ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2020. Випуск 90. С. 105–112
Visnyk of the Lviv Univ. Series Mech. Math. 2020. Issue 90. P. 105–112
http://publications.lnu.edu.ua/bulletins/index.php/mmf
doi: http://dx.doi.org/10.30970/vmm.2020.90.105-112

УДК 517.95

OPTIMAL AGE CONTROL OF BIO-POPULATION DYNAMICS

Volodymyr KYRYLYCH, Olha MILCHENKO

Ivan Franko National University of Lviv,
Universytetska Str., 1, 79000, Lviv, Ukraine
e-mail: vkyrylych@ukr.net, olga.milchenko@lnu.edu.ua

The object of this paper is the problem of optimal control of a semi-linear system of two first-order hyperbolic equations with integral constraints. Using the method of characteristics and the maximum principle the necessary conditions for optimality are obtained.

Key words: optimal control, hyperbolic system, method of characteristics, needle variation, bio-population theory

1. Introduction

Based on the classical McKendrick-von Foerster [9, 12] model of bio-population dynamics, mathematical models for the age structure of the population itself are widely used in modern conditions. Such problems also arise in studies of the reproduction and spread of epidemics, drug addiction [1]–[4], socio-economic and economic processes [7, 8, 10], etc.

In this paper we consider an optimal control problem for the competing populations dynamics [11], which is described by a system of two hyperbolic equations [5], for which the control function is subject to an integral constraint [2, 5]. To obtain a necessary condition of the optimal control, the method of characteristics and the formula for the increment of the target functional on the needle variation of the admissible control have been used.

2. Statement of the problem

In the domain $(x,t) \in \Pi = (0,l) \times (0,T)$, consider the process of population development, which evolution in space and time is described by the system of two hyperbolic

2020 Mathematics Subject Classification: 18A22 first-order equations

$$\tag{1} \begin{cases} \dfrac{\partial y_1(x,t)}{\partial t} + \dfrac{\partial y_1(x,t)}{\partial x} & = -\mu(x) y_1(x,t), \\ \dfrac{\partial y_2(x,t)}{\partial t} + \dfrac{\partial y_2(x,t)}{\partial x} & = -y_1(x,t) y_2(x,t), \end{cases}$$

where $y = (y_1, y_2) : \overline{\Pi} \to \mathbb{R}^2$.

For system (1) set the initial and boundary conditions:

$$\tag{2} y_1(x,0) = \nu_1(x), \;\; y_2(x,0) = \nu_2(x), \;\; x \in [0,l],$$

$$\tag{3} y_1(0,t) = \lambda(t), \;\; t \in [0,T],$$

$$\tag{4} y_2(0,t) = \beta(t) \int\limits_{x_1}^{x_2} K(x) u(x) y_2(x,t) dx, \;\; t \in [0,T].$$

Here $\mu(x)$, $\nu_1(t)$, $\nu_2(t)$, $\beta(t)$, $K(s)$ are standard bio-population parameters [2, 5]. For example, when (4) describes the individuals fertility process, then $K(x)$ is the fraction of females in the population process. The control function here is $u = u(x)$, which sets the age distribution of the reproductive period of females ($0 < x_1 < x_2 = l$) [2].

Let us choose minimization of the functional

$$\tag{5} I[u] = \int\limits_0^l \varphi(y_1(x,T), y_2(x,T), x) dx.$$

as the target.

The problem (1)-(5) will be investigated under the following assumptions:

A1) $u, K \in C^1[x_1, x_2]$; in addition, we will assume that $u(x) \equiv 0$, $K(x) \equiv 0$, if $x \notin [x_1, x_2]$;

A2) $\nu_i \in C^1[0,l], i = 1,2$, and $\lambda \in C^1[0,T]$;

A3) $\mu \in C[0,l]$;

A4) $\varphi, \varphi'_{y_i} \in C(\overline{\Pi} \times \overline{\Pi} \times [0,l]), i = 1,2$;

A5) zero-order agreement conditions $\nu_1(0) = \lambda(0)$,

$$\nu_2(0) = \beta(0) \int\limits_{x_1}^{x_2} K(s) u(s) \nu_2(s) ds$$

are fulfilled;

A6) $\int\limits_0^l \mu(x) dx = +\infty$, $\int\limits_{x_1}^{x_2} u(x) dx = 1$.

(the fulfillment of conditions **A6**) ensures the natural parameters of the population, e.g. the first of conditions of **A6**) guarantees the zero density of individuals in the population if their age exceeds the maximal limit l [3]).

Remark 1. Usually, in applied economic problems the function $\beta(t)$ is understood as $\beta(t) = e^{-\rho t} g(t)$, where ρ is a discount rate, g is a bounded function on $[0, T]$.

3. Generalized solution of the problem

Let $\xi = \tau + x - t$ be the characteristic equation of the system (1), i.e. the solution of the Cauchy problem $\dfrac{d\xi}{d\tau} = 1$, $\xi(t) = x$, $(x,t) \in \overline{\Pi}$. Then, integrating (1) in the corresponding boundaries along the characteristics, we obtain

$$y_1(x,t) = \begin{cases} \lambda(t-x)\exp\Big(-\int\limits_0^x \mu(\sigma)d\sigma\Big), & x \leqslant t, \\ \nu_1(x-t)\exp\Big(-\int\limits_{x-t}^x \mu(\sigma)d\sigma\Big), & x > t, \end{cases} \tag{6}$$

$$y_2(x,t) = \begin{cases} \beta(t-x)\int\limits_{x_1}^{x_2} K(r)u(r)y_2(r,t-x)dr \cdot \exp\Bigg(-\lambda(t- \\ -x)\int\limits_0^x \exp\Big(-\int\limits_0^\rho \mu(\sigma)d\sigma\Big)\Bigg)d\rho, & x \leqslant t, \\ \nu_2(x-t)\exp\Bigg(-\nu_1(x-t)\int\limits_0^t \exp\Big(-\int\limits_{x-t}^\rho \mu(\sigma)d\sigma\Big)\Bigg)d\rho, & x > t. \end{cases} \tag{7}$$

Definition 1. By generalized solution of the problem (1)–(4), that corresponds to the control u, we mean the vector function $y = (y_1, y_2) \in \Big(C(\overline{\Pi})\Big)^2$ whose components satisfy relations (6), (7) in $\overline{\Pi}$, if the conditions **A1**)–**A6**) are fulfilled.

Remark 2. The fulfillment of conditions **A5**) guarantees the continuity of solution of problem (1)–(5) when passing through the characteristic $x = t$, besides, the second condition of **A5**) represents an additional integral constraint on the control u.

4. The optimality condition

Consider the formula of the increment of the functional (5) $\triangle I(u) = I(\tilde{u}) - I(u)$ on admissible processes $\{u, y_1, y_2\}$ and $\{\tilde{u} = u + \triangle u, \tilde{y}_1 = y_1 + \triangle y_1, \tilde{y}_2 = y_2 + \triangle y_2\}$ [2, 6, 10]. The functions $\triangle y_i = \triangle y_i(x,t), i = 1,2$ are the solution of mixed problem

$$\frac{\partial \triangle y_1}{\partial t} + \frac{\partial \triangle y_1}{\partial x} = -\mu(x)\triangle y_1, \;\; (x,t) \in \Pi,$$

$$\triangle y_1(x,0) = 0, \;\; 0 \leqslant x \leqslant l, \tag{8}$$

$$\triangle y_1(0,t) = 0, \;\; 0 \leqslant t \leqslant T,$$

$$\frac{\partial \triangle y_2}{\partial t} + \frac{\partial \triangle y_2}{\partial x} = -\triangle y_1 \triangle y_2, \;\; (x,t) \in \Pi,$$

$$\triangle y_2(0,t)=\beta(t)\int\limits_{x_1}^{x_2}K(r)\{\tilde u(r)\tilde x_2(r,t)-u(r)x_2(r,t)\}dr,\ \ 0\leqslant t\leqslant T. \tag{9}$$

Then, taking into account (8)–(9), we write the formula of the increment of the functional

$$\triangle I(u)=I(\tilde u)-I(u)=\int\limits_0^l \triangle\varphi(y_1(x,T),y_2(x,t),x)dx+$$
$$+\int\limits_0^l\int\limits_0^T\left\{\psi_1(x,t)\left[\frac{\partial\triangle y_1}{\partial t}+\frac{\partial\triangle y_1}{\partial x}+\mu(x)\triangle y_1\right]+\psi_2(x,t)\left[\frac{\partial\triangle y_2}{\partial t}+\frac{\partial\triangle y_2}{\partial x}+\triangle y_1\triangle y_2\right]\right\}dtdx,$$

where $\triangle\varphi=\varphi(\tilde y_1(x,T),\tilde y_2(x,T),x)-\varphi(y_1(x,T),y_2(x,T),x)$, and $\psi_i(x,t),\ i=1,2$, are now arbitrary smooth in $\overline{\Pi}$ functions.

Let us perform the standard transformations for such cases (using Taylor formula, integration by parts, etc.).

First of all, we develop $\triangle\varphi$ by the Taylor formula into a series, singling out the linear part, namely

$$\triangle\varphi(y_1(x,T),y_2(x,T),x)=\frac{\partial\varphi}{\partial y_1}\triangle y_1(x,T)+\frac{\partial\varphi}{\partial y_2}\triangle y_2(x,T)+o_\varphi(|\triangle y_1|,|\triangle y_2|),$$

where $o_\varphi(\cdot,\cdot)$ stands for higher orders of smallness with respect to $\triangle y_1$ and $\triangle y_2$.

Then

$$\triangle I(u)=\int\limits_0^l\Big\{\varphi'_{y_1}(y_1(x,T),y_2(x,T),x)\triangle y_1(x,T)+\varphi'_{y_2}(y_1(x,T),y_2(x,T),x)\triangle y_2(x,T)+$$
$$+o_\varphi(|\triangle y_1|,|\triangle y_2|)\Big\}dx-\int\limits_0^l\int\limits_0^T\Big\{\frac{\partial\psi_1}{\partial t}+\frac{\partial\psi_1}{\partial x}-\mu(x)\psi_1(x,t)\Big\}\triangle y_1(x,t)dtdx-$$
$$-\int\limits_0^l\int\limits_0^T\Big\{\frac{\partial\psi_2}{\partial t}+\frac{\partial\psi_2}{\partial x}-\psi_2(x,t)\triangle y_1(x,t)\Big\}\triangle y_2(x,t)dtdx-$$
$$-\int\limits_0^l\int\limits_0^T\beta(t)\psi_2(0,t)K(r)u(r)\triangle y_2(r,t)dtds+\int\limits_0^l\psi_1(x,T)\triangle y_1(x,T)dx+$$
$$+\int\limits_0^T\psi_1(l,T)\triangle y_1(l,T)dt+\int\limits_0^l\psi_2(s,T)\triangle y_2(x,T)dx-$$
$$-\int\limits_0^l\int\limits_0^T\beta(t)\psi_2(0,t)K(r)\triangle u(r)y_2(r,t)dtds-\int\limits_0^l\int\limits_0^T\beta(t)\psi_2(0,t)K(r)\triangle u(r)\triangle y_2(r,t)dtdr.$$

Taking into account the values for expressions of states of the system (8), (9), we can write down the relations

$$\triangle y_1(x,t) = 0, \text{ for all } (x,t) \in \overline{\Pi}, \tag{10}$$

$$\triangle y_2(x,t) = \begin{cases} \beta(t-x)\int\limits_0^l K(r)\{\tilde{u}(r)\tilde{y}_2(r,t-x) - u(r)y_2(r,t-x)\}\times \\ \times \exp\Big(-\lambda(t-x)\exp\Big(-\int\limits_0^\rho \mu(\sigma)d\sigma\Big)\Big)d\rho, & x \leqslant t, \\ 0, & x > t. \end{cases} \tag{11}$$

These transformations allow us to formulate a conjugate problem for the functions $\psi_i(x,t),\ i = 1,2$:

$$\begin{gathered} \frac{\partial \psi_1}{\partial t} + \frac{\partial \psi_1}{\partial x} = \mu(x)\psi_1(x,t), \quad (x,t) \in \Pi, \\ \psi_1(x,T) = -\frac{\partial \varphi(y_1(x,T), y_2(x,T), x)}{\partial y_1}, \quad x \in [0,l], \\ \psi_1(l,t) = 0, \quad t \in [0,T], \end{gathered} \tag{12}$$

$$\begin{gathered} \frac{\partial \psi_2}{\partial t} + \frac{\partial \psi_2}{\partial x} = -\beta(t)\psi_2(0,t)K(x)u(x), \quad (x,t) \in \Pi, \\ \psi_2(x,T) = -\frac{\partial \varphi(y_1(x,T), y_2(x,T), x)}{\partial y_2}, \quad x \in [0,l], \\ \psi_2(l,t) = 0, \quad t \in [0,T]. \end{gathered} \tag{13}$$

For an arbitrarily fixed control $u = u(x)$, the hyperbolic systems (12), (13) with the corresponding initial conditions at $t = T$ and boundary conditions at $x = l$ have a unique generalized solution [6]. Here, by the generalized solution for arbitrary control u we also mean $\psi_1 \in C(\overline{\Pi})$ and $\psi_2 \in C(\overline{\Pi})$, and for ψ_2 as a continuous solution of the corresponding second order Volterra integro-operator equation. Now, given (12), (13), we can rewrite the modified expression for the increment of the functional as

$$\triangle I(u) = -\int\limits_0^l K(r)\triangle u(r)\int\limits_0^T \beta(t)\psi_2(0,t)y_2(r,t)dr - \gamma, \tag{14}$$

where

$$\gamma = -\int\limits_0^l K(r)\triangle u(r)\int\limits_0^T \beta(t)\psi_2(0,t)\triangle y_2(x,t)dtdr + \int\limits_0^l o_\varphi(|\triangle y(x,T)|)dx. \tag{15}$$

The obtained ratio (14) for the increment of the functional (5) is valid for two admissible processes $\{u, y_1, y_2\}$ and $\{\tilde{u}, \tilde{y}_1, \tilde{y}_2\}$.

Further investigation of the problem (1)–(5) is based on the application of the nonclassical variation of admissible controls described, for example, in [2, p. 123].

Let $u = u(x)$ be an admissible control and construct its variation by rule

$$u_{\varepsilon,\delta}(x) = \left(1 + \varepsilon \frac{d\delta(x)}{dx}\right) u(x + \varepsilon\delta(x)), \tag{16}$$

where $\varepsilon \in [0,1]$ is a parameter that characterizes the smallness of the variation, $\delta(x)$ is a continuously differentiable function that satisfies conditions

$$0 \leqslant x + \delta(x) \leqslant l, \ \ \frac{d\delta(x)}{dx} \geqslant -1, \ \ x \in [0,l], \ \ \delta(0) = \delta(l) = 0.$$

Let us emphasize some properties of variation (16). First, the control is smooth, and the range of values of the function $u_{\varepsilon,\delta}(x)$ is determined by the range of values of the control $u(x)$. Moreover, to satisfy the second condition of **A6**), by substituting $s = x + \varepsilon\delta(x)$, we obtain

$$\int_0^l u_{\varepsilon,\delta}(x)dx = \int_0^l \left(1 + \varepsilon \frac{d\delta(x)}{dx}\right) u(x + \varepsilon\delta(x))dx = \int_0^l u(s)ds.$$

Let us return to the increment of the functional (14) and make an estimate of the remainder term γ for admissible processes $\{u, y_1, y_2\}$ and $\{u_{\varepsilon,\delta} = u + \triangle u, y_{1\varepsilon} = y_1 + \triangle y_1, y_{2\varepsilon} = y_2 + \triangle y_2\}$. That is, given (16), we obtain

$$\begin{aligned}\triangle u(s) = \left(1 + \varepsilon \frac{d\delta(x)}{dx}\right) u(x + \varepsilon\delta(x)) - u(x) = \\ = u(x + \varepsilon\delta(x)) - u(x) + \varepsilon \frac{d\delta(x)}{dx} u(x + \varepsilon\delta(x)) = \\ = \varepsilon \frac{d}{dx}\Big(\delta(x)u(x)\Big) + o(\varepsilon),\end{aligned}$$

which indicates the increment of control as a value of order ε [2].

Given the introduced variation (16), let us rewrite (11) the increment of state $y_2(x,t)$, namely

$$\triangle y_2(x,t) = \begin{cases} 0, & x \geqslant t, \\ \beta(t-x) \int\limits_0^l K(r)\{u_{\varepsilon,\delta}(r) y_{2\varepsilon}(r, t-x) - u(r) y_2(r, t-x)\} \times & \\ \times \exp\left(-\lambda(t-x) \exp\Big(-\int\limits_0^\rho \mu(\sigma)d\sigma\Big)\right) d\rho, & x < t. \end{cases}$$

Based on assumptions **A1**)–**A6**), that is, given the constraint of the initial data in $\overline{\Pi}$ and making transformations

$$\begin{aligned}u_{\varepsilon,\delta}(x) y_{2\varepsilon}(x, t-x) - u(x) y_2(x, t-x) = \\ = u_{\varepsilon,\delta}(x) y_{2\varepsilon}(x, t-x) - u(x) y_{2\varepsilon}(x, t-x) + u(x) y_{2\varepsilon}(x, t-x) - \\ - u(x) y_2(x, t-x) = \triangle u(x) y_{2\varepsilon}(x, t-x) + u(x) \triangle y_2(x, t-x),\end{aligned}$$

we obtain an estimate

$$|\triangle y_2(x,t)| \leqslant L_1 \int_{x_1}^{x_2} |\triangle y_2(x,t-x)|dx + L_2 \int_{x_1}^{x_2} |\triangle u(x)|dx, \tag{17}$$

where $L_i = \text{const} \geqslant 0$, $i = 1,2$, or given that $|\triangle u| \sim \varepsilon$, estimating, for example, (17) through the resolvent of the corresponding kernel, it is easy to obtain that

$$|\triangle y_2(x,t)| \sim \varepsilon \text{ for } (x,t) \in \overline{\Pi}.$$

As a result, the increment of the functional (14) can be written as

$$\triangle I(u) = I(u_{\varepsilon,\delta}) - I(u) = \tag{18}$$

$$= -\varepsilon \int_{x_1}^{x_2} K(x) \frac{d}{dx}(\delta(x)u(x)) \int_0^T \psi_2(0,t)\beta(t)y_2(x,t)dtdx + o(\varepsilon),$$

or

$$\triangle I(u) = -\varepsilon \Big\{ \delta(x)u(x) \int_0^T K(x)\psi_2(0,t)\beta(t)y_2(x,t)dt \Big|_{x_1}^{x_2} -$$

$$- \int_{x_1}^{x_2} \delta(x)u(x) \frac{d}{dx} \int_0^T K(x)\psi_2(0,t)\beta(t)y_2(x,t)dt \Big\} =$$

$$= \varepsilon \int_{x_1}^{x_2} \delta(x)u(x) \int_0^T \psi_2(0,t)\beta(t) \frac{d}{dx}\Big(K(x)y_2(x,t)\Big)dt.$$

Then, based on an arbitrary choice of $\delta(x)$ and the main lemma of the calculus of variations, we obtain the necessary optimality condition.

Theorem 1. *If the process $\{y^*, u^*\}$ is optimal in problem (1)–(5), then condition*

$$u^*(x) \int_0^T \psi_2^*(0,t)\beta(t) \frac{d}{dx}\Big(K(x)y_2^*(x,t)\Big)dt = 0, \quad x \in [0,l],$$

is fulfilled, where $y_2^*(x,t)$ is expressed by (7), and $\psi_2^*(x,t)$ is the generalized solution of the conjugate problem (13), at $u = u^*(x)$, $y = y^*(x,t)$.

The obtained result can serve for the construction of numerical algorithms for solving optimal control problems [2, p. 125].

References

1. C. Almeder, J. P. Caulkins, G. Feichtinger, and G. Tragler, *An age-structured single-state drug initiation model-cycles of grug epidemics and optimal prevention programs*, Socio-Economic Planning Sciences **38** (2004), no. 1, 91–109. DOI: 10.1184/R1/6471008.v1
2. A. V. Arguchintsev, *Optimal control of hyperbolic systems*, Fizmatlit Publ, Moscow, 2007 (in Russian).

3. W. L. Chan and B. Z. Guo, *Optimal birth control of population dynamics*, J. Math. Anal. Appl. **144** (1989), no. 2, 532–552. DOI: 10.1016/0022-247x(89)90350-8
4. S. Chawla and S. Lenhart, *Application of optimal control theory to bioremediation*, J. Comput. Appl. Math. 114 (2000), no. 1, 81–102. DOI: 10.1016/S0377-0427(99)00290-3
5. A. L. Dawidowicz, A. Poskrobko, *Matematyczne modele dynamiki populacji zależne od wieku*, Metody matematyczne w zastosowaniach, 2014, Tom **2**, pp. 39–58.
6. Н. В. Дерев'янко, В. М. Кирилич, *Оптимальне керування гіперболічною системою напівлінійних рівнянь першого порядку з нескінченним горизонтом планування*, Укр. мат. ж. **67** (2015), no. 2, 185–201; **English version:** T. O. Derev'yanko and V. M. Kyrylych, *Problem of optimal control for a semilinear hyperbolic system of equations of the first order with infinite horizon planning*, Ukr. Math. J. **67** (2015), no. 2, 211–229. DOI: 10.1007/s11253-015-1075-3
7. G. Feichtinger, G. Tragler, and V. M. Veliov, *Optimality conditions for age-stractured control systems*, J. Math. Anal. Appl. **288** (2003), no. 1, 47–68. DOI: 10.1016/j.jmaa.2003.07.001
8. N. Hritoneko and Y. Yatsenko, *Mathematical modelling in economics, ccology and the environment*, 2nd Ed., Springer Optimization and Its Applications **88**, Springer 2013. DOI: 10.1007/978-1-4614-9311-2
9. A. G. M'Kendrick, *Applications of mathematics to medical problems*, Proc. Edinb. Math. Soc. **40** (1926), 98–130. DOI: 10.1017/S0013091500034428
10. S. P. Sethi and G. L. Thompson, *Optimal control theory. Applications to Management Science and Economics*, 2nd ed., New York, Springer, 2005.
11. V. Volterra, *Variations and fluctuations of the number of individuals in animal species living together*, In: Chapman, R. N., Ed., Animal Ecology, New York, McGraw-Hill, 1931. pp. 409–448. DOI: 10.1086/284409
12. J. von Foerster, *Some remarks on changing populations*, In: The Kinetics of Cellular Proliferation, ed. by F. Stohlman, Jr., Grune and Stratton, New York, 1955, pp. 382—407.

Стаття: надійшла до редколегії 06.07.2020
прийнята до друку 17.11.2020

ОПТИМАЛЬНЕ КЕРУВАННЯ ВІКОВОЮ ДИНАМІКОЮ БІОПОПУЛЯЦІЇ

Володимир КИРИЛИЧ, Ольга МІЛЬЧЕНКО

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, 79000, Львів
e-mail: vkyrylych@ukr.net, olga.milchenko@lnu.edu.ua

Об'єкт дослідження — задача оптимального керування напівлінійною системою двох гіперболічних рівнянь першого порядку з інтегральними обмеженнями. За допомогою методу характеристик і принципу максимуму одержано необхідні умови оптимальності.

Ключові слова: оптимальне керування, гіперболічна система, метод характеристик, голкова варіація, теорія біопопуляцій.