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A NOTE ON FEEBLY COMPACT SEMITOPOLOGICAL SYMMETRIC INVERSE SEMIGROUPS OF A BOUNDED FINITE RANK

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We study feebly compact shift-continuous T_1-topologies on the symmetric inverse semigroup $\mathscr{I}_\lambda^n$ of finite transformations of the rank $\leqslant n$. It is proved that such T_1-topology is sequentially pracompact if and only if it is feebly compact. Also, we show that every shift-continuous feebly ω-bounded T_1-topology on $\mathscr{I}_\lambda^n$ is compact.

Key words: semigroup, inverse semigroup, semitopological semigroup, compact, sequentially pracompact, totally countably pracompact, ω-bounded-pracompact, feebly ω-bounded, feebly compact, Δ-system, the Sunflower Lemma, product, Σ-product.

1. Introduction and preliminaries

We follow the terminology of the monographs [4, 6, 10, 29, 32, 33]. If X is a topological space and $A \subseteq X$, then by $\operatorname{cl}_X(A)$ and $\operatorname{int}_X(A)$ we denote the topological closure and interior of A in X, respectively. By $|A|$ we denote the cardinality of a set A, by $A\triangle B$ the symmetric difference of sets A and B, by $\mathbb{N}$ the set of positive integers, and by ω the first infinite cardinal. By $\mathfrak{D}(\omega)$ and $\mathbb{R}$ we denote an infinite countable discrete space and the real numbers with the usual topology, respectively.

A semigroup S is called *inverse* if every a in S possesses a unique inverse a^{-1}, i.e., if there exists a unique element a^{-1} in S such that

$$aa^{-1}a = a \qquad \text{and} \qquad a^{-1}aa^{-1} = a^{-1}.$$

A map which associates to any element of an inverse semigroup its inverse is called the *inversion*.

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If S is a semigroup, then by $E(S)$ we denote the subset of all idempotents of S. On the set of idempotents $E(S)$ there exists a natural partial order: $e \leqslant f$ *if and only if* $ef = fe = e$. A *semilattice* is a commutative semigroup of idempotents. We observe that the set of idempotents of an inverse semigroup is a semilattice [34].

Every inverse semigroup S admits a partial order:

$$a \preccurlyeq b \qquad \text{if and only if there exists} \qquad e \in E(S) \quad \text{such that} \quad a = eb.$$

We shall say that $\preccurlyeq$ is the *natural partial order* on S (see [4, 34]).

Let λ be an arbitrary nonzero cardinal. A map α from a subset D of λ into λ is called a *partial transformation* of λ. In this case the set D is called the *domain* of α and is denoted by $\operatorname{dom}\alpha$. The image of an element $x \in \operatorname{dom}\alpha$ under α is denoted by $x\alpha$. Also, the set $\{x \in \lambda \colon y\alpha = x \text{ for some } y \in Y\}$ is called the *range* of α and is denoted by $\operatorname{ran}\alpha$. For convenience we denote by $\varnothing$ the empty transformation, a partial mapping with $\operatorname{dom}\varnothing = \operatorname{ran}\varnothing = \varnothing$.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of λ together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \quad \text{if} \quad x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \colon y\alpha \in \operatorname{dom}\beta\}, \qquad \text{for} \quad \alpha, \beta \in \mathscr{I}_\lambda.$$

The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the cardinal λ (see [6]). For any $\alpha \in \mathscr{I}_\lambda$ the cardinality of $\operatorname{dom}\alpha$ is called the *rank* of α and it is denoted by $\operatorname{rank}\alpha$. The symmetric inverse semigroup was introduced by V. V. Wagner [34] and it plays a major role in the theory of semigroups.

Put $\mathscr{I}_\lambda^n = \{\alpha \in \mathscr{I}_\lambda \colon \operatorname{rank}\alpha \leqslant n\}$, for $n = 1, 2, 3, \ldots$. Obviously, $\mathscr{I}_\lambda^n$ $(n = 1, 2, 3, \ldots)$ are inverse semigroups, $\mathscr{I}_\lambda^n$ is an ideal of $\mathscr{I}_\lambda$, for each $n = 1, 2, 3, \ldots$. The semigroup $\mathscr{I}_\lambda^n$ is called the *symmetric inverse semigroup of finite transformations of the rank* $\leqslant n$ [21]. By

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

we denote a partial one-to-one transformation which maps x_1 onto y_1, x_2 onto y_2, $\ldots$, and x_n onto y_n. Obviously, in such case we have $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$ $(i, j = 1, 2, 3, \ldots, n)$. The empty partial map $\varnothing \colon \lambda \rightharpoonup \lambda$ is denoted by $\mathbf{0}$. It is obvious that $\mathbf{0}$ is zero of the semigroup $\mathscr{I}_\lambda^n$.

Let λ be a nonzero cardinal. On the set $B_\lambda = (\lambda \times \lambda) \cup \{0\}$, where $0 \notin \lambda \times \lambda$, we define the semigroup operation "$\,\cdot\,$" as follows

$$(a, b) \cdot (c, d) = \begin{cases} (a, d), & \text{if } \ b = c; \\ 0, & \text{if } \ b \neq c, \end{cases}$$

and $(a, b) \cdot 0 = 0 \cdot (a, b) = 0 \cdot 0 = 0$ for $a, b, c, d \in \lambda$. The semigroup B_λ is called the *semigroup of* $\lambda \times \lambda$*-matrix units* (see [6]). Obviously, for any cardinal $\lambda > 0$, the semigroup of $\lambda \times \lambda$-matrix units B_λ is isomorphic to $\mathscr{I}_\lambda^1$.

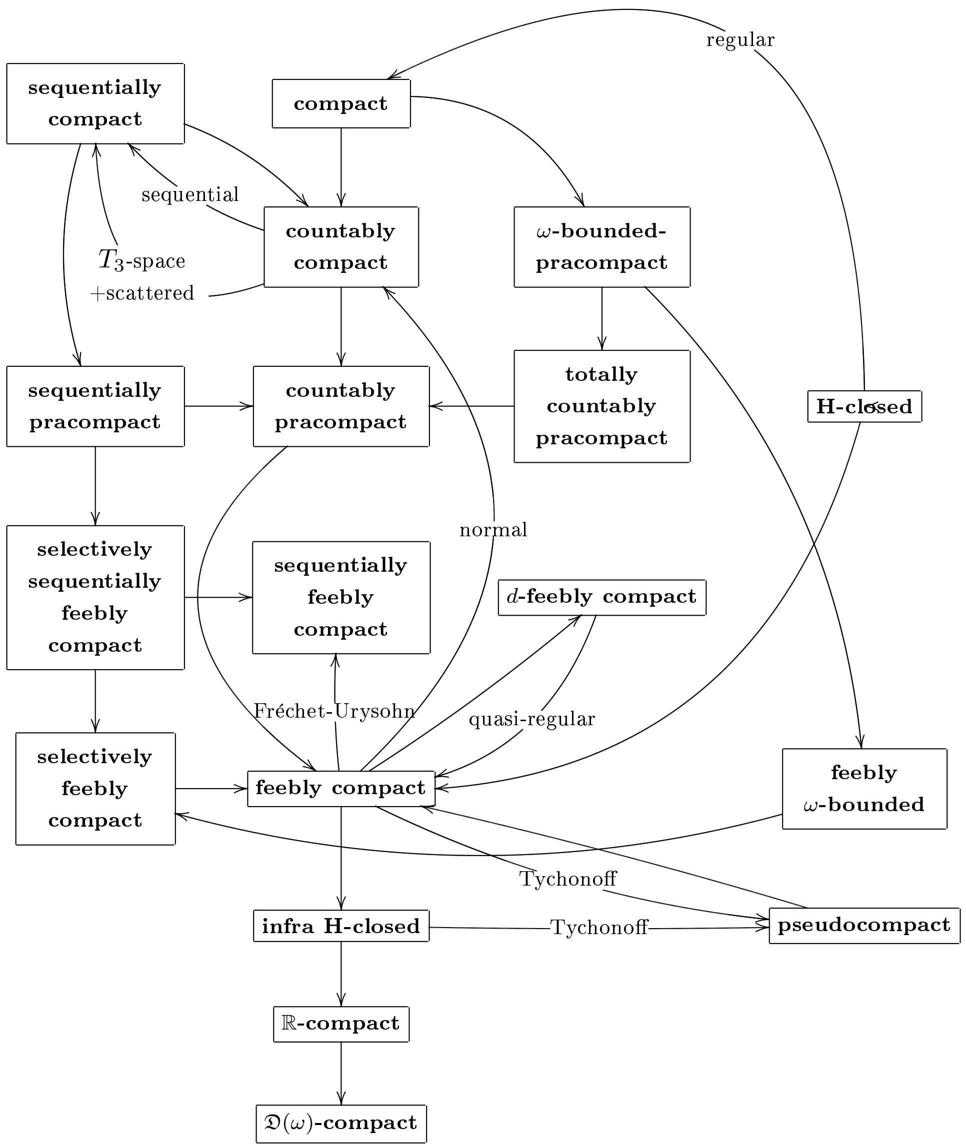
A subset A of a topological space X is called *regular open* if $\operatorname{int}_X(\operatorname{cl}_X(A)) = A$.

We recall that a topological space X is said to be

- *semiregular* if X has a base consisting of regular open subsets;
- *compact* if each open cover of X has a finite subcover;
- *sequentially compact* if each sequence $\{x_i\}_{i \in \mathbb{N}}$ of X has a convergent subsequence in X;

- *countably compact* if each open countable cover of X has a finite subcover;
- *H-closed* if X is a closed subspace of every Hausdorff topological space in which it is contained;
- *ω-bounded-pracompact* if X contains a dense subset D such that each countable subset of D has the compact closure in X [20];
- *infra H-closed* provided that any continuous image of X into any first countable Hausdorff space is closed (see [27]);
- *totally countably pracompact* if there exists a dense subset D of the space X such that each sequence of points of the set D has a subsequence with the compact closure in X [20];
- *sequentially pracompact* if there exists a dense subset D of the space X such that each sequence of points of the set D has a convergent subsequence [20];
- *countably compact at a subset* $A \subseteq X$ if every infinite subset $B \subseteq A$ has an accumulation point x in X [1];
- *countably pracompact* if there exists a dense subset A in X such that X is countably compact at A [1];
- *feebly ω-bounded* if for each sequence $\{U_n\}_{n\in\mathbb{N}}$ of nonempty open subsets of X there is a compact subset K of X such that $K \cap U_n \neq \varnothing$ for each n [20];
- *selectively sequentially feebly compact* if for every family $\{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of X, one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence ([8]);
- *sequentially feebly compact* if for every family $\{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of X, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J : W \cap U_n = \varnothing\}$ is finite for every open neighborhood W of x (see [9]);
- *selectively feebly compact* for each sequence $\{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of X, one can choose a point $x \in X$ and a point $x_n \in U_n$ for each $n \in \mathbb{N}$ such that the set $\{n \in \mathbb{N} : x_n \in W\}$ is infinite for every open neighborhood W of x ([8]);
- *feebly compact* (or *lightly compact*) if each locally finite open cover of X is finite [3];
- *d-feebly compact* (or *DFCC*) if every discrete family of open subsets in X is finite (see [31]);
- *pseudocompact* if X is Tychonoff and each continuous real-valued function on X is bounded;
- *Y-compact* for some topological space Y, if $f(X)$ is compact, for any continuous map $f : X \to Y$.

According to Theorem 3.10.22 of [10], a Tychonoff topological space X is feebly compact if and only if X is pseudocompact. Also, a Hausdorff topological space X is feebly compact if and only if every locally finite family of nonempty open subsets of X is finite. Every compact space and every sequentially compact space are countably compact, every countably compact space is countably pracompact, every countably pracompact space is feebly compact (see [1]), every H-closed space is feebly compact too (see [19]). Also, every space feebly compact is infra H-closed by Proposition 2 and Theorem 3 of [27]. Using results of other authors we get that the following diagram which describes relations between the above defined classes of topological spaces.



A *topological* (*semitopological*) *semigroup* is a topological space together with a continuous (separately continuous) semigroup operation. If S is a semigroup and τ is a topology on S such that (S,τ) is a semitopological semigroup, then we shall call τ a *shift-continuous topology* on S. An inverse topological semigroup with the continuous inversion is called a *topological inverse semigroup*.

Topological properties of an infinite (semi)topological semigroup $\lambda\times\lambda$-matrix units were studied in [15, 17]. In [15] it was shown that on the infinite semitopological semigroup of $\lambda\times\lambda$-matrix units B_λ there exists a unique compact shift-continuous Hausdorff topology τ_c and also it is shown that every pseudocompact Hausdorff shift-continuous topology τ on B_λ is compact. Also, in [15] it is proved that every nonzero element of a Hausdorff semitopological semigroup of $\lambda\times\lambda$-matrix units B_λ is an isolated point in the topological space B_λ. In [15] it is shown that the infinite semigroup of $\lambda\times\lambda$-matrix units B_λ cannot be embedded into a compact Hausdorff topological semigroup, every Hausdorff topological inverse semigroup S that contains B_λ as a subsemigroup, contains B_λ as a closed subsemigroup, i.e., B_λ is *algebraically complete* in the class of Hausdorff topological inverse semigroups. This result in [14] is extended onto the called inverse semigroups with *tight ideal series* and, as a corollary, onto the semigroup $\mathscr{I}_\lambda^n$. Also, in [21] it was proved that for every positive integer n the semigroup $\mathscr{I}_\lambda^n$ is *algebraically h-complete* in the class of Hausdorff topological inverse semigroups, i.e., every homomorphic image of $\mathscr{I}_\lambda^n$ is algebraically complete in the class of Hausdorff topological inverse semigroups. In the paper [22] this result is extended onto the class of Hausdorff semitopological inverse semigroups and it is shown therein that for an infinite cardinal λ the semigroup $\mathscr{I}_\lambda^n$ admits a unique Hausdorff topology τ_c such that $(\mathscr{I}_\lambda^n,\tau_c)$ is a compact semitopological semigroup. Also, it was proved in [22] that every countably compact Hausdorff shift-continuous topology τ on B_λ is compact. In [17] it was shown that a topological semigroup of finite partial bijections $\mathscr{I}_\lambda^n$ with a compact subsemigroup of idempotents is absolutely H-closed (i.e., every homomorphic image of $\mathscr{I}_\lambda^n$ is algebraically complete in the class of Hausdorff topological semigroups) and any Hausdorff countably compact topological semigroup does not contain $\mathscr{I}_\lambda^n$ as a subsemigroup for an arbitrary infinite cardinal λ and any positive integer n. In [17] there were given sufficient conditions onto a topological semigroup $\mathscr{I}_\lambda^1$ to be non-H-closed. Also in [11] it is proved that an infinite semitopological semigroup of $\lambda\times\lambda$-matrix units B_λ is H-closed in the class of semitopological semigroups if and only if the space B_λ is compact. In the paper [12] we studied feebly compact shift-continuous T_1-topologies on the semigroup $\mathscr{I}_\lambda^n$. For any positive integer $n\geqslant 2$ and any infinite cardinal λ a Hausdorff countably pracompact non-compact shift-continuous topology on $\mathscr{I}_\lambda^n$ is constructed there. In [12] it is shown that for an arbitrary positive integer n and an arbitrary infinite cardinal λ for a shift–continuous T_1-topology τ on $\mathscr{I}_\lambda^n$ the following conditions are equivalent: (i) τ is countably pracompact; (ii) τ is feebly compact; (iii) τ is d-feebly compact; (iv) $(\mathscr{I}_\lambda^n,\tau)$ is H-closed; (v) $(\mathscr{I}_\lambda^n,\tau)$ is $\mathfrak{D}(\omega)$-compact; (vi) $(\mathscr{I}_\lambda^n,\tau)$ is $\mathbb{R}$-compact; (vii) $(\mathscr{I}_\lambda^n,\tau)$ is infra H-closed. Also in [12] we proved that for an arbitrary positive integer n and an arbitrary infinite cardinal λ every shift-continuous semiregular feebly compact T_1-topology τ on $\mathscr{I}_\lambda^n$ is compact. Similar results were obtained for a semitopological semilattice $(\exp_n\lambda,\cap)$ in [23, 24, 25]. Also, in [26, 30] it is proved that feeble compactness implies compactness for semitopological bicyclic extensions.

In this paper we study feebly compact shift-continuous T_1-topologies on the symmetric inverse semigroup $\mathscr{I}_\lambda^n$ of finite transformations of the rank $\leqslant n$. It is proved that such T_1-topology is sequentially pracompact if and only if it is feebly compact. Also, we show that every shift-continuous feebly ω-bounded T_1-topology on $\mathscr{I}_\lambda^n$ is compact. The results of this paper are announced in [13].

2. On feebly compact shift continuous topologies on the semigroup $\mathscr{I}_\lambda^n$

Later we shall assume that n is an arbitrary positive integer.

For every element α of the semigroup $\mathscr{I}_\lambda^n$ we put

$$\uparrow_l\alpha=\left\{\beta\in\mathscr{I}_\lambda^n\colon \alpha\alpha^{-1}\beta=\alpha\right\} \qquad \text{and} \qquad \uparrow_r\alpha=\left\{\beta\in\mathscr{I}_\lambda^n\colon \beta\alpha^{-1}\alpha=\alpha\right\}.$$

Then Proposition 5 of [22] implies that $\uparrow_l\alpha=\uparrow_r\alpha$ and by Lemma 6 of [29, Section 1.4] we have that $\alpha\preccurlyeq\beta$ if and only if $\beta\in\uparrow_l\alpha$ for $\alpha,\beta\in\mathscr{I}_\lambda^n$. Hence we put $\uparrow_{\preccurlyeq}\alpha=\uparrow_l\alpha=\uparrow_r\alpha$ for any $\alpha\in\mathscr{I}_\lambda^n$.

Remark 1. Later we identify every element α of the semigroup $\mathscr{I}_\lambda^n$ with the graph $\mathsf{graph}(\alpha)$ of the partial map $\alpha\colon\lambda\rightharpoonup\lambda$ (see [29]). Then according to this identification we have that $\alpha\preccurlyeq\beta$ if and only if $\alpha\subseteq\beta$.

Lemma 1. *Let n be an arbitrary positive integer and λ be any infinite cardinal. Let α be any nonzero element of the semigroup $\mathscr{I}_\lambda^n$ with* $\operatorname{rank}\alpha=m\leqslant n$. *Then the poset $(\uparrow_{\preccurlyeq}\alpha,\preccurlyeq)$ is order isomorphic to the poset $(\mathscr{I}_\lambda^{n-m},\preccurlyeq)$.*

Proof. Suppose that

$$\alpha=\left(\begin{array}{ccc} x_1 & \cdots & x_m\\ y_1 & \cdots & y_m\end{array}\right)$$

for some $x_1,\ldots,x_m,y_1,\ldots,y_m\in\lambda$. If $m=n$ then the inequality $\alpha\preccurlyeq\beta$ in $(\mathscr{I}_\lambda^n,\preccurlyeq)$ implies $\alpha=\beta$, and hence later we assume that $m<n$. Then for any $\beta\in\mathscr{I}_\lambda^n$ such that $\alpha\preccurlyeq\beta$ by Remark 1 we have that

$$\beta=\left(\begin{array}{cccccc} x_1 & \cdots & x_m & x_{m+1} & \cdots & x_n\\ y_1 & \cdots & y_m & y_{m+1} & \cdots & y_n\end{array}\right)$$

for some $x_{m+1},\ldots,x_n,y_{m+1},\ldots,y_n\in\lambda$. Since λ is infinite,

$$|\lambda|=|\lambda\setminus\{x_1,\ldots,x_m\}|=|\lambda\setminus\{y_1,\ldots,y_m\}|,$$

and hence there exist bijective maps $\mathfrak{u}\colon\lambda\setminus\{x_1,\ldots,x_m\}\to\lambda$ and $\mathfrak{v}\colon\lambda\setminus\{y_1,\ldots,y_m\}\to\lambda$. Simple verifications show that the map $\mathfrak{I}\colon(\uparrow_{\preccurlyeq}\alpha,\preccurlyeq)\to(\mathscr{I}_\lambda^{n-m},\preccurlyeq)$ defined in the following way $\alpha\mapsto\mathbf{0}$ and

$$\left(\begin{array}{cccccc} x_1 & \cdots & x_m & x_{m+1} & \cdots & x_n\\ y_1 & \cdots & y_m & y_{m+1} & \cdots & y_n\end{array}\right)\mapsto\left(\begin{array}{ccc} (x_{m+1})\mathfrak{u} & \cdots & (x_n)\mathfrak{u}\\ (y_{m+1})\mathfrak{v} & \cdots & (y_n)\mathfrak{v}\end{array}\right)$$

is an order isomorphism. $\square$

Later we need the following technical lemma from [12].

Lemma 2 ([12, Lemma 3])**.** *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Let τ be a feebly compact shift-continuous T_1-topology on the semigroup $\mathscr{I}_\lambda^n$. Then for every $\alpha\in\mathscr{I}_\lambda^n$ and any open neighbourhood $U(\alpha)$ of α in $(\mathscr{I}_\lambda^n,\tau)$ there exist finitely many $\alpha_1,\ldots,\alpha_k\in\uparrow_{\preccurlyeq}\alpha\setminus\{\alpha\}$ such that*

$$\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}\cap\uparrow_{\preccurlyeq}\alpha\subseteq U(\alpha)\cup\uparrow_{\preccurlyeq}\alpha_1\cup\cdots\cup\uparrow_{\preccurlyeq}\alpha_k.$$

Lemma 3. *Let τ be a feebly compact topology on $\mathscr{I}_\lambda^1$ such that $\uparrow_{\preccurlyeq}\alpha$ is closed-and-open for any $\alpha\in\mathscr{I}_\lambda^1$. Then τ is compact.*

The statement of Lemma 3 follows from the fact that all nonzero elements of the semigroup $\mathscr{I}_\lambda^1$ are closed-and-open in $(\mathscr{I}_\lambda^1,\tau)$.

A family of non-empty sets $\{A_i\colon i\in\mathscr{I}\}$ is called a *Δ-system* (a *sunflower* or a *Δ-family*) if the pairwise intersections of its members are the same, i.e., $A_i\cap A_j=S$ for some set S (for $i\neq j$ in $\mathscr{I}$) [28]. The following statement is well known as the *Sunflower Lemma* or the *Lemma about a Δ-system* (see [28, p. 107]).

Lemma 4. *Every infinite family of n-element sets ($n<\omega$) contains an infinite Δ-subfamily.*

Proposition 1. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then every feebly compact shift-continuous T_1-topology τ on $\mathscr{I}_\lambda^n$ is sequentially pracompact.*

Proof. Suppose to the contrary that there exists a feebly compact shift-continuous T_1-topology τ on $\mathscr{I}_\lambda^n$ which is not sequentially countably pracompact. Then every dense subset D of $(\mathscr{I}_\lambda^n,\tau)$ contains a sequence of points from D which has no a convergent subsequence.

By Proposition 2 of [12] the subset $\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}$ is dense in $(\mathscr{I}_\lambda^n,\tau)$ and by Lemma 2 from [12] every point of the set $\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}$ is isolated in $(\mathscr{I}_\lambda^n,\tau)$. Then the set $\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}$ contains an infinite sequence of points $\{\chi_p\colon p\in\mathbb{N}\}$ which has no a convergent subsequence. If we identify elements of the semigroups with their graphs then by Lemma 4 the sequence $\{\chi_p\colon p\in\mathbb{N}\}$ contains an infinite Δ-subfamily, that is an infinite subsequence $\{\chi_{p_i}\colon i\in\mathbb{N}\}$ such that there exists $\chi\in\mathscr{I}_\lambda^n$ such that $\chi_{p_i}\cap\chi_{p_j}=\chi$ for any distinct $i,j\in\mathbb{N}$.

Suppose that $\chi=\mathbf{0}$ is the zero of the semigroup $\mathscr{I}_\lambda^n$. Since the sequence $\{\chi_{p_i}\colon i\in\mathbb{N}\}$ is an infinite Δ-subfamily, the intersection $\{\chi_{p_i}\colon i\in\mathbb{N}\}\cap{\uparrow_{\preccurlyeq}}\gamma$ contains at most one set for every non-zero element $\gamma\in\mathscr{I}_\lambda^n$. Thus $(\mathscr{I}_\lambda^n,\tau)$ contains an infinite locally finite family of open non-empty subsets which contradicts the feeble compactness of $(\mathscr{I}_\lambda^n,\tau)$.

If χ is a non-zero element of the semigroup $\mathscr{I}_\lambda^n$ then by Lemma 2 from [12], ${\uparrow_{\preccurlyeq}}\chi$ is an open-and-closed subspace of $(\mathscr{I}_\lambda^n,\tau)$, and hence by Theorem 14 from [3] the space ${\uparrow_{\preccurlyeq}}\chi$ is feebly compact. We observe that the element χ is the minimum of the poset ${\uparrow_{\preccurlyeq}}\chi$. Since the sequence $\{\chi_{p_i}\colon i\in\mathbb{N}\}$ is an infinite Δ-subfamily, the intersection $\{\chi_{p_i}\colon i\in\mathbb{N}\}\cap{\uparrow_{\preccurlyeq}}\gamma$ contains at most one set for every element $\gamma\in{\uparrow_{\preccurlyeq}}\chi\setminus\{\chi\}$. Thus the subspace ${\uparrow_{\preccurlyeq}}\chi$ of $(\mathscr{I}_\lambda^n,\tau)$ contains an infinite locally finite family of open non-empty subsets which contradicts the feeble compactness of $(\mathscr{I}_\lambda^n,\tau)$. □

Proposition 2. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then every feebly compact shift-continuous T_1-topology τ on $\mathscr{I}_\lambda^n$ is totally countably pracompact.*

Proof. By Proposition 2 of [12] the subset $\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}$ is dense in $(\mathscr{I}_\lambda^n,\tau)$ and by Lemma 2 from [12] every point of the set $\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}$ is isolated in $(\mathscr{I}_\lambda^n,\tau)$. We put $D=\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}$. Fix an arbitrary sequence $\{\chi_p\colon p\in\mathbb{N}\}$ of points of D.

It is obvious that at least one of the following conditions holds:

(1) for any $\eta\in\mathscr{I}_\lambda^n\setminus\{\mathbf{0}\}$ the set ${\uparrow_{\preccurlyeq}}\eta\cap\{\chi_p\colon p\in\mathbb{N}\}$ is finite;
(2) there exists $\eta\in\mathscr{I}_\lambda^n\setminus\{\mathbf{0}\}$ such that the set ${\uparrow_{\preccurlyeq}}\eta\cap\{\chi_p\colon p\in\mathbb{N}\}$ is infinite.

Suppose that case **(1)** holds. By Lemma 2 of [12] for every point $\alpha \in \mathscr{I}_\lambda^n \setminus \{0\}$ there exists an open neighbourhood $U(\alpha)$ of α in $(\mathscr{I}_\lambda^n, \tau)$ such that $U(\alpha) \subseteq {\uparrow_{\preccurlyeq}}\alpha$ and hence our assumption implies that zero $\mathbf{0}$ is a unique accumulation point of the sequence $\{\chi_p \colon p \in \mathbb{N}\}$. By Lemma 2 for an arbitrary open neighbourhood $W(\mathbf{0})$ of zero $\mathbf{0}$ in $(\mathscr{I}_\lambda^n, \tau)$ there exist finitely many nonzero elements $\eta_1, \ldots, \eta_k \in \mathscr{I}_\lambda^n$ such that

$$\left(\mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1}\right) \subseteq W(\mathbf{0}) \cup {\uparrow_{\preccurlyeq}}\eta_1 \cup \cdots \cup {\uparrow_{\preccurlyeq}}\eta_k,$$

and hence we get that $\{\mathbf{0}\} \cup \{\chi_p \colon p \in \mathbb{N}\}$ is a compact subset of $(\mathscr{I}_\lambda^n, \tau)$.

Suppose that case **(2)** holds: there exists $\eta^1 {\in} \mathscr{I}_\lambda^n \setminus \{\mathbf{0}\}$ such that the set ${\uparrow_{\preccurlyeq}}\eta^1 \cap \{\chi_p \colon p{\in}\mathbb{N}\}$ is infinite. Then by Lemma 2 of [12], ${\uparrow_{\preccurlyeq}}y^1$ is an open-and-closed subset of $(\mathscr{I}_\lambda^n, \tau)$ and hence by Theorem 14 from [3] the subspace ${\uparrow_{\preccurlyeq}}\eta^1$ of $(\mathscr{I}_\lambda^n, \tau)$ is feebly compact. By Lemma 1 the poset $({\uparrow_{\preccurlyeq}}\eta^1, \preccurlyeq)$ is order isomorphic to the poset $(\mathscr{I}_\lambda^{m_1}, \preccurlyeq)$ for some positive integer $m_1 = 2, \ldots, n-1$.

Let $\{\chi_p^1 \colon p \in \mathbb{N}\}$ be a subsequence of $\{\chi_p \colon p \in \mathbb{N}\}$ such that

$$\left\{\chi_p^1 \colon p \in \mathbb{N}\right\} = {\uparrow_{\preccurlyeq}}\eta^1 \cap \{\chi_p \colon p \in \mathbb{N}\}.$$

Then for the feebly compact poset $({\uparrow_{\preccurlyeq}}\eta^1, \preccurlyeq)$ and the sequence $\{\chi_p^1 \colon p \in \mathbb{N}\}$ at least one of the following conditions holds:

- $\mathbf{(1)}_*$ for any $\eta \in {\uparrow_{\preccurlyeq}}\eta^1 \setminus \{\eta^1\}$ the set ${\uparrow_{\preccurlyeq}}\eta \cap \{\chi_p^1 \colon p \in \mathbb{N}\}$ is finite;
- $\mathbf{(2)}_*$ there exists $\eta \in {\uparrow_{\preccurlyeq}}\eta^1 \setminus \{\eta^1\}$ such that the set ${\uparrow_{\preccurlyeq}}\eta \cap \{\chi_p^1 \colon p \in \mathbb{N}\}$ is infinite.

Since every chain in the poset $({\uparrow_{\preccurlyeq}}\eta^1, \preccurlyeq)$ is finite, repeating finitely many times our above procedure we obtain two chains of the length $s \leqslant n$:

- (i) the chain $\mathbf{0} \preccurlyeq \eta^1 \preccurlyeq \cdots \preccurlyeq \eta^s$ of distinct elements of the poset $({\uparrow_{\preccurlyeq}}\eta^1, \preccurlyeq)$; and
- (ii) the chain
$$\{\chi_p \colon p \in \mathbb{N}\} \supseteq \left\{\chi_p^1 \colon p \in \mathbb{N}\right\} \supseteq \cdots \supseteq \left\{\chi_p^s \colon p \in \mathbb{N}\right\}$$
of infinite subsequences of the sequence $\{\chi_p \colon p \in \mathbb{N}\}$,

such that the following conditions hold:

- (a) $\{\chi_p^j \colon p \in \mathbb{N}\} \subseteq {\uparrow_{\preccurlyeq}}\eta^j$ for every $j = 1, \ldots, s$;
- (b) either $\{\chi_p^s \colon p \in \mathbb{N}\} \cup \{\eta^s\}$ is a compact subset of the poset $({\uparrow_{\preccurlyeq}}\eta^1, \preccurlyeq)$ or the poset $({\uparrow_{\preccurlyeq}}\eta^s, \preccurlyeq)$ is order isomorphic to the poset $(\mathscr{I}_\lambda^1, \preccurlyeq)$.

If $\{\chi_p^s \colon p \in \mathbb{N}\} \cup \{\eta^s\}$ is a compact subset of $(\mathscr{I}_\lambda^n, \tau)$ then our above part of the proof implies that the sequence $\{\chi_p \colon p \in \mathbb{N}\}$ has the subsequence $\{\chi_p^s \colon p \in \mathbb{N}\}$ with the compact closure.

If the poset $({\uparrow_{\preccurlyeq}}\eta^s, \preccurlyeq)$ is order isomorphic to the poset $(\mathscr{I}_\lambda^1, \preccurlyeq)$, then by Lemma 2 of [12] the subspace ${\uparrow_{\preccurlyeq}}\eta^s$ of $(\mathscr{I}_\lambda^n, \tau)$ is open-and-closed and hence by Lemmas 1 and 3 the poset $({\uparrow_{\preccurlyeq}}\eta^s, \preccurlyeq)$ is compact. Then the inclusion $\{\chi_p^s \colon p \in \mathbb{N}\} \subseteq {\uparrow_{\preccurlyeq}}\eta^s$ implies that the sequence $\{\chi_p \colon p \in \mathbb{N}\}$ has the subsequence $\{\chi_p^s \colon p \in \mathbb{N}\}$ with the compact closure. This completes the proof of the proposition. □

We summarise our results in the following theorem.

Theorem 1. *Let n be any positive integer and λ be any infinite cardinal. Then for any T_1-semitopological semigroup $\mathscr{I}_\lambda^n$ the following conditions are equivalent:*

- (i) *$\mathscr{I}_\lambda^n$ is sequentially pracompact;*

(*ii*) $\mathscr{I}_{\lambda}^{n}$ *is totally countably pracompact;*
(*iii*) $\mathscr{I}_{\lambda}^{n}$ *is feebly compact.*

Proof. Implications $(i)\Rightarrow(iii)$ and $(ii)\Rightarrow(iii)$ are trivial. The corresponding their converse implications $(iii)\Rightarrow(i)$ and $(iii)\Rightarrow(ii)$ follow from Propositions 1 and 2, respectively. □

It is well known that the (Tychonoff) product of pseudocompact spaces is not necessarily pseudocompact (see [10, Section 3.10]). On the other hand Comfort and Ross in [7] proved that the Tychonoff product of an arbitrary family of pseudocompact topological groups is a pseudocompact topological group. The Comfort–Ross Theorem is generalized in [2] and it is proved that a Tychonoff product of an arbitrary non-empty family of feebly compact paratopological groups is feebly compact. Also, a counterpart of the Comfort–Ross Theorem for pseudocompact primitive topological inverse semigroups and primitive inverse semiregular feebly compact semitopological semigroups with closed maximal subgroups was proved in [16] and [18], respectively.

Since the Tychonoff product of H-closed spaces is H-closed (see [5, Theorem 3] or [10, 3.12.5 (d)]) Theorem 1 implies a counterpart of the Comfort–Ross Theorem for feebly compact semitopological semigroups $\mathscr{I}_{\lambda}^{n}$:

Corollary 1. *Let* $\left\{\mathscr{I}_{\lambda_i}^{n_i}\colon i\in\mathscr{J}\right\}$ *be a family of non-empty feebly compact* T_1-*semitopological semigroups and* $n_i\in\mathbb{N}$ *for all* $i\in\mathscr{I}$. *Then the Tychonoff product* $\prod\left\{\mathscr{I}_{\lambda_i}^{n_i}\colon i\in\mathscr{I}\right\}$ *is feebly compact.*

Definition 1. If $\{X_i\colon i\in\mathscr{I}\}$ is an uncountable family of sets, $X=\prod\{X_i\colon i\in\mathscr{J}\}$ is their Cartesian product and p is a point in X, then the subset

$$\Sigma(p,X)=\{x\in X\colon\ |\{i\in\mathscr{J}\colon x(i)\neq p(i)\}|\leqslant\omega\}$$

of X is called the Σ-*product* of $\{X_i\colon i\in\mathscr{J}\}$ with the basis point $p\in X$. In the case when $\{X_i\colon i\in\mathscr{J}\}$ is a family of topological spaces we assume that $\Sigma(p,X)$ is a subspace of the Tychonoff product $X=\prod\{X_i\colon i\in\mathscr{J}\}$.

It is obvious that if $\{X_i\colon i\in\mathscr{J}\}$ is a family of semigroups then $X=\prod\{X_i\colon i\in\mathscr{J}\}$ is a semigroup as well. Moreover $\Sigma(p,X)$ is a subsemigroup of X for arbitrary idempotent $p\in X$. Theorem 1 and Proposition 2.2 of [20] imply the following corollary.

Corollary 2. *Let* $\left\{\mathscr{I}_{\lambda_i}^{n_i}\colon i\in\mathscr{J}\right\}$ *be a family of non-empty feebly compact* T_1-*semitopological semigroups and* $n_i\in\mathbb{N}$ *for all* $i\in\mathscr{J}$. *Then for every idempotent* p *of the product* $X=\prod\left\{\mathscr{I}_{\lambda_i}^{n_i}\colon i\in\mathscr{J}\right\}$ *the* Σ-*product* $\Sigma(p,X)$ *is feebly compact.*

3. On compact shift continuous topologies on the semigroup $\mathscr{I}_{\lambda}^{n}$

The following example implies that there exists a countable feebly compact Hausdorff semitopological semigroup $\left(\mathscr{I}_{\omega}^{2},\right)$ which is not ω-bounded-pracompact.

Example 1. The following family

$$\mathscr{B}_{\mathsf{c}}=\big\{U_\alpha(\alpha_1,\ldots,\alpha_k)={\uparrow_{\preccurlyeq}}\alpha\setminus({\uparrow_{\preccurlyeq}}\alpha_1\cup\cdots\cup{\uparrow_{\preccurlyeq}}\alpha_k)\colon\\ \alpha_i\in{\uparrow_{\preccurlyeq}}\alpha\setminus\{\alpha\},\alpha,\alpha_i\in\mathscr{I}_{\omega}^{2},i=1,\ldots,k\big\}$$

determines a base of the topology τ_{c} on $\mathscr{I}_{\omega}^{2}$. By Proposition 10 from [22], $(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{c}})$ is a Hausdorff compact semitopological semigroup with continuous inversion.

We construct a stronger topology τ_{fc}^{2} on $\mathscr{I}_{\lambda}^{2}$ in the following way. For every nonzero element $x\in\mathscr{I}_{\lambda}^{2}$ we assume that the base $\mathscr{B}_{\mathsf{fc}}^{2}(x)$ of the topology τ_{fc}^{2} at the point x coincides with the base of the topology τ_{c}^{2} at x, and

$$\mathscr{B}_{\mathsf{fc}}^{2}(0)=\left\{U_B(\mathbf{0})=U(\mathbf{0})\setminus\left(\mathscr{I}_{\lambda}^{2}\setminus\{\mathbf{0}\}\right\}:U(0)\in\mathscr{B}_{\mathsf{c}}^{2}(0)\right\}$$

form a base of the topology τ_{fc}^{2} at zero $\mathbf{0}$ of the semigroup $\mathscr{I}_{\lambda}^{2}$. Since $(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{fc}}^{2})$ is a variant of the semitopological semigroup defined in Example 3 of [12], τ_{fc}^{2} is a Hausdorff topology on $\mathscr{I}_{\lambda}^{2}$. Moreover, by Proposition 1 of [12], $(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{fc}}^{2})$ is a countably pracompact semitopological semigroup with continuous inversion.

Proposition 3. *The space* $(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{fc}}^{2})$ *is not* ω*-bounded-pracompact.*

Proof. Since the space $(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{fc}}^{2})$ is feebly compact and Hausdorff, by Proposition 2 of [12] the subset $\mathscr{I}_{\lambda}^{2}\setminus\mathscr{I}_{\lambda}^{1}$ is dense in $(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{fc}}^{2})$, and by Lemma 2 from [12] every point of the set $\mathscr{I}_{\lambda}^{2}\setminus\mathscr{I}_{\lambda}^{1}$ is isolated in $(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{fc}}^{2})$. This implies that every dense subset D of $(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{fc}}^{2})$ contains the set $\mathscr{I}_{\lambda}^{n}\setminus\mathscr{I}_{\lambda}^{n-1}$. Then

$$\operatorname{cl}_{\left(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{fc}}^{2}\right)}(D)=\operatorname{cl}_{\left(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{fc}}^{2}\right)}(\mathscr{I}_{\lambda}^{2}\setminus\mathscr{I}_{\lambda}^{1})=\mathscr{I}_{\omega}^{2}$$

for every dense subset D of $(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{fc}}^{2})$. Since $\mathscr{I}_{\omega}^{2}$ is countable, so is D, and hence the space $(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{fc}}^{2})$ is not ω-bounded-pracompact, because $(\mathscr{I}_{\omega}^{2},\tau_{\mathsf{fc}}^{2})$ is not compact. □

Proposition 4. *Let n be any positive integer and λ be any infinite cardinal. If $\mathscr{I}_{\lambda}^{n}$ is a T_1-semitopological semigroup then the following statements hold:*

(1) *$\mathscr{I}_{A}^{n}$ is a closed subsemigroup of $\mathscr{I}_{\lambda}^{n}$ for any subset $A\subseteq\lambda$;*
(2) *the band $E(\mathscr{I}_{\lambda}^{n})$ is a closed subset of $\mathscr{I}_{\lambda}^{n}$.*

Proof. (1) Fix an arbitrary $\gamma\in\mathscr{I}_{\lambda}^{n}\setminus\mathscr{I}_{A}^{n}$. Then $\operatorname{dom}\gamma\nsubseteq A$ or $\operatorname{ran}\gamma\nsubseteq A$. Since $\eta\preccurlyeq\delta$ if and only if $\mathsf{graph}(\eta)\subseteq\mathsf{graph}(\delta)$ for $\eta,\delta\in\mathscr{I}_{\lambda}^{n}$, the above arguments imply that ${\uparrow_{\preccurlyeq}}\gamma\cap\mathscr{I}_{A}^{n}=\varnothing$. By Lemma 2 of [12] the set ${\uparrow_{\preccurlyeq}}\gamma$ is open in $\mathscr{I}_{\lambda}^{n}$, which implies statement (1).

(2) Fix an arbitrary $\gamma\in\mathscr{I}_{\lambda}^{n}\setminus E(\mathscr{I}_{\lambda}^{n})$. Since $\mathscr{I}_{\lambda}^{n}$ is an inverse subsemigroup of the symmetric inverse monoid $\mathscr{I}_{\lambda}$, all idempotents of $\mathscr{I}_{\lambda}^{n}$ is are partial identity maps of rank $\leqslant n$. Then similar arguments as in statement (1) imply that $E(\mathscr{I}_{\lambda}^{n})$ is a closed subset of $\mathscr{I}_{\lambda}^{n}$. □

Proposition 4 implies the following corollary.

Corollary 3. *Let n be any positive integer, λ be any infinite cardinal and A be an arbitrary infinite subset of λ. If $\mathscr{I}_{\lambda}^{n}$ is a compact T_1-semitopological semigroup then $\mathscr{I}_{A}^{n}$ with the induced topology from $\mathscr{I}_{\lambda}^{n}$ is a compact semitopological semigroup.*

Lemma 5. *Let n be any positive integer, λ be any infinite cardinal and A be an arbitrary infinite countable subset of λ. If $\mathscr{I}_{\lambda}^{n}$ is an ω-bounded-pracompact T_1-semitopological semigroup then $\mathscr{I}_{A}^{n}\setminus\mathscr{I}_{A}^{n-1}$ is a dense subset of $\mathscr{I}_{A}^{n}$, and hence $\mathscr{I}_{A}^{n}$ is compact.*

Proof. For any $\alpha\in\mathscr{I}_A^n$ we denote $\uparrow_{\preccurlyeq}^A\alpha=\uparrow_{\preccurlyeq}\alpha\cap\mathscr{I}_A^n$.

By induction we shall show that the set $\uparrow_{\preccurlyeq}^A\alpha\cap(\mathscr{I}_A^n\setminus\mathscr{I}_A^{n-1})$ is dense in $\uparrow_{\preccurlyeq}^A\alpha$ for any $\alpha\in\mathscr{I}_A^n$. In the case when $\operatorname{rank}\alpha=n-1$ by Lemmas 1 and 3 we have that the set $\uparrow_{\preccurlyeq}\alpha$ is compact, and hence by Proposition 4(1), $\uparrow_{\preccurlyeq}^A\alpha$ is compact as well. Since all points of $\mathscr{I}_A^n\setminus\mathscr{I}_A^{n-1}$ are isolated in $\mathscr{I}_\lambda^n$, the set $\uparrow_{\preccurlyeq}^A\alpha\cap(\mathscr{I}_A^n\setminus\mathscr{I}_A^{n-1})$ is dense in $\uparrow_{\preccurlyeq}^A\alpha$.

Next we show that the statement $\uparrow_{\preccurlyeq}^A\alpha\cap(\mathscr{I}_A^n\setminus\mathscr{I}_A^{n-1})$ *is dense in* $\uparrow_{\preccurlyeq}^A\alpha$ *for any* $\alpha\in\mathscr{I}_A^n$ *with* $\operatorname{rank}\alpha=n-k$*, for all* $k<m$ implies that the same is true for any $\beta\in\mathscr{I}_A^n$ with $\operatorname{rank}\beta=n-m$, where $m\leqslant n$. Fix an arbitrary $\beta\in\mathscr{I}_A^n$ with $\operatorname{rank}\beta=n-m$. Suppose to the contrary that the set $\uparrow_{\preccurlyeq}^A\beta\cap(\mathscr{I}_A^n\setminus\mathscr{I}_A^{n-1})$ is not dense in $\uparrow_{\preccurlyeq}^A\beta$. The assumption of induction implies that $\gamma\in\operatorname{cl}_{\mathscr{I}_A^n}(\uparrow_{\preccurlyeq}^A\beta\cap(\mathscr{I}_A^n\setminus\mathscr{I}_A^{n-1}))$ for any $\gamma\in\uparrow_{\preccurlyeq}^A\beta\setminus\{\beta\}$, and hence $\beta\notin\operatorname{cl}_{\mathscr{I}_A^n}(\uparrow_{\preccurlyeq}^A\beta\cap(\mathscr{I}_A^n\setminus\mathscr{I}_A^{n-1}))$. Then there exists an open neighbourhood $U(\beta)$ of β in $\mathscr{I}_A^n$ such that $U(\beta)\cap(\uparrow_{\preccurlyeq}^A\beta\cap(\mathscr{I}_A^n\setminus\mathscr{I}_A^{n-1}))=\varnothing$. By Lemma 2 from [12] for any $\delta\in\mathscr{I}_\lambda^n$ the set $\uparrow_{\preccurlyeq}\delta$ is open-and-closed in $\mathscr{I}_\lambda^n,\tau$, and hence $\uparrow_{\preccurlyeq}^A\delta$ is open-and-closed in $\mathscr{I}_A^n$ as well. Hence we get that

$$\operatorname{cl}_{\mathscr{I}_A^n}(\uparrow_{\preccurlyeq}^A\beta\cap(\mathscr{I}_A^n\setminus\mathscr{I}_A^{n-1}))=\uparrow_{\preccurlyeq}^A\beta\setminus\{\beta\}$$

but the family $\mathscr{U}=\left\{\uparrow_{\preccurlyeq}^A\delta\colon\delta\in\uparrow_{\preccurlyeq}^A\beta\setminus\{\beta\}\right\}$ is an open cover of $\uparrow_{\preccurlyeq}^A\beta$ which has no a finite subcover. This contradicts the condition that $\mathscr{I}_\lambda^n$ is a ω-bounded-pracompact space, which completes the proof of the first statement of the lemma. The last statement immediately follows from the first statement and the definition of the ω-bounded-pracompact space. □

Theorem 2 describes feebly ω-bounded shift-continuous T_1-topologies on the semigroup $\mathscr{I}_\omega^n$.

Theorem 2. *Let n be any positive integer and λ be any infinite cardinal. Then for any T_1-semitopological semigroup $\mathscr{I}_\lambda^n$ the following conditions are equivalent:*

- *(i) $\mathscr{I}_\lambda^n$ compact;*
- *(ii) $\mathscr{I}_\lambda^n$ is ω-bounded-pracompact;*
- *(iii) $\mathscr{I}_\lambda^n$ is feebly ω-bounded.*

Proof. Implications $(i)\Rightarrow(iii)$ and $(ii)\Rightarrow(iii)$ are trivial.

$(iii)\Rightarrow(ii)$ Let $\mathscr{I}_\lambda^n$ be a feebly ω-bounded T_1-semitopological semigroup. By Proposition 2 of [12] the set $\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}$ is dense in $\mathscr{I}_\lambda^n$. Fix an arbitrary infinite countable subset $D=\{\alpha_i\colon i\in\mathbb{N}\}$ in $\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}$. By Lemma 2 from [12] every point of D is isolated in $\mathscr{I}_\omega^n$, and hence by feeble ω-boundedness of $\mathscr{I}_\lambda^n$ we get that there exists a compact subset $K\subseteq\mathscr{I}_\lambda^n$ such that $D\subseteq K$. Since the closure of a subset in compact space is compact, so is the closure of D. Hence the space $\mathscr{I}_\lambda^n$ is ω-bounded-pracompact.

$(ii)\Rightarrow(i)$ Suppose the contrary: there exists a noncompact ω-bounded-pracompact T_1-semitopological semigroup $\mathscr{I}_\lambda^n$. By Theorem 1 of [12] the space $\mathscr{I}_\lambda^n$ is not countably compact. Then by Theorem 3.10.3 of [10] the space $\mathscr{I}_\lambda^n$ has an infinite countable closed discrete subspace D. We put

$$A=\{x\in\lambda\colon x\in\operatorname{dom}\alpha\cup\operatorname{ran}\alpha\text{ for some }\alpha\in D\}.$$

Since the set D is countable, $\bigcup\limits_{\alpha\in D}(\operatorname{dom}\alpha\cup\operatorname{ran}\alpha)$ is countable, and hence A is countable, too. Then $\mathscr{I}_A^n$ contains D. By Proposition 4(1), $\mathscr{I}_A^n$ is a closed subspace of $\mathscr{I}_\lambda^n$, which implies that D is an infinite countable closed discrete subspace of $\mathscr{I}_A^n$. This contradicts Lemma 5, and hence $\mathscr{I}_\lambda^n$ is compact. □

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ЗАУВАЖЕННЯ ПРО СЛАБКО КОМПАКТНІ НАПІВТОПОЛОГІЧНІ СИМЕТРИЧНІ ІНВЕРСНІ НАПІВГРУПИ ОБМЕЖЕНОГО СКІНЧЕННОГО РАНГУ

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Вивчаємо слабко компактні трансляційно-неперервні T_1-топології на симетричній інверсній напівгрупі $\mathscr{I}_\lambda^n$ скінченних перетворень кардинала λ обмеженого рангу $\leqslant n$. Доведено, що така T_1-топологія секвенціально пракомпакна тоді і тільки тоді, коли вона слабко компактна. Також, ми довели, що кожна трансляційно-неперервна слабко ω-обмежена T_1-топологія на напівгрупі $\mathscr{I}_\lambda^n$ компактна.

Ключові слова: напівгрупа, інверсна напівгрупа, напівтопологічна напівгрупа, компактний, секвенціально пракомпактний, цілком зліченно пракомпактний, ω-обмежений-пракомпактний, слабко ω-обмежений, слабко компактний, Δ-система, лема про соняшник, добуток, Σ-добуток.