+\pi)$ we obtain ($\tau \to +\infty$)

$$w\frac{g'(w)}{g(w)} = w\left(\frac{g_1'(w)}{g_1(w)} - \frac{\widetilde{g}_2'(w)}{\widetilde{g}_2(w)}\right) = \\ = (1+o(1))\,\frac{\pi\rho\Delta}{\sin\pi\rho}\left(e^{i\rho(\theta-\psi-\pi)} - e^{i\rho(\theta+\psi)}\right)\tau^{\rho},$$

and for $\theta \in (-\psi-\pi, \psi)$ we have that

$$w\frac{g'(w)}{g(w)} = (1+o(1))\,\frac{\pi\rho\Delta}{\sin\pi\rho}\left(e^{i\rho(\theta-\psi+\pi)} - e^{i\rho(\theta+\psi)}\right)\tau^{\rho},$$

namely, for $\theta \in (-\psi-\pi, -\psi+\pi)\setminus\{\psi\}$

$$w\frac{g'(w)}{g(w)} = (1+o(1))\,\frac{\pi\rho\Delta}{\sin\pi\rho}h(\theta;\psi)\tau^{\rho}, \quad \tau\to+\infty, \tag{7}$$

moreover, the equality (7) holds uniformly for any $\delta > 0$ on the set $(-\psi-\pi, -\psi+\pi)\setminus(\psi-\delta, \psi+\delta)$.

Since $-\psi-\pi < -\pi/2$, $-\psi+\pi > \pi/2$, $w = 1/(1-z)$,

$$g'(w) = B'\left(\frac{w-1}{w}\right)\frac{1}{w^2} = (1-z)^2 B'(z)$$

and for $z = 1 - re^{-i\varphi}$ we have

$$\theta = \arg w = \arg\left(\frac{1}{1-z}\right) = \arg\left(\frac{1}{r}e^{i\varphi}\right) = \varphi,$$

then by (7) we obtain (4), and hence, lemma 1 is proved. □

Remark 1. In fact, the relation (4) holds uniformly relative to φ on the set

$$[\psi+\delta, -\psi+\pi-\delta] \cup [-\psi-\pi+\delta, \psi-\delta],$$

$0 < \delta < 1$.

Lemma 2. *Let $0 < \rho < 1$, $0 < \Delta < +\infty$, $-\pi/2 < \psi < \pi/2$ and let the zeros $a_n = 1 - r_n e^{-i\psi}$ of the product B lie on the ray l_ψ. If for $z = 1 - re^{-i\varphi}$, $\varphi \in (-\pi/2;\pi/2)\setminus\{\psi\}$ the following relation*

$$(1-z)\frac{B'(z)}{B(z)} = (1+o(1))\,h(\varphi;\psi)r^{-\rho}, \; r\to 0+$$

holds, moreover, for any $\delta > 0$ the above relation holds uniformly relative to φ on the set $(-\pi/2, \pi/2)\setminus(\psi-\delta, \psi+\delta)$, then $n(t;B) = (1+o(1))(1-t)^{-\rho}$, $t \to 1-$.

Proof. Without loss of generality, we assume that $\psi = 0$. After replacing z by $z = (w-1)/w$ we get

$$g(w) = B\left(\frac{w-1}{w}\right) = \prod_{n=1}^{+\infty}\frac{1}{a_n}\prod_{n=1}^{+\infty}\left(1-\frac{w}{b_n}\right)/\prod_{n=1}^{+\infty}\left(1-\frac{w}{c_n}\right) = p_0\frac{g_1(w)}{g_2(w)},$$

where $b_n = 1/r_n > 0$ are zeros of the entire function g_1, $c_n = -1/r_n + 1 < 0$ are zeros of g_2 and $n(\tau, 0, g_1) = n(1-1/\tau; B)$.

Since (see the proof of Lemma 1) $wg'(w)/g(w) = (1-z)B'(z)/B(z)$, then under the conditions of Lemma 2 for $w = \tau e^{i\theta}$, $0 < |\theta| < \pi$, we have

$$w\frac{g'(w)}{g(w)} = (1+o(1))\,\frac{\pi\rho\Delta}{\sin\pi\rho}h(\theta;0)\tau^{-\rho},\ \tau\to+\infty. \tag{8}$$

Set $\Omega = \{b_n\colon n\in\mathbb{N}\}$, $-\pi<\alpha<0<\beta<\pi$,

$$S_r(\alpha,\beta) = \{w\colon |w|\leqslant r, \alpha\leqslant \arg w\leqslant\beta\},$$

$r\notin\Omega$, and denote by $\partial S_r^+(\alpha,\beta) = I_r(\alpha)\cup C_r(\alpha,\beta)\cup I_r^-(\beta)$ a positive orientation of the boundary of the sector $S_r(\alpha,\beta)$, where $I_r(\alpha) = \{z_1(t) = te^{i\alpha}, 0\leqslant t\leqslant r\}$, $I_r(\beta) = \{z_2(t) = te^{i\beta}, 0\leqslant t\leqslant r\}$, $C_r(\alpha,\beta) = \{z_3(t) = re^{it}, \alpha\leqslant t\leqslant\beta\}$. Then

$$2\pi i n(r,0,g) = \int\limits_{\partial S_r^+(\alpha,\beta)}\frac{g'(z)}{g(z)}dz = \left(\int\limits_{I_r(\alpha)} + \int\limits_{I_r^-(\beta)} + \int\limits_{C_r(\alpha,\beta)}\right)\frac{g'(z)}{g(z)}dz = J_1+J_2+J_3.$$

By (8), just like in the proof of Theorem 2 from [9], as $r\to+\infty$ we obtain

$$J_1 = \int\limits_0^r \frac{g'(te^{i\alpha})}{g(te^{i\alpha})}e^{i\alpha}dt = (1+o(1))\frac{\pi\rho\Delta}{\sin\pi\rho}\int\limits_0^r h(\alpha;0)t^{\rho-1}dt = (1+o(1))\frac{\pi\Delta}{\sin\pi\rho}h(\alpha;0)r^\rho$$

and, similarly,

$$J_2 = -(1+o(1))\frac{\pi\Delta}{\sin\pi\rho}h(\beta;0)r^\rho,$$

$$\begin{aligned}J_3 &= \int\limits_\alpha^\beta \frac{g'(re^{it})}{g(re^{it})}ire^{it}dt = (1+o(1))\frac{\pi\rho\Delta}{\sin\pi\rho}ir^\rho\int\limits_\alpha^\beta h(t;0)dt\\ &= (1+o(1))\frac{\pi\rho\Delta}{\sin\pi\rho}ir^\rho\left(\left(e^{i\pi\rho}-1\right)\int\limits_\alpha^0 e^{i\rho t}dt + \left(e^{-i\pi\rho}-1\right)\int\limits_0^\beta e^{i\rho t}dt\right)\\ &= (1+o(1))\frac{\pi\Delta}{\sin\pi\rho}r^\rho\big(\left(e^{i\pi\rho}-1\right)\left(1-e^{i\rho\alpha}\right) + \left(e^{-i\pi\rho}-1\right)\left(e^{i\rho\beta}-1\right)\big)\\ &= (1+o(1))\frac{\pi\Delta}{\sin\pi\rho}r^\rho\big(2i\sin\pi\rho - h(\alpha;0)+h(\beta;0)\big).\end{aligned}$$

Consequently,

$$2\pi i n(r,0,g) = (1+o(1))2\pi i\Delta r^\rho,\ r\to+\infty.$$

Since $n(r,0,g) = n(1-1/r;B)$, we conclude that $n(t;B) = (1+o(1))\Delta(1-t)^{-\rho}$, $t\to 1-$. □

4. Proof of theorems

Let $B\in\mathscr{B}(\rho,\Gamma_m)$ and suppose that for all $j=\overline{1,m}$ (2) holds. Represent B in the form

$$B(z) = B_1(z)\cdot B_2(z)\cdot\ldots\cdot B_m(z),$$

where $B_j(z)$ is a Blaschke product constructed by zeros of B lying on the ray l_{ψ_j}.

Let $\widetilde{l}_{\psi_j} = \{z\colon \widetilde{r}_j \leqslant |z| < 1, \arg(1-z) = -\psi_j\}$, where $\widetilde{r}_j$ is the smallest modulus of zero of B lying on the ray l_{ψ_j}, $j = \overline{1,m}$; $G = \mathbb{D} \setminus \bigcup_{j=1}^{m} \widetilde{l}_{\psi_j}$; let $\ln B$ be an univalent branch in G of the multivalued function $\operatorname{Ln} B(z) = \ln|B(z)| + i \operatorname{Arg} B(z)$ such that $\ln B(0) < 0$.

From the equality

$$\ln B(z) = \ln B_1(z) + \ln B_2(z) + \ldots + \ln B_m(z), \quad z \in G,$$

owing to (4), for $z = 1 - re^{-i\varphi}$, $\varphi \in (-\pi/2; \pi/2) \setminus \bigcup_{j=1}^{m} \psi_j$, we have

$$(1-z)\frac{B'(z)}{B(z)} = (1-z)\sum_{j=1}^{m} \frac{B_j'(z)}{B_j(z)} = \tag{9}$$

$$= (1 + o(1))\,\frac{\pi\rho}{\sin \pi\rho} \sum_{j=1}^{m} \Delta_j h(\varphi; \psi_j) r^{-\rho}, \quad r \to 0+,$$

which is equivalent to (1) in the case of zeros of B lying on a finite system of rays Γ_m. The transition from Γ_m to the general case of the location of zeros of $B \in \mathscr{B}(\rho)$, which have an angular density at the point 1, is carried out according to a known scheme (see for example [3, p. 69-70]. So Theorem 1 is proved.

Let the conditions of Theorem 2 be satisfied. Then by (9) and Lemma 2 for each $j = \overline{1,m}$ we obtain

$$n(t; \psi_j) = (1 + o(1))\, \Delta_j (1-t)^{-\rho}, \quad t \to 1-,$$

that proves Theorem 2.

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ЛОГАРИФМІЧНА ПОХІДНА ТА КУТОВА ЩІЛЬНІСТЬ НУЛІВ ДОБУТКУ БЛЯШКЕ

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Добутки Бляшке є важливим підкласом функцій, аналітичних в одиничному крузі з обмеженою характеристикою Неванлінни і мероморфних у $\mathbb{C}$ за винятком точок скупчення нулів $B(z)$. Асимптотика й оцінки логарифмічної похідної мероморфних функцій зовні виняткових множин відіграють важливу роль в різних галузях математики. Зокрема, Гольдберг А.А., Хейман В.К. і Майлс Дж. досліджували ці питання в неванліннівській теорії розподілу значень, Гундерсен Г.Г. і Стрейліц Ш.І. – в аналітичній теорії диференційних рівнянь. Нехай $z_0 = 1$ єдина гранична точка нулів (a_n) добутку Бляшке $B(z)$,

$$\Gamma_m = \bigcup_{j=1}^{m} \{z : |z| < 1, \arg(1-z) = -\theta_j\} = \bigcup_{j=1}^{m} l_{\theta_j},$$

$-\pi/2 + \eta < \theta_1 < \theta_2 < \ldots < \theta_m < \pi/2 - \eta$, – скінченна система променів, $0 < \eta < 1$. За умови існування кутової щільності нулів $B(z)$ знайдено асимптотику логарифмічної похідної $B(z)$ при $z = 1 - re^{-i\varphi} \to 1$, $-\pi/2 < \varphi < \pi/2$. Також розглядається обернена задача для добутків Бляшке $B(z)$, нулі яких розташовані на Γ_m.

Ключові слова: логарифмічна похідна, добуток Бляшке, кутова щільність нулів.