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ON THE UNIVALENCE RADII OF SUCCESSIVE GELFOND-LEONT'EV-SĂLĂGEAN AND GELFOND-LEONT'EV-RUSCHEWEYH DERIVATIVES

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For an analytic in the disk $\{z : |z| < 1\}$ function $f(z) = z + \sum\limits_{k=1}^{\infty} f_k z^k$ and formal power series $l(z) = 1 + \sum\limits_{k=1}^{\infty} l_k z^k$ with $l_k > 0$ the operator

$$D^n_{l,[S]f(z)} = z + \sum_{k=2}^{\infty} \left(\frac{l_1 l_{k-1}}{l_k}\right)^n f_k z^k$$

is called the Gelfond-Leont'ev-Sălăgean derivative and the operator

$$D^n_{l,[R]} f(z) = z + \sum_{k=2}^{\infty} \frac{l_{k-1} l_n}{l_{n+k-1}} f_k z^k$$

is called the Gelfond-Leont'ev-Ruscheweyh derivative. By $\varrho[f]$ we denote the radius of the univalence of the function f. It is proved, for example, that for each $n \geq 1$

$$\frac{\sqrt{2}-1}{\sqrt{2}} \left|\frac{f_1}{f_2}\right| \left(\frac{l_2}{l_1^2}\right)^n \leq \varrho[D^n_{l,[S]} f] \leq 2 \left|\frac{f_1}{f_2}\right| \left(\frac{l_2}{l_1^2}\right)^n$$

and

$$\frac{\sqrt{2}-1}{\sqrt{2}} \left|\frac{f_1}{f_2}\right| \frac{l_{n+1}}{l_1 l_n} \leq \varrho[D^n_{l,[R]} f] \leq 2 \left|\frac{f_1}{f_2}\right| \frac{l_{n+1}}{l_1 l_n}.$$

Key words: analytic function, Gelfond-Leont'ev-Sălăgean derivative, Gelfond-Leont'ev-Ruscheweyh derivative, radius of the univalence.

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1. Introduction

For a formal power series

$$f(z) = f_0 + \sum_{k=1}^{\infty} f_k z^k, \quad z = re^{i\theta}, \tag{1}$$

and $l(z) = 1 + \sum_{k=1}^{\infty} l_k z^k$ ($l_k > 0$) the formal power series $D_l^n f(z) = \sum_{k=0}^{\infty} \frac{l_k}{l_{k+n}} f_{k+n} z^k$ is called the Gelfond-Leont'ev derivative [1]. If $l(z) = e^z$ (i.e. $l_k = 1/k!$) then $D_l^n f = f^{(n)}$ is the usual derivative.

If the function $f(z) = z + \sum_{k=2}^{\infty} f_k z^k$ is analytic in the disk $\{z : |z| < 1\}$ then the operator $D_{[S]}^n f$ ($n \geq 0$) defined by

$$D_{[S]}^0 f(z) = f(z), \quad D_{[S]}^1 f(z) = D_{[S]} f(z) = z f'(z),$$

$$D_{[S]}^n f(z) = D_{[S]}(D_{[S]}^{n-1} f(z)) = z + \sum_{k=2}^{\infty} k^n f_k z^k$$

is known as the Sălăgean derivative [2]. The operator

$$D_{[R]}^n f(z) = \frac{z}{n!} \frac{d^n}{dz^n} \{z^{n-1} f(z)\} = z + \sum_{k=2}^{\infty} \frac{(k+n-1)!}{n!(k-1)!} f_k z^k$$

is called [3] the Ruscheweyh derivative.

In [4], combining the definitions of Gelfond-Leont'ev derivative with Sălăgean derivative and Ruscheweyh derivative, the operator

$$D_{l,[S]}^n f(z) = l_1 z D_l^1 (D_{l,[S]}^{n-1} f(z)) = z + \sum_{k=2}^{\infty} \left(\frac{l_1 l_{k-1}}{l_k} \right)^n f_k z^k$$

is called the Gelfond-Leont'ev-Sălăgean derivative and the operator

$$D_{l,[R]}^n f(z) = z l_n D_l^n \{z^{n-1} f(z)\} = z + \sum_{k=2}^{\infty} \frac{l_{k-1} l_n}{l_{n+k-1}} f_k z^k$$

is called the Gelfond-Leont'ev-Ruscheweyh derivative. In [4] the behavior of maximal terms of successive Gelfond-Leont'ev-Sălăgen and Gelfond-Leont'ev-Ruscheweyh derivatives and in [5] the properties of Hadamard compositions of such derivatives are studied.

The radius $\varrho[f]$ of univalence of a function f is defined as follows: if $f'(0) = 0$ then we put $\varrho[f] = 0$, and if $f'(0) \neq 0$ then $\varrho[f]$ is a radius of the largest disk with the center at a point $z = 0$, in which the function f is univalent. The asymptotic behavior of the sequence of radii of the univalence of ordinary derivatives of the function f has been studied by many authors. The most significant contribution was made by S.M. Shah and S.Y. Trimble [6–8]. The asymptotic behavior of the sequence of radii of the univalence of Gelfond-Leont'ev derivatives is investigated in [9–12].

Here we consider a similar problem for Gelfond-Leont'ev-Sălăgen and Gelfond-Leont'ev-Ruscheweyh derivatives. Our research is based on the following lemmas [11].

Lemma 1. *Let* $\alpha(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ *and* $\varrho \in (0, +\infty)$. *If the function* α *is univalent in* $\mathbb{D}_\varrho = \{z: \, |z| < \varrho\}$ *then* $|\alpha_k|\varrho^{k-1} \le k|\alpha_1|$ *for all* $k \ge 1$.

Lemma 2. *Let* $\alpha(z) = \sum_{k=0}^{\infty} \alpha_k z^k$. *If* $\sum_{k=2}^{\infty} k|\alpha_k|\varrho^{k-1} \le |\alpha_1|$ *for some* $\varrho \in (0, +\infty)$ *then the function* α *is univalent in* $\mathbb{D}_\varrho$.

2. Gelfond-Leont'ev-Gelfond-Leont'ev-Sălăgean derivatives

Here for a function (1) we will consider a more general form of the Gelfond-Leont'ev-Sălăgean derivative $D^n_{l,[S]} f$.

We remark that if a function f is given by a gap power series

$$f(z) = f_0 + \sum_{j=0}^{\infty} f_{k_j+1} z^{k_j+1}, \quad f_{k_j+1} \ne 0 \, (j \ge 0), \tag{2}$$

where $0 \le k_j \uparrow \infty$ as $0 \le j \to \infty$, then

$$D^{k_n}_{l,[S]} f(z) = \sum_{j=0}^{\infty} \left(\frac{l_1 l_{k_j}}{l_{k_j+1}}\right)^{k_n} f_{k_j+1} z^{k_j+1}. \tag{3}$$

Theorem 1. *If the function* f *is given by a gap power series* (2) *then for the radius of univalence of Gelfond-Leont'ev-Sălăgean derivative* $D^{k_n}_{l,[S]} f$ *the estimates*

$$\frac{\sqrt{2}-1}{\sqrt{2}} \sqrt[k_j]{\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right|} \left(\frac{l_{k_0} l_{k_j+1}}{l_{k_j} l_{k_0+1}}\right)^{k_n/k_j} \le \varrho[D^{k_n}_{l,[S]} f] \le \sqrt[k_j]{(k_j+1)\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right|} \left(\frac{l_{k_0} l_{k_j+1}}{l_{k_j} l_{k_0+1}}\right)^{k_n/k_j} \tag{4}$$

hold for every $j \ge 1$.

Proof. If function (3) is univalent in $\mathbb{D}_\varrho$ then by Lemma 1

$$\left|\left(\frac{l_1 l_{k_j}}{l_{k_j+1}}\right)^{k_n} f_{k_j+1}\right| \varrho^{k_j} \le (k_j+1)\left(\frac{l_1 l_{k_0}}{l_{k_0+1}}\right)^{k_n} |f_{k_0+1}|,$$

i.e.

$$\varrho[D^{k_n}_{l,[S]} f]^{k_j} \le (k_j+1)\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right| \left(\frac{l_{k_0} l_{k_j+1}}{l_{k_j} l_{k_0+1}}\right)^{k_n},$$

whence we obtain the right hand side of (4).

Now let $0 < x \le (\sqrt{2}-1)/\sqrt{2}$. Then

$$\sum_{j=1}^{\infty} (k_j+1) x^{k_j} \le \sum_{j=1}^{\infty} (j+1) x^j = \frac{1}{(1-x)^2} - 1 \le 1.$$

Therefore, if

$$\varrho = \frac{\sqrt{2}-1}{\sqrt{2}} \sqrt[k_j]{\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right|} \left(\frac{l_{k_0} l_{k_j+1}}{l_{k_j} l_{k_0+1}}\right)^{k_n/k_j} \tag{5}$$

then

$$x = \varrho \sqrt[k_j]{\left|\frac{f_{k_j+1}}{f_{k_0+1}}\right| \left(\frac{l_{k_j} l_{k_0+1}}{l_{k_0} l_{k_j+1}}\right)^{k_n/k_j}} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

and, thus,

$$\sum_{j=1}^{\infty}(k_j+1)\left(\frac{l_{k_j} l_{k_0+1}}{l_{k_0} l_{k_j+1}}\right)^{k_n} \left|\frac{f_{k_j+1}}{f_{k_0+1}}\right| \varrho^{k_j} = \sum_{j=1}^{\infty}(k_j+1)x^{k_j} \le 1,$$

whence

$$\sum_{j=1}^{\infty}(k_j+1)\left(\frac{l_1 l_{k_j}}{l_{k_j+1}}\right)^{k_n} |f_{k_j+1}|\varrho^{k_j} \le \left(\frac{l_1 l_{k_0}}{l_{k_0+1}}\right)^{k_n} |f_{k_0+1}|.$$

Hence by Lemma 2 the function $D^{k_n}_{l,[S]f}$ is univalent in $\mathbb{D}_\varrho$ and, thus, in view of (5)

$$\varrho[D^{k_n}_{l,[S]}f] \ge \frac{\sqrt{2}-1}{\sqrt{2}} \sqrt[k_j]{\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right| \left(\frac{l_{k_0} l_{k_j+1}}{l_{k_j} l_{k_0+1}}\right)^{k_n/k_j}} \tag{6}$$

for every $j \ge 1$, i.e. the left hand side of (4) is correct. Theorem 1 is proved. □

If series (2) has a radius of convergence $R \in (0, +\infty)$ then

$$\varliminf_{j\to\infty} \sqrt[k_j]{(k_j+1)\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right|} = R$$

and, since $\varliminf\limits_{j\to\infty} a_j b_j \le \varliminf\limits_{j\to\infty} a_j \varlimsup_{j\to\infty} b_j$, from (4) we get

$$\varrho[D^{k_n}_{l,[S]}f] \le \varliminf_{j\to\infty} \sqrt[k_j]{(k_j+1)\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right|} \left(\frac{l_{k_0} l_{k_j+1}}{l_{k_j} l_{k_0+1}}\right)^{k_n/k_j} \le R\left(\varlimsup_{j\to\infty} \sqrt[k_j]{\frac{l_{k_0} l_{k_j+1}}{l_{k_j} l_{k_0+1}}}\right)^{k_n}.$$

Similarly, since $\varlimsup\limits_{j\to\infty} a_j b_j \ge \varliminf\limits_{j\to\infty} a_j \varlimsup_{j\to\infty} b_j$, from (6) we get

$$\varrho[D^{k_n}_{l,[S]}f] \ge \frac{\sqrt{2}-1}{\sqrt{2}} \varlimsup_{j\to\infty} \sqrt[k_j]{\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right|} \left(\frac{l_{k_0} l_{k_j+1}}{l_{k_j} l_{k_0+1}}\right)^{k_n/k_j} \ge \frac{\sqrt{2}-1}{\sqrt{2}} R\left(\varlimsup_{j\to\infty} \sqrt[k_j]{\frac{l_{k_0} l_{k_j+1}}{l_{k_j} l_{k_0+1}}}\right)^{k_n}.$$

Therefore, we obtain the following statement.

Proposition 1. *If a function f is given by the gap power series* (2) *with the radius of convergence $R \in (0, +\infty)$ then for the radius of univalence of Gelfond-Leont'ev-Sălăgean derivative $D^{k_n}_{l,[S]}f$ the estimates*

$$\frac{\sqrt{2}-1}{\sqrt{2}} R\left(\varlimsup_{j\to\infty} \sqrt[k_j]{\frac{l_{k_0} l_{k_j+1}}{l_{k_j} l_{k_0+1}}}\right)^{k_n} \le \varrho[D^{k_n}_{l,[S]}f] \le R\left(\varlimsup_{j\to\infty} \sqrt[k_j]{\frac{l_{k_0} l_{k_j+1}}{l_{k_j} l_{k_0+1}}}\right)^{k_n}$$

hold.

Theorem 1 and Proposition 1 imply the following statement.

Corollary 1. *For the radius of univalence of Gelfond-Leont'ev-Sălăgean derivative $D^n_{l,[S]}f$ of function* (1) *the estimates*

$$\frac{\sqrt{2}-1}{\sqrt{2}}\sqrt[j]{\left|\frac{f_1}{f_{j+1}}\right|}\left(\frac{l_{j+1}}{l_jl_1}\right)^{n/j}\le\varrho[D^n_{l,[S]}f]\le\sqrt[j]{(j+1)\left|\frac{f_1}{f_{j+1}}\right|}\left(\frac{l_{j+1}}{l_jl_1}\right)^{n/j}\tag{7}$$

hold for every $j\ge1$. If, moreover, series (1) *has the radius of convergence $R\in(0,+\infty)$ then*

$$\frac{\sqrt{2}-1}{\sqrt{2}}R\left(\varlimsup_{j\to\infty}\sqrt[j]{\frac{l_{j+1}}{l_j}}\right)^n\le\varrho[D^n_{l,[S]}f]\le R\left(\varlimsup_{j\to\infty}\sqrt[j]{\frac{l_{j+1}}{l_j}}\right)^n.$$

We remark also that for $j=1$ from (7) we get

$$\frac{\sqrt{2}-1}{\sqrt{2}}\left|\frac{f_1}{f_2}\right|\left(\frac{l_2}{l_1^2}\right)^n\le\varrho[D^n_{l,[S]}f]\le2\left|\frac{f_1}{f_2}\right|\left(\frac{l_2}{l_1^2}\right)^n.$$

3. Gelfond-Leont'ev-Ruscheweyh derivatives

Here for a function (1) we will consider also a slightly more general form of the Gelfond-Leont'ev-Ruscheweyh derivative

$$D^n_{l,[R]}(z)=\sum_{k=1}^{\infty}\frac{l_{k-1}l_n}{l_{n+k-1}}f_kz^k.$$

We remark that if a function f is given by a gap power series (2) then

$$D^{k_n}_{l,[R]}(z)=\sum_{j=0}^{\infty}\frac{l_{k_n}l_{k_j}}{l_{k_j+k_n}}f_{k_j+1}z^{k_j+1}.\tag{8}$$

Theorem 2. *If a function f is given by a gap power series* (2) *then for the radius of univalence of Gelfond-Leont'ev-Ruscheweyh derivative $D^{k_n}_{l,[R]}f$ the estimates*

$$\frac{\sqrt{2}-1}{\sqrt{2}}\sqrt[k_j]{\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right|\frac{l_{k_0}l_{k_j+k_n}}{l_{k_j}l_{k_0+k_n}}}\le\varrho[D^{k_n}_{l,[R]}f]\le\sqrt[k_j]{(k_j+1)\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right|\frac{l_{k_0}l_{k_j+k_n}}{l_{k_j}l_{k_0+k_n}}}\tag{9}$$

hold for every $j\ge1$.

Proof. If the function (8) is univalent in $\mathbb{D}_\varrho$ then by Lemma 1

$$\left|\frac{l_{k_n}l_{k_j}}{l_{k_j+k_n}}f_{k_j+1}\right|\varrho^{k_j}\le(k_j+1)\frac{l_{k_n}l_{k_0}}{l_{k_0+k_n}}|f_{k_0+1}|,$$

i.e.

$$\varrho[D^{k_n}_{l,[R]}f]^{k_j}\le(k_j+1)\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right|\frac{l_{k_j+k_n}l_{k_0}}{l_{k_0+k_n}l_{k_j}},$$

whence we obtain the right hand side of (9).

Now we put

$$\varrho=\frac{\sqrt{2}-1}{\sqrt{2}}\sqrt[k_j]{\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right|\frac{l_{k_0}l_{k_j+k_n}}{l_{k_j}l_{k_0+k_n}}}.\tag{10}$$

Then as in the proof of Theorem 1 we obtain

$$\sum_{j=1}^{\infty}(k_j+1)\frac{l_{k_n}l_{k_j}}{l_{k_j+k_n}}|f_{k_j+1}|\varrho^{k_j}\le\frac{l_{k_n}l_{k_0}}{l_{k_0+k_n}}|f_{k_0+1}|.$$

Hence by Lemma 2 the function $D^{k_n}_{l,[R]f}$ is univalent in $\mathbb{D}_\varrho$ and, thus, in view of (10)

$$\varrho[D^{k_n}_{l,[R]f}]\ge\frac{\sqrt{2}-1}{\sqrt{2}}\sqrt[k_j]{\left|\frac{f_{k_0+1}}{f_{k_j+1}}\right|\frac{l_{k_0}l_{k_j+k_n}}{l_{k_j}l_{k_0+k_n}}}\tag{11}$$

for every $j\ge1$, i.e. the left hand side of (9) is correct. Theorem 2 is proved. □

Using (9), (11) and repeating the proof of Proposition 1 we come to the next statement.

Proposition 2. *If a function f is given by the gap power series* (2) *with the radius of convergence $R\in(0,+\infty)$ then for the radius of univalence of Gelfond-Leont'ev-Ruscheweyh derivative $D^{k_n}_{l,[R]}f$ the following estimates we have*

$$\frac{\sqrt{2}-1}{\sqrt{2}}R\varlimsup_{j\to\infty}\sqrt[k_j]{\frac{l_{k_0}l_{k_j+k_n}}{l_{k_j}l_{k_0+k_n}}}\le\varrho[D^{k_n}_{l,[R]}f]\le R\varlimsup_{j\to\infty}\sqrt[k_j]{\frac{l_{k_0}l_{k_j+k_n}}{l_{k_j}l_{k_0+k_n}}}.$$

Theorem 2 and Proposition 2 imply the following statement.

Corollary 2. *For the radius of univalence of Gelfond-Leont'ev–Ruscheweyh derivative $D^n_{l,[R]}f$ of a function* (1) *the estimates*

$$\frac{\sqrt{2}-1}{\sqrt{2}}\sqrt[j]{\left|\frac{f_1}{f_{j+1}}\right|\frac{l_{j+n}}{l_jl_n}}\le\varrho[D^n_{l,[R]}f]\le\sqrt[j]{(j+1)\left|\frac{f_1}{f_{j+1}}\right|\frac{l_{j+n}}{l_jl_n}}\tag{12}$$

hold for every $j\ge1$. If, moreover, series (1) *has the radius of convergence $R\in(0,+\infty)$ then*

$$\frac{\sqrt{2}-1}{\sqrt{2}}R\varlimsup_{j\to\infty}\sqrt[j]{\frac{l_{j+n}}{l_jl_n}}\le\varrho[D^n_{l,[S]}f]\le R\varlimsup_{j\to\infty}\sqrt[j]{\frac{l_{j+n}}{l_jl_n}}.$$

For $j=1$ from (12) we get

$$\frac{\sqrt{2}-1}{\sqrt{2}}\left|\frac{f_1}{f_2}\right|\frac{l_{n+1}}{l_1l_n}\le\varrho[D^n_{l,[R]}f]\le2\left|\frac{f_1}{f_2}\right|\frac{l_{n+1}}{l_1l_n}.$$

Choosing a function l in one or another way , we obtain the corresponding estimates for the radius of univalence.

Example 1. Let $l_k=\exp\{ak^2\}$. Then $\dfrac{l_{j+1}}{l_j}=\exp\{2aj+1\}$ and $\dfrac{l_{j+n}}{l_jl_n}=\exp\{2ajn\}$. Therefore, if series (1) has the radius of convergence $R\in(0,+\infty)$ then by Corollaries 1 and 2

$$\frac{\sqrt{2}-1}{\sqrt{2}}Re^{2an}\le\varrho[D^n_{l,[S]}f],\ \varrho[D^n_{l,[R]}f]\le Re^{2an}.$$

Example 2. If $l_k = q^k$ then $\dfrac{l_{j+1}}{l_j} = q$ and $\dfrac{l_{j+n}}{l_j l_n} = 1$. Therefore, if series (1) has the radius of convergence $R \in (0, +\infty)$ then

$$\frac{\sqrt{2}-1}{\sqrt{2}} R \le \varrho[D^n_{l,[S]} f],\ \varrho[D^n_{l,[R]} f] \le R. \tag{13}$$

Example 3. If $l_k = \dfrac{1}{k!}$ then $\dfrac{l_{j+1}}{l_j} = \dfrac{1}{j+1}$ and $\dfrac{l_{j+n}}{l_j l_n} = \dfrac{j!n!}{(j+n)!}$. Therefore, if series (1) has the radius of convergence $R \in (0, +\infty)$ then estimates (13) hold.

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ПРО РАДІУСИ ОДНОЛИСТОСТІ ПОСЛІДОВНИХ ПОХІДНИХ ГЕЛЬФОНДА-ЛЕОНТЬЄВА-САЛАГЕНА І ГЕЛЬФОНДА-ЛЕОНТЬЄВА-РУШЕВЕЯ

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Для аналітичної в крузі $\{z : |z| < 1\}$ функції $f(z) = z + \sum_{k=1}^{\infty} f_k z^k$ і формального степеневого ряду $l(z) = 1 + \sum_{k=1}^{\infty} l_k z^k$ з $l_k > 0$ оператор $D^n_{l,[S]} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{l_1 l_{k-1}}{l_k}\right)^n f_k z^k$ називається похідною Гельфонда-Леонтьєва-Салагена, а оператор $D^n_{l,[R]} f(z) = z + \sum_{k=2}^{\infty} \frac{l_{k-1} l_n}{l_{n+k-1}} f_k z^k$ називається похідною Гельфонда-Леонтьєва-Рушевея. Через $\varrho[f]$ позначимо радіус однолистості функції f. Доведено, наприклад, що для кожного $n \geq 1$

$$\frac{\sqrt{2}-1}{\sqrt{2}} \left|\frac{f_1}{f_2}\right| \left(\frac{l_2}{l_1^2}\right)^n \leq \varrho[D^n_{l,[S]} f] \leq 2 \left|\frac{f_1}{f_2}\right| \left(\frac{l_2}{l_1^2}\right)^n$$

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$$\frac{\sqrt{2}-1}{\sqrt{2}} \left|\frac{f_1}{f_2}\right| \frac{l_{n+1}}{l_1 l_n} \leq \varrho[D^n_{l,[R]} f] \leq 2 \left|\frac{f_1}{f_2}\right| \frac{l_{n+1}}{l_1 l_n}.$$

Ключові слова: аналітична функція, похідна Гельфонда-Леонтьєва-Салагена, похідна Гельфонда-Леонтьєва-Рушевея, радіус однолистості.