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CLOSE-TO-CONVEXITY OF POLYNOMIAL SOLUTIONS OF A DIFFERENTIAL EQUATION OF THE SECOND ORDER WITH POLYNOMIAL COEFFICIENTS OF THE SECOND DEGREE

Myroslav SHEREMETA, Yuriy TRUKHAN

Ivan Franko National University of Lviv,
Universytetska Str., 1, 79000, Lviv, Ukraine
e-mails: m.m.sheremeta@gmail.com,
yurkotrukhan@gmail.com

An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function f is said to be convex if $f(\mathbb{D})$ is a convex domain and is said to be close-to-convex if there exists a convex in $\mathbb{D}$ function Φ such that $\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0\,(z \in \mathbb{D})$. We indicate conditions on real parameters β_0, β_1, γ_0, γ_1, γ_2 and α_0, α_1, α_2 of the differential equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = \alpha_0 z^2 + \alpha_1 z + \alpha_2,$$

under which this equation has a polynomial solution

$$f(z) = \sum_{n=0}^{p} f_n z^n \,(\deg f = p \geq 2)$$

close-to-convex in $\mathbb{D}$ together with all its derivatives $f^{(j)}$ $(1 \leq j \leq p-1)$.

Key words: linear non-homogeneous differential equation of the second order, polynomial coefficient, polynomial solution, close-to-convex function.

1. Introduction and auxiliary results

An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known [1, p. 203] (see also [2, p. 8]) that the condition $\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the convexity of f. A function f is said to be close-to-convex in $\mathbb{D}$ (W. Kaplan [3], see also [1, p. 583], [2, p. 11]) if there exists a convex in $\mathbb{D}$ function Φ such that

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$\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0\,(z \in \mathbb{D})$. Any close-to-convex function f has a characteristic property that the complement G of the domain $f(\mathbb{D})$ can be filled with rays which start from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0) \neq 0$. Hence it follows that any function f is close-to-convex in $\mathbb{D}$ if and only if the function $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ is close-to-convex in $\mathbb{D}$, where $g_n = f_n/f_1$.

S. M. Shah [4] indicated conditions on real parameters β_0, β_1, γ_0, γ_1, γ_2 of the differential equation

$$z^2 w'' + (\beta_0 z^2 + \beta_1 z) w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2) w = 0, \tag{2}$$

under which there exists an entire transcendental solution (1) such that f and all its derivatives are close-to-convex in $\mathbb{D}$. The investigations are continued in the papers [5–10], but in all of these papers the case of polynomial solutions of (2) was not investigated. In the papers [11–14] properties of entire solutions of a linear differential equation of n-th order with polynomial coefficients of n-th degree are investigated. Some results from these papers are published also in monograph [2].

Here we consider a differential equation

$$z^2 w'' + (\beta_0 z^2 + \beta_1 z) w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2) w = \alpha_0 z^2 + \alpha_1 z + \alpha_2 \tag{3}$$

with real parameters and study the existence and closeness-to-convexity of its polynomial solutions.

At first we remark that a function (1) is a solution of the differential equation (3) if and only if

$$\sum_{n=2}^{\infty} n(n-1) f_n z^n + \beta_0 \sum_{n=2}^{\infty} (n-1) f_{n-1} z^n + \gamma_0 \sum_{n=2}^{\infty} f_{n-2} z^n +$$
$$+\beta_1 \sum_{n=1}^{\infty} n f_n z^n + \gamma_1 \sum_{n=1}^{\infty} f_{n-1} z^n + \gamma_2 \sum_{n=0}^{\infty} f_n z^n = \alpha_0 z^2 + \alpha_1 z + \alpha_2,$$

i. e.

$$\gamma_2 f_0 = \alpha_2, \quad (\beta_1 + \gamma_2) f_1 + \gamma_1 f_0 = \alpha_1, \quad (2 + 2\beta_1 + \gamma_2) f_2 + (\beta_0 + \gamma_1) f_1 + \gamma_0 f_0 = \alpha_0 \tag{4}$$

and for $n \geq 3$

$$(n(n + \beta_1 - 1) + \gamma_2) f_n + (\beta_0 (n-1) + \gamma_1) f_{n-1} + \gamma_0 f_{n-2} = 0. \tag{5}$$

Clearly, by some condition differential equation (3) may have a linear solution, which obviously is convex function in $\mathbb{D}$. We are going to investigate a solutions of degree ≥ 2. In this case the following statement is true.

Lemma 1. *In order that the polynomial*

$$f(z) = \sum_{n=0}^{p} f_n z^n, \quad \deg f = p \geq 2, \tag{6}$$

be a solution of the differential equation (3), *it is necessary that* $\gamma_0 = p\beta_0 + \gamma_1 = 0$.

Proof. Indeed, for $n = p+2$ from (5) we get

$$((p+2)(p+\beta_1+1)+\gamma_2)f_{p+2} + ((p+1)\beta_0+\gamma_1)f_{p+1} + \gamma_0 f_p = 0.$$

If f has the form (6) then $f_{p+2} = f_{p+1} = 0$ and $f_p \neq 0$. Therefore, $\gamma_0 = 0$ and from (5) for $n = p+1$ we obtain

$$((p+1)(p+\beta_1)+\gamma_2)f_{p+1} + (p\beta_0+\gamma_1)f_p = 0.$$

Since $f_{p+1} = 0$ and $f_p \neq 0$, it follows that $p\beta_0 + \gamma_1 = 0$. Lemma 1 is proved. □

By the condition $\gamma_0 = p\beta_0 + \gamma_1 = 0$ from (4) and (5) we get

$$\gamma_2 f_0 = \alpha_2, \quad (\beta_1+\gamma_2)f_1 = \alpha_1 + p\beta_0 f_0, \quad (2+2\beta_1+\gamma_2)f_2 = \alpha_0 + (p-1)\beta_0 f_1 \tag{7}$$

and for $3 \leq n \leq p$

$$(n(n+\beta_1-1)+\gamma_2)f_n = (p-n+1)\beta_0 f_{n-1}. \tag{8}$$

We remark that the condition $\gamma_0 = p\beta_0 + \gamma_1 = 0$ is not sufficient in order that a solution of differential equation (3) has the form (6). Indeed, although in view of (8) we have

$$((p+3)(p+\beta_1+2)+\gamma_2)f_{p+3} = 0,$$

it does not follow from here that $f_{p+3} = 0$, since $(p+3)(p+\beta_1+2)+\gamma_2$ can be equal to zero. Therefore, further we assume that

$$n(n+\beta_1-1)+\gamma_2 \neq 0, \quad 3 \leq n \leq p.$$

This condition allows us to rewrite the equality (8) in the form

$$f_n = \frac{(p-n+1)\beta_0}{n(n+\beta_1-1)+\gamma_2} f_{n-1}, \quad n \geq 3, \tag{9}$$

whence it follows that $f_p = 0$, if $\beta_0 = 0$. Therefore, further we assume also that $\beta_0 \neq 0$.

To study the closeness-to-convexity of the polynomial (6), we will use the following criterion of Alexander [15, 16] (see also [2, p. 11]).

Lemma 2. *If*

$$1 \geq 2g_2 \geq 3g_3 \geq \cdots \geq pg_p > 0$$

then the polynomial $g(z) = \sum_{n=0}^{p} g_n z^n$ *is close-to-convex in* $\mathbb{D}$.

In view of (4) and (5) it is clear that the existence of a close-to-convex solution (6) of differential equation (3) depends on the equality to zero of the parameter γ_2. Therefore, we will consider two cases $\gamma_2 \neq 0$ and $\gamma_2 = 0$.

2. Closeness-to-convexity provided $\gamma_2 \neq 0$

From the first equality of (7) it follows that $f_0 = \alpha_2/\gamma_2$, and the second equality of (7) implies

$$(\beta_1+\gamma_2)f_1 = \alpha_1 + p\beta_0\alpha_2/\gamma_2.$$

Since the condition $f_1 \neq 0$ is necessary for a closeness-to-convexity of f, from the last equality it follows that either $\beta_1 + \gamma_2 \neq 0$ and $\alpha_1 + p\beta_0\alpha_2/\gamma_2 \neq 0$ or

$$\beta_1 + \gamma_2 = \alpha_1 + p\beta_0\alpha_2/\gamma_2 = 0.$$

In the first case we have $f_1 = \dfrac{\alpha_1\gamma_2 + p\beta_0\alpha_2}{\gamma_2(\beta_1+\gamma_2)}$, and if $2+2\beta_1+\gamma_2 \neq 0$ from the third equality (7) we obtain

$$f_2 = \frac{(p-1)\beta_0(\alpha_1\gamma_2 + p\beta_0\alpha_2) + \alpha_0\gamma_2(\beta_1+\gamma_2)}{\gamma_2(\beta_1+\gamma_2)(2+2\beta_1+\gamma_2)}.$$

Using these equalities and equality (9) we prove the following theorem.

Theorem 1. *Let* $p \geq 3$, $\gamma_2 \neq 0$, $\gamma_0 = p\beta_0 + \gamma_1 = 0$, $\beta_1+\gamma_2 \neq 0$, $\alpha_1\gamma_2 + p\beta_0\alpha_2 \neq 0$ *and*

$$0 < \frac{(p-1)\beta_0(\alpha_1\gamma_2 + p\beta_0\alpha_2) + \alpha_0\gamma_2(\beta_1+\gamma_2)}{(\gamma_2\alpha_1 + p\beta_0\alpha_2)(2+2\beta_1+\gamma_2)} \leq \frac{1}{2}. \tag{10}$$

If for all $3 \leq n \leq p$

$$0 < \frac{(p-n+1)\beta_0}{n(n+\beta_1-1)+\gamma_2} \leq \frac{n-1}{n} \tag{11}$$

then differential equation (3) *has a close-to-convex in* $\mathbb{D}$ *polynomial solution*

$$\begin{aligned} f(z) = \frac{\alpha_2}{\gamma_2} + \frac{\alpha_1\gamma_2 + p\beta_0\alpha_2}{\gamma_2(\beta_1+\gamma_2)} z + \\ + \frac{(p-1)\beta_0(\alpha_1\gamma_2 + p\beta_0\alpha_2) + \alpha_0\gamma_2(\beta_1+\gamma_2)}{\gamma_2(\beta_1+\gamma_2)(2+2\beta_1+\gamma_2)} z^2 + \sum_{n=3}^{p} f_n z^n \end{aligned} \tag{12}$$

where the coefficients f_n *satisfy* (9).

If $\beta_0 > 0$, $2+\beta_1 > 0$ *and either* $\gamma_2 > 0$ *and* $(p-2)\beta_0 \leq 2+\beta_1$ *or* $-3(2+\beta_1) < \gamma_2 < 0$ *and* $3(p-2)\beta_0 \leq 3(2+\beta_1) + \gamma_2$ *then differential equation* (3) *has a polynomial solution* (12) *close-to-convex in* $\mathbb{D}$ *together with all its derivatives* $f^{(j)}$ *(*$1 \leq j \leq p-1$*).*

Proof. Let $g(z) = z + \sum_{n=2}^{p} g_n z^n$, where $g_n = f_n/f_1$. In view of (9), (10) and (11) $f_2/f_1 > 0$ and $f_n/f_1 > 0$ for all $3 \leq n \leq p$, i. e. $g_n > 0$ for all $2 \leq n \leq p$. From (10) it follows also that $2g_2 \leq 1$, and (9) and (11) imply $ng_n \leq (n-1)g_{n-1}$ for all $3 \leq n \leq p$. Therefore, by Lemma 2 the function g and, thus, the function f are close-to-convex in $\mathbb{D}$. The first part of Theorem 1 is proved.

Now suppose that the condition

$$0 < \frac{(p-(n+j)+1)\beta_0}{(n+j)(n+j+\beta_1-1)+\gamma_2} \leq \frac{n-1}{n+j} \tag{13}$$

holds for some $1 \leq j \leq p-2$ and all $2 \leq n \leq p-j$ and show that the derivative $f^{(j)}$ of function (12) is close-to-convex in $\mathbb{D}$.

Indeed, for $1 \leq j \leq p-2$ the derivative

$$f^{(j)}(z) = j!f_j + (j+1)!f_{j+1}z + \sum_{n=2}^{p-j}(n+1)(n+2)\ldots(n+j)f_{n+j}z^n.$$

is close-to-convex in $\mathbb{D}$ if and only if the function

$$g_j(z) = z + \sum_{n=2}^{p-j} g_{n,j}z^n, \quad g_{n,j} = \frac{(n+1)(n+2)\ldots(n+j)f_{n+j}}{(j+1)!f_{j+1}},$$

is close-to-convex in $\mathbb{D}$. For the function g_j the inequality $2g_{2,j} \le 1$ is equivalent to the inequality

$$\frac{(p-j-1)\beta_0}{(j+2)(j+\beta_1+1)+\gamma_2} \le \frac{1}{j+2}$$

which follows from the condition (13) with $n=2$. If $3 \le n \le p-j$ then the inequality $ng_{n,j} \le (n-1)g_{n-1,j}$ is equivalent to condition (13). Therefore, by Lemma 2 the function g_j and, thus, the function $f^{(j)}$ are close-to-convex in $\mathbb{D}$.

Now suppose that $\beta_0 > 0$, $2+\beta_1 > 0$ and $\gamma_2 > 0$. Then condition (11) holds for all $3 \le n \le p$ if $\dfrac{(p-n+1)\beta_0}{n(n+\beta_1-1)} \le \dfrac{n-1}{n}$, i. e. $\dfrac{(p-n+1)\beta_0}{(n-1)(n+\beta_1-1)} \le 1$. Since the left part of the last inequality decreases, this inequality holds if $\dfrac{(p-2)\beta_0}{2(2+\beta_1)} \le 1$, i. e. $(p-2)\beta_0 \le 2+\beta_1$. Similarly, condition (13) holds for all $1 \le j \le p-2$ and $2 \le n \le p-j$ if $\dfrac{(p-(n+j)+1)\beta_0}{(n-1)(n+j+\beta_1-1)} \le 1$ and the last inequality is true if $\dfrac{(p-2)\beta_0}{2+\beta_1} \le 1$, i. e. $(p-2)\beta_0 \le 2(2+\beta_1)$.

Finally, let $\beta_0 > 0$, $2+\beta_1 > 0$ and $\gamma_2 < 0$. Then for all $3 \le n \le p$

$$\frac{(p-n+1)\beta_0}{n(n+\beta_1-1)+\gamma_2} = \frac{(p-n+1)\beta_0}{n(n+\beta_1-1-|\gamma_2|/n)} \le \frac{(p-n+1)\beta_0}{n(n+\beta_1-1-|\gamma_2|/3)}.$$

Therefore, (11) holds for all $3 \le n \le p$ if

$$\frac{(p-n+1)\beta_0}{(n-1)(n+\beta_1-1-|\gamma_2|/3)} \le 1,$$

whence as above it follows that (11) holds for all $3 \le n \le p$ if $\dfrac{(p-2)\beta_0}{2(2+\beta_1+\gamma_2/3)} \le 1$, i. e. $-3(2+\beta_1) < \gamma_2 < 0$ and $\frac{3}{2}(p-2)\beta_0 \le 3(2+\beta_1)+\gamma_2$. Similarly we prove that condition (13) holds for all $1 \le j \le p-2$ and $2 \le n \le p-j$ if $\dfrac{(p-2)\beta_0}{2+\beta_1+\gamma_2/3} \le 1$, i. e. $-3(2+\beta_1) < \gamma_2$ and $3(p-2)\beta_0 \le 3(2+\beta_1)+\gamma_2$. Thus, for all $1 \le j \le p-2$ the derivative $f^{(j)}$ is close-to-convex in $\mathbb{D}$. Since the derivative $f^{(p-1)}$ is a linear function, the proof of Theorem 1 is complete. □

Now we consider the case

$$\beta_1 + \gamma_2 = \alpha_1 + p\beta_0\alpha_2/\gamma_2 = 0.$$

From the second equality (7) it follows that f_1 may be arbitrary. If we choose $f_1 = 1$ then under the condition $2+\beta_1 \ne 0$ in view of the third equality (7) we get

$$f_2 = \frac{\alpha_0 + (p-1)\beta_0}{2+\beta_1}.$$

From (8) under the condition $n+\beta_1 \ne 0$ we obtain

$$f_n = \frac{(p-n+1)\beta_0}{(n-1)(n+\beta_1)} f_{n-1}, \quad 3 \le n \le p. \tag{14}$$

Theorem 2. *Let $p \geq 3$, $\gamma_2 \neq 0$, $\gamma_0 = p\beta_0 + \gamma_1 = \beta_1 + \gamma_2 = \alpha_1\gamma_2 + p\beta_0\alpha_2 = 0$ and*

$$0 < \frac{\alpha_0 + (p-1)\beta_0}{2+\beta_1} \leq \frac{1}{2}. \tag{15}$$

If for all $3 \leq n \leq p$

$$0 < \frac{(p-n+1)\beta_0}{(n-1)(n+\beta_1)} \leq \frac{n-1}{n} \tag{16}$$

then differential equation (3) *has a close-to-convex in $\mathbb{D}$ polynomial solution*

$$f(z) = \frac{\alpha_2}{\gamma_2} + z + \frac{\alpha_0 + (p-1)\beta_0}{2+\beta_1} z^2 + \sum_{n=3}^{p} f_n z^n \tag{17}$$

where the coefficients f_n satisfy (14)*.*

If $\beta_0 > 0$, $2 + \beta_1 > 0$ and $3(p-2)\beta_0 \leq 2(3+\beta_1)$ then differential equation (3) *has polynomial solution* (17)*, which together with its derivatives $f^{(j)}$ $(1 \leq j \leq p-1)$ are close-to-convex in $\mathbb{D}$.*

Proof. From (14) and (16) the inequality $f_n > 0$ follows for all n. Condition (15) implies the inequality $2f_2 \leq 1$ and condition (16) implies $nf_n \leq (n-1)f_{n-1}$ for all $3 \leq n \leq p$. Therefore, by Lemma 2 the function f is close-to-convex in $\mathbb{D}$. The first part of Theorem 2 is proved.

Now we suppose the condition

$$0 < \frac{(p-(n+j)+1)\beta_0}{(n+j-1)(n+j+\beta_1)} \leq \frac{n-1}{n+j} \tag{18}$$

holds for some $1 \leq j \leq p-2$ and all $2 \leq n \leq p-j$. The proof of the closeness-to-convexity of the derivative $f^{(j)}$ $(1 \leq j \leq p-2)$ is the same as the proof in Theorem 1. Note only that the inequality $2g_{2,j} \leq 1$ is equivalent to the inequality

$$\frac{(p-j-1)\beta_0}{j+2+\beta_1} \leq \frac{j+1}{j+2},$$

which follows from condition (18) for $n = 2$, and the inequality $ng_{n,j} \leq (n-1)g_{n-1,j}$ coincides with condition (18).

Let $\beta_0 > 0$ and $2 + \beta_1 > 0$. Since the values

$$\frac{(p-n+1)\beta_0}{n+\beta_1}, \quad \frac{n}{(n-1)^2}, \quad \frac{n+j}{(n-1)(n+j-1)}$$

decrease with the increasing of n and the value

$$\frac{(2+j)(p-j-1)\beta_0}{(j+1)(j+2+\beta_1)}$$

decreases with the increasing of j, conditions (16) and (18) hold if $\frac{3(p-2)\beta_0}{2(3+\beta_1)} \leq 1$, i. e. $3(p-2)\beta_0 \leq 2(3+\beta_1)$. Thus, for all $1 \leq j \leq p-2$ the derivative $f^{(j)}$ is close-to-convex in $\mathbb{D}$. Since the derivative $f^{(p-1)}$ is a linear function, the proof of Theorem 2 is complete. □

3. Closeness-to-convexity provided $\gamma_2 = 0$

Now (7) implies $\alpha_2 = 0$ and, thus, f_0 may be arbitrary. If we choose $f_0 = 0$ then from (7) and (9) we get

$$\beta_1 f_1 = \alpha_1, \quad 2(1+\beta_1)f_2 = \alpha_0 + (p-1)\beta_0 f_1, \tag{19}$$

and for $3 \le n \le p$

$$f_n = \frac{(p-n+1)\beta_0}{n(n+\beta_1-1)} f_{n-1}. \tag{20}$$

Since for the close-to-convex function $f_1 \ne 0$, from the first equality of (19) it follows that either $\beta_1 \ne 0$ and $\alpha_1 \ne 0$ or $\beta_1 = \alpha_1 = 0$. In the first of these cases the following theorem holds.

Theorem 3. *Let $p \ge 3$, $\gamma_2 = \alpha_2 = \gamma_0 = \gamma_1 + p\beta_0 = 0$, $\beta_1 \ne 0$,$\alpha_1 \ne 0$ and*

$$0 < \frac{(p-1)\beta_0\alpha_1 + \alpha_0\beta_1}{\alpha_1(1+\beta_1)} \le 1 \tag{21}$$

If for all $3 \le n \le p$

$$0 < \frac{(p-n+1)\beta_0}{(n-1)(n+\beta_1-1)} \le 1 \tag{22}$$

then differential equation (3) has a close-to-convex in $\mathbb{D}$ polynomial solution

$$f(z) = \frac{\alpha_1}{\beta_1} z + \frac{(p-1)\beta_0\alpha_1 + \beta_1\alpha_0}{2\beta_1(1+\beta_1)} z^2 + \sum_{n=3}^{p} f_n z^n \tag{23}$$

where the coefficients f_n satisfy (20).

If $\beta_0 > 0$, $2 + \beta_1 > 0$ and $(p-2)\beta_0 \le 2 + \beta_1$ then differential equation (3) has a polynomial solution (23) close-to-convex in $\mathbb{D}$ together with its derivatives $f^{(j)}$ $(1 \le j \le$ $\le p-1)$.

Proof. Suppose that the function g is defined as in the proof of Theorem 1. In view of (20), (21) and (22) $f_2/f_1 > 0$ and $f_n/f_1 > 0$ for all $3 \le n \le p$, i. e. $g_n > 0$ for all $2 \le n \le p$. From (21) it follows also that $2g_2 \le 1$, and (22) and (20) imply $ng_n \le (n-1)g_{n-1}$ for all $3 \le n \le p$. Therefore, by Lemma 2 the function g and, thus, the function (23) are close-to-convex in $\mathbb{D}$. The first part of Theorem 3 is proved.

Now we suppose that

$$0 < \frac{(p-(n+j)+1)\beta_0}{(n-1)(n+j+\beta_1-1)} \le 1 \tag{24}$$

holds for some $1 \le j \le p-2$ and for all $2 \le n \le p-j$. Then the proof of the close-to-convexity of the derivative $f^{(j)}$ is the same as the proof in Theorem 1. Note only the inequality $2g_{2,j} \le 1$ is equivalent to the inequality $\dfrac{(p-j-1)\beta_0}{j+1+\beta_1} \le 1$, which follows from condition (24) for $n = 2$, and the inequality $ng_{n,j} \le (n-1)g_{n-1,j}$ coincides with condition (24).

It is easy to check that if $\beta_0 > 0$, $2 + \beta_1 > 0$ then condition (22) holds for all $3 \le n \le p$ if $\dfrac{(p-2)\beta_0}{2(2+\beta_1)} \le 1$, and (24) holds for all $1 \le j \le p-2$ and all $2 \le n \le p-1$ if $\dfrac{(p-2)\beta_0}{2+\beta_1} \le 1$, i.e. $(p-2)\beta_0 \le 2+\beta_1$. The proof of Theorem 3 is complete. □

In the second case the following theorem is true.

Theorem 4. *Let $p \ge 3$, $\gamma_2 = \alpha_2 = \gamma_0 = \gamma_1 + p\beta_0 = \beta_1 = \alpha_1 = 0$ and $0 < \alpha_0 + (p-1)\beta_0 \le \le 1$. If $0 < (p-n+1)\beta_0 < (n-1)^2$ for all $3 \le n \le p$ then differential equation* (3) *has a close-to-convex in $\mathbb{D}$ polynomial solution*

$$f(z) = z + \frac{(p-1)\beta_0 + \alpha_0}{2} z^2 + \sum_{n=3}^{p} f_n z^n \tag{25}$$

where the coefficients f_n satisfy (20) *with $\beta_1 = 0$.*

If $0 < (p-2)\beta_0 \le 2$ then differential equation (3) *has polynomial solution* (25) *close-to-convex in $\mathbb{D}$ together with its derivatives $f^{(j)}$ $(1 \le j \le p-1)$.*

Proof. From the conditions $0 < \alpha_0 + (p-1)\beta_0 \le 1$ and $0 < (p-n+1)\beta_0 < (n-1)^2$ for all $3 \ge n \ge p$ in view of (20) with $\beta_1 = 0$ it follows as above that all $f_n > 0$, $2f_2 \le 1$ and $nf_n \le (n-1)f_{n-1}$ for all $3 \le n \le p$. Therefore, by Lemma 2 the function (25) is close-to-convex in $\mathbb{D}$. The first part of Theorem 4 is proved.

Now we suppose that

$$0 < (p-(n+j)+1)\beta_0 \le (n-1)(n+j-1)$$

for some $1 \le j \le p-2$ and all $2 \le n \le p-j$. Then the proof of the close-to-convexity of the derivative $f^{(j)}$ is the same as the proof in Theorem 1. Note only the inequality $2g_{2,j} \le 1$ is equivalent to the inequality $\dfrac{(p-j-1)\beta_0}{j+1} \le 1$, which follows from condition

$$0 < (p-(n+j)+1)\beta_0 \le (n-1)(n+j-1)$$

for $n = 2$, and the inequality $ng_{n,j} \le (n-1)g_{n-1,j}$ coincides with this condition. Hence as in the proof of Theorem 3 we get the second part of Theorem 4. □

4. Other results

The condition $p \ge 3$ in the proved theorems is not significant. Repeating the proofs of these theorems one can show that the following analogues of these theorems are hold for $p = 2$.

Proposition 1. *Let $\gamma_2 \ne 0$, $\gamma_0 = 2\beta_0 + \gamma_1 = 0$, $\beta_1 + \gamma_2 \ne 0$, $\alpha_1\gamma_2 + 2\beta_0\alpha_2 \ne 0$, $2 + 2\beta_1 + +\gamma_2 \ne 0$ and*

$$0 < \frac{\beta_0(\alpha_1\gamma_2 + 2\beta_0\alpha_2) + \alpha_0\gamma_2(\beta_1+\gamma_2)}{(\gamma_2\alpha_1 + 2\beta_0\alpha_2)(2 + 2\beta_1 + \gamma_2)} \le \frac{1}{2}.$$

Then differential equation (3) *has a polynomial solution*

$$f(z) = \frac{\alpha_2}{\gamma_2} + \frac{\alpha_1\gamma_2 + 2\beta_0\alpha_2}{\gamma_2(\beta_1+\gamma_2)} z + \frac{\beta_0(\alpha_1\gamma_2 + 2\beta_0\alpha_2) + \alpha_0\gamma_2(\beta_1+\gamma_2)}{\gamma_2(\beta_1+\gamma_2)(2+2\beta_1+\gamma_2)} z^2$$

close-to-convex in $\mathbb{D}$.

Proposition 2. *Let* $\gamma_2 \neq 0$, $\gamma_0 = 2\beta_0 + \gamma_1 = \beta_1 + \gamma_2 = \alpha_1\gamma_2 + 2\beta_0\alpha_2 = 0$, $2 + \beta_1 \neq 0$ *and* $0 < \dfrac{\alpha_0 + \beta_0}{2 + \beta_1} \leq \dfrac{1}{2}$. *Then differential equation* (3) *has a polynomial solution*

$$f(z) = \frac{\alpha_2}{\gamma_2} + z + \frac{\alpha_0 + \beta_0}{2 + \beta_1} z^2$$

close-to-convex in $\mathbb{D}$.

Proposition 3. *Let* $\gamma_2 = \alpha_2 = \gamma_0 = \gamma_1 + 2\beta_0 = 0$, $\beta_1 \neq 0$, $\alpha_1 \neq 0$, $1 + \beta_1 \neq 0$ *and* $0 < \dfrac{\beta_0\alpha_1 + \alpha_0\beta_1}{\alpha_1(1 + \beta_1)} \leq 1$. *Then differential equation* (3) *has a polynomial solution*

$$f(z) = \frac{\alpha_1}{\beta_1} z + \frac{\beta_0\alpha_1 + \beta_1\alpha_0}{2\beta_1(1 + \beta_1)} z^2$$

close-to-convex in $\mathbb{D}$.

Proposition 4. *Let* $\gamma_2 = \alpha_2 = \gamma_0 = \gamma_1 + 2\beta_0 = \beta_1 = \alpha_1 = 0$ *and* $0 < \alpha_0 + \beta_0 \leq 1$. *Then differential equation* (3) *has a polynomial solution*

$$f(z) = z + \frac{\beta_0 + \alpha_0}{2} z^2$$

close-to-convex in $\mathbb{D}$.

Recall that before obtaining the above results we demanded the fulfillment of conditions (9) and $\beta_0 \neq 0$. Now suppose that $\beta_0 = 0$. Then by Lemma 2 $\gamma_0 = \gamma_1 = 0$, and thus, from (7) and (8) we get

$$\gamma_2 f_0 = \alpha_2, \quad (\beta_1 + \gamma_2) f_1 = \alpha_1, \quad (2 + 2\beta_1 + \gamma_2) f_2 = \alpha_0 \tag{26}$$

and for $3 \leq n \leq p$

$$(n(n + \beta_1 - 1) + \gamma_2) f_n = 0. \tag{27}$$

From (27) it follows that if $p(p + \beta_1 - 1) + \gamma_2 = 0$ then $f_p \neq 0$ may be arbitrary. Two cases are possible: 1) $n(n + \beta_1 - 1) + \gamma_2 \neq 0$ for all $3 \leq n \leq p$ and 2) there is $3 \leq p_1 < p$ such that $p_1(p_1 + \beta_1 - 1) + \gamma_2 = 0$.

In the first case we have $f_{p-1} = 0$ provided $p > 3$ and it is impossible to use Alexander's criterion. In the second case we have $p_1 p = \gamma_2$ and $p_1 + p = 1 - \beta_1$. Therefore, if either $p_1 > 3$ or $p > p_1 + 1$ then again we cannot apply Alexander's criterion. Thus, we can apply Alexander's criterion if either $n(n + \beta_1 - 1) + \gamma_2 \neq 0$ for all $3 \leq n \leq p$ and $p = 3$ or $p_1(p_1 + \beta_1 - 1) + \gamma_2 = 0$ for some $3 \leq p_1 < p$ and $p_1 = 3$, $p = 4$.

Given the possible value of the parameter γ_2, using (26) and choosing $f_3 = 2f_2/3$, you can prove the following statement.

Proposition 5. *Let* $\beta_0 = \gamma_0 = \gamma_1 = 0$, $p = 3$ *and* $3(2 + \beta_1) + \gamma_2 = 0$. *Then:*

1) *if* $\gamma_2 \neq 0$, $\gamma_2 \neq 3$ *and* $\gamma_2 \neq 6$, $\alpha_1 \neq 0$ *and* $0 < \dfrac{\alpha_0(\gamma_2 - 3)}{\alpha_1(\gamma_2 - 6)} \leq \dfrac{1}{4}$ *then differential equation* (3) *has a polynomial solution*

$$f(z) = \frac{\alpha_2}{\gamma_2} + \frac{3\alpha_1}{2(\gamma_2 - 3)} z + \frac{3\alpha_0}{\gamma_2 - 6} z^2 + \frac{2\alpha_0}{\gamma_2 - 6} z^3$$

close-to-convex in $\mathbb{D}$;

2) *if* $\gamma_2 = 3$, $\alpha_1 = 0$ *and* $-\frac{1}{2} \le \alpha_0 < 0$ *then differential equation* (3) *has a polynomial solution*
$$f(z) = \alpha_2/3 + z - \alpha_0 z^2 - 2\alpha_0 z^3/3$$
close-to-convex in $\mathbb{D}$*;*
3) *if* $\gamma_2 = 6$, $\alpha_0 = 0$ *and* $\alpha_1 \ne 0$ *then differential equation* (3) *has a polynomial solution*
$$f(z) = \frac{\alpha_2}{6} + \frac{\alpha_1}{2} z + \frac{\alpha_1}{4} z^2 + \frac{\alpha_1}{6} z^3$$
close-to-convex in $\mathbb{D}$*;*
4) *if* $\gamma_2 = \alpha_2 = 0$ *and* $0 < \alpha_0/\alpha_1 \le 1/2$ *then differential equation* (3) *has a polynomial solution*
$$f(z) = -\alpha_1 z/2 - \alpha_0 z^2/2 - \alpha_0 z^3/3$$
close-to-convex in $\mathbb{D}$*.*

In the case when $3(2 + \beta_1) + \gamma_2 = 0$ and $4(3 + \beta_1) + \gamma_2 = 0$ (i. e. $p_1 = 3$, $p = 4$) from (26) we get $f_0 = \alpha_2/12$, $f_1 = \alpha_1/6$, $f_2 = \alpha_0/2$, and choosing $f_3 = \alpha_0/3$, $f_4 = \alpha_0/4$ we obtain the following statement.

Proposition 6. *If* $\gamma_2 = 12$, $\beta_0 = \gamma_0 = \gamma_1 = 0$, $0 < \alpha_0/\alpha_1 < 1/6$,
$$3(2 + \beta_1) + \gamma_2 = 4(3 + \beta_1) + \gamma_2 = 0$$
then differential equation (3) *has a polynomial solution*
$$f(z) = \frac{\alpha_2}{12} + \frac{\alpha_1}{6} z + \frac{\alpha_0}{2} z^2 + \frac{\alpha_0}{3} z^3 + \frac{\alpha_0}{4} z^4$$
close-to-convex in $\mathbb{D}$*.*

Finally, we remark that polynomial (6) can be close-to-convex in the case when $f_2 = \cdots = f_{p-1} = 0$. Since each starlike function is close-to-convex, it follows from such a lemma.

Lemma 3. *If* $|\alpha| \le 1/p$ *then the polynomial* $f(z) = z + \alpha z^p$ *is a starlike function.*

Proof. Recall that an analytic univalent in $\mathbb{D}$ function $f(z) = z + \sum_{n=2}^{\infty} f_n z^n$ is said to be starlike if $f(\mathbb{D})$ is starlike domain with respect to the origin. It is well known [1, p. 202] (see also [2, p. 9]) that the condition $\mathrm{Re}\,\{zf'(z)/f(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the starlikeness of f. If $f(z) = z + \alpha z^p$ then for $|\alpha| \le 1/p$ and $|z| < 1$ we have
$$\mathrm{Re}\frac{zf'(z)}{f(z)} = \mathrm{Re}\left\{1 + \frac{(p-1)\alpha z^{p-1}}{1 + \alpha z^{p-1}}\right\} \ge 1 - \left|\frac{(p-1)\alpha z^{p-1}}{1 + \alpha z^{p-1}}\right| > 1 - \frac{(p-1)|\alpha|}{1 - |\alpha|} \ge 0,$$
i. e. the function $f(z) = z + \alpha z^p$ starlike and, thus, close-to-convex. Lemma 3 is proved.

Suppose that $\gamma_2 \ne 0$,
$$\alpha_2 = \alpha_0 + (p-1)\beta_0 = p(p + \beta_1 - 1) + \gamma_2 = 0,$$
$\beta_1 + \gamma_2 = \alpha_1$ and $n(n + \beta_1 - 1) + \gamma_2 \ne 0$ for all $n = 1, 2, \ldots, p-1$. Then in view of (7) $f_0 = 0$, $f_1 = 1$, $f_2 = 0$ and in view of (8) $f_3 = \cdots = f_{p-1} = 0$. Choosing $f_p = 1/p$ and using Lemma 3 we get the following statement.

Proposition 7. *If* $\gamma_2 \neq 0$,

$$\alpha_2 = \alpha_0 + (p-1)\beta_0 = p(p+\beta_1-1) + \gamma_2 = 0$$

and $n(n+\beta_1-1)+\gamma_2 \neq 0$ *for all* $n = 1, 2, \ldots, p-1$ *then differential equation* (3) *has a polynomial solution* $f(z) = z + z^p/p$ *close-to-convex in* $\mathbb{D}$.

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БЛИЗЬКІСТЬ ДО ОПУКЛОСТІ МНОГОЧЛЕННИХ РОЗВ'ЯЗКІВ ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ ДРУГОГО ПОРЯДКУ З МНОГОЧЛЕННИМИ КОЕФІЦІЄНТАМИ ДРУГОГО СТЕПЕНЯ

Мирослав ШЕРЕМЕТА, Юрій ТРУХАН

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, 79000, Львів
e-mail: m.m.sheremeta@gmail.com, yurkotrukhan@gmail.com

Аналітична однолиста в $\mathbb{D} = \{z : |z| < 1\}$ функція $f(z) = \sum\limits_{n=0}^{\infty} f_n z^n$ називається опуклою, якщо $f(\mathbb{D})$ - опукла область, і називається близькою до опуклої, якщо існує така опукла в $\mathbb{D}$ функція Φ, що $\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0$ $(z \in \mathbb{D})$. Кожна близька до опуклої в $\mathbb{D}$ функція f є однолистою в $\mathbb{D}$, і отже, $f'(0) \neq 0$. Тому функція f є близькою до опуклої в $\mathbb{D}$ тоді і тільки тоді, коли функція $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ близька до опуклої в $\mathbb{D}$, де $g_n = f_n/f_1$. С.М. Шах визначив умови на дійсні параметри β_0, β_1, γ_0, γ_1, γ_2, за яких диференціальне рівняння $z^2 w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0$ має цілі розв'язки, які разом зі своїми похідними близькі до опуклих в $\mathbb{D}$ функціями. Багато авторів продовжили ці дослідження. Тут розглядається неоднорідне рівняння Шаха $z^2 w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = \alpha_0 z^2 + \alpha_1 z + \alpha_2$ з дійсними параметрами і вивчається існування близьких до опуклих його многочленних розв'язків. Неважко довести, що для того, щоб многочлен $f(z) = \sum\limits_{n=0}^{p} f_n z^n$, $(\deg f = p \geq 2)$ був розв'язком цього рівняння, необхідно, щоб $\gamma_0 = p\beta_0 + \gamma_1 = 0$. Основні такі результати:
1) якщо $p \geq 3$, $\gamma_0 = p\beta_0 + \gamma_1 = 0$, $\beta_1 + \gamma_2 \neq 0$, $\alpha_1\gamma_2 + p\beta_0\alpha_2 \neq 0$, $\beta_0 > 0$, $2 + \beta_1 > 0$,

$$0 < \frac{(p-1)\beta_0(\alpha_1\gamma_2 + p\beta_0\alpha_2) + \alpha_0\gamma_2(\beta_1 + \gamma_2)}{(\gamma_2\alpha_1 + p\beta_0\alpha_2)(2 + 2\beta_1 + \gamma_2)} \leq \frac{1}{2}$$

і або $\gamma_2 > 0$ та $(p-2)\beta_0 \leq 2 + \beta_1$, або $-3(2 + \beta_1) < \gamma_2 < 0$ та $3(p-2)\beta_0 \leq 3(2+\beta_1)+\gamma_2$, то неоднорідне рівняння Шаха має многочленний розв'язок $f(z) = \frac{\alpha_2}{\gamma_2} + \frac{\alpha_1\gamma_2 + p\beta_0\alpha_2}{\gamma_2(\beta_1 + \gamma_2)}z + \frac{(p-1)\beta_0(\alpha_1\gamma_2 + p\beta_0\alpha_2) + \alpha_0\gamma_2(\beta_1 + \gamma_2)}{\gamma_2(\beta_1 + \gamma_2)(2 + 2\beta_1 + \gamma_2)}z^2 + \sum\limits_{n=3}^{p} f_n z^n$, де $f_n = \frac{(p-n+1)\beta_0}{n(n+\beta_1-1)+\gamma_2}f_{n-1}$ для $3 \leq n \leq p$, який разом з усіма своїми похідними $f^{(j)}$ $(1 \leq j \leq p-1)$, близькими до опуклих в $\mathbb{D}$

функціями;
2) якщо $p \geq 3$, $\gamma_2 \neq 0$, $\gamma_0 = p\beta_0 + \gamma_1 = \beta_1 + \gamma_2 = \alpha_1\gamma_2 + p\beta_0\alpha_2 = 0$, $\beta_0 > 0$, $2 + \beta_1 > 0$, $3(p-2)\beta_0 \leq 2(3+\beta_1)$ і $0 < \frac{\alpha_0 + (p-1)\beta_0}{2+\beta_1} \leq \frac{1}{2}$, то неоднорідне рівняння Шаха має многочленний розв'язок $f(z) = \frac{\alpha_2}{\gamma_2} + z + \frac{\alpha_0 + (p-1)\beta_0}{2+\beta_1} z^2 + \sum_{n=3}^{p} f_n z^n$, де $f_n = \frac{(p-n+1)\beta_0}{(n-1)(n+\beta_1)} f_{n-1}$ для $3 \leq n \leq p$, який разом з усіма своїми похідними $f^{(j)}$ $(1 \leq j \leq p-1)$ близькими до опуклих в $\mathbb{D}$ функціями;
3) якщо $p \geq 3$, $\gamma_2 = \alpha_2 = \gamma_0 = \gamma_1 + p\beta_0 = 0$, $\beta_1 \neq 0$,$\alpha_1 \neq 0$, $\beta_0 > 0$, $2+\beta_1 > 0$, $(p-2)\beta_0 \leq 2 + \beta_1$ і $0 < \frac{(p-1)\beta_0\alpha_1 + \alpha_0\beta_1}{\alpha_1(1+\beta_1)} \leq 1$, то неоднорідне рівняння Шаха має многочленний розв'язок $f(z) = \frac{\alpha_1}{\beta_1} z + \frac{(p-1)\beta_0\alpha_1 + \beta_1\alpha_0}{2\beta_1(1+\beta_1)} z^2 + \sum_{n=3}^{p} f_n z^n$, де $f_n = \frac{(p-n+1)\beta_0}{n(n+\beta_1-1)} f_{n-1}$ для $3 \leq n \leq p$, який разом з усіма своїми похідними $f^{(j)}$ $(1 \leq j \leq p-1)$ близькими до опуклих в $\mathbb{D}$ функціями;
4) якщо $p \geq 3$, $\gamma_2 = \alpha_2 = \gamma_0 = \gamma_1 + p\beta_0 = \beta_1 = \alpha_1 = 0$, $(p-2)\beta_0 \leq 2$ і $0 < \alpha_0 + (p-1)\beta_0 \leq 1$, то неоднорідне рівняння Шаха має многочленний розв'язок $f(z) = z + \frac{(p-1)\beta_0 + \alpha_0}{2} z^2 + \sum_{n=3}^{p} f_n z^n$, де $f_n = \frac{(p-n+1)\beta_0}{n(n-1)} f_{n-1}$ для $3 \leq n \leq p$, який разом з усіма своїми похідними $f^{(j)}$ $(1 \leq j \leq p-1)$ близькими до опуклих в $\mathbb{D}$ функціями.

Ключові слова: лінійне неоднорідне диференціальне рівняння другого порядку, многочленні коефіцієнти, многочленний розв'язок, близька до опуклої функція.