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ON SPACES OF $*$-MEASURES ON ULTRAMETRIC SPACES

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The notion of $*$-measure on a compact Hausdorff space is introduced and investigated in a previous publication of the first named author. In the present note we consider the set of all $*$-measures of compact support on an ultrametric space. An ultrametrization of this set is provided, which determines a functor in the category of ultrametric spaces and non-expanding maps. We prove that this functor is locally non-expanding and preserves the class of complete ultrametric spaces.

Key words: ultrametric space, non-expanding map, $*$-measure.

1. Introduction

A metric d on a set X is called an ultrametric if it satisfies the strong triangle inequality:

$$d(x,y) \le \max\{d(x,z), d(z,y)\}, \ x,y,z \in X.$$

The ultrametric spaces were first introduced by Hausdorff in 1934. They find numerous applications not only in mathematics but also in another disciplines, e.g. biology, physics [2, 14], computer science [6], logic programming and artificial intelligence [9], linguistics [10].

In [15] an ultrametric is defined on the set of probability measures of compact support on an ultrametric space. It is shown that this construction determines a locally nonexpansive functor in the category of ultrametric spaces and nonexpanding maps, and this functor "makes a useful building block for the definition of metric domains for probabilistic program constructs."

Some categorical properties of this construction are established in [3]. In particular, it is proved therein that the probability measure functor determines a monad on the category of ultrametric spaces and nonexpanding maps.

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The notion of $*$-measure is introduced by the first-named author [13]. The aim of the present note is to define an ultrametric on the set of of $*$-measures of compact support defined on ultrametric spaces. We prove that the obtained construction determines a functor on the category of ultrametric spaces and non-expanding maps. Also, we show that this construction preserves completeness of ultrametric spaces.

2. Results

By $\mathbb{I}$ we denote the unit segment $[0,1]$. Recall that a triangular norm (a t-norm) is a continuous function $(a,b)\mapsto a*b\colon \mathbb{I}\times\mathbb{I}\to\mathbb{I}$ satisfying the conditions

(1) $*$ is associative;
(2) $*$ is commutative;
(3) $*$ is monotone, i.e. $a\le a'$ and $b\le b'$ both imply $a*b\le a'*b'$ for all $a,a',b,b'\in\mathbb{I}$;
(4) 1 is a unit.

See. e.g., [4] for the details. The following are examples of t-norms: $\cdot$ (multiplication), min, $(a,b)\mapsto\max\{a+b-1,0\}$ (Łukasiewicz t-norm).

Let us recall the notion of $*$-measure (see [13] for details). Given topological spaces X,Y, by $C(X,Y)$ we denote the set of continuous functions from X to Y. By $\vee$ we denote the operation of maximum of numbers as well as pointwise maximum of real-valued functions.

Definition 1. Let $*$ be a t-norm. A functional $\mu\colon C(X,\mathbb{I})\to\mathbb{I}$ is called a $*$-*measure* on a compact Hausdorff space X if the following is satisfied:

(1) $\mu(c_X)=c$, where c_X denotes the constant function on X taking value c;
(2) $\mu(\lambda*\varphi)=\lambda*\mu(\varphi)$;
(3) $\mu(\varphi\vee\psi)=\mu(\varphi)\vee\mu(\psi)$.

The set of all $*$-measures on X is denoted by $M^*(X)$. It is known [13] that the set $M^*(X)$ is compact being endowed with the weak* topology. This construction determines a functor in the category **Comp** of compact Hausdorff spaces and continuous maps. This functor satisfies some natural properties. In particular, the notion of support is defined for any element $\mu\in M^*(X)$. By the definition, the support of μ is the minimal (with respect to the inclusion) closed subset A of X satisfying the following condition: for every $\varphi,\psi\in C(X,\mathbb{I})$,

$$\varphi|A=\psi|A \implies \mu(\varphi)=\mu(\psi).$$

Given an ultrametric space (X,d) and $r>0$, denote by $\mathcal{F}_r(X)$ the set of all functions from X to $\mathbb{I}$ constant on all balls of radius r. We keep the notation $M^*(X)$ for the set of all $*$-measures on some Hausdorff compactification $bX\supset X$ whose support is a compact subset of X. Note that this is nothing but Chigogidze's extension of the normal functors [1].

Given $\mu,\nu\in M^*(X)$, we let

$$\tilde d(\mu,\nu)=\inf\{r>0|\mu(\varphi)=\nu(\varphi)\text{ for all }\varphi\in\mathcal{F}_r\}.$$

Theorem 1. *The function $\tilde d$ is an ultrametric on $M^*(X)$.*

Proof. First we show that the function $\tilde{d}$ is well defined. Since the sets $\mathrm{supp}(\mu)$, $\mathrm{supp}(\nu)$ are compact, they are bounded. The latter means that there exist $r > 0$ and x_0 such that $\mathrm{supp}(\mu) \cup \mathrm{supp}(\nu) \subset B_r(x_0)$.

Consider the set $\mathcal{F}_r(X)$. Let $\varphi \in \mathcal{F}_r(X)$, then $\varphi|B_r(x_0) \equiv c$ is constant and $\mu(\varphi) = c = \nu(\varphi)$, for some $c \in \mathbb{I}$. It follows that the set of which we consider the infimum is nonempty and therefore the formal definition makes sense.

By the definition $\tilde{d}(\mu,\nu) \geqslant 0$. Furthermore, $\tilde{d}(\mu,\mu) = 0$.

Now let $\tilde{d}(\mu,\nu) = 0$. We have to show that $\mu = \nu$.

Note that for every $r > 0$ and for every $\varphi \in \mathcal{F}_r$ we have $\mu(\varphi) = \nu(\varphi)$. We need to show that $\mu(\varphi) = \nu(\varphi)$ for all $\varphi \in C(X,\mathbb{I})$.

Suppose the contrary, i.e. that there exists $\varphi \in C(X,\mathbb{I})$ such that $\mu(\varphi) \neq \nu(\varphi)$. Note that each $\mu \in M^*(X)$ is a continuous map with respect to the sup-metric on $C(X,\mathbb{I})$ for any zero-dimensional space X (see [13]).

Since each ultrametric space is a zero-dimensional space [16], we see that if $\varphi_i \underset{i\to\infty}{\longrightarrow} \varphi$ with respect to the sup-metric, then $\mu(\varphi_i) \underset{i\to\infty}{\longrightarrow} \mu(\varphi)$.

Construct a sequence of functions $\varphi_i \in \mathcal{F}_{r_i}(X)$ converging to φ. Since $supp\mu \cup supp\nu$ is a zero-dimensional compactum, we can choose for each $i \in \mathcal{N}$ a number $r_i > 0$ and a function $\varphi_i \in \mathcal{F}_{r_i}(X)$ such that $\|\varphi - \varphi_r\| \leqslant \varepsilon$. Therefore, choosing $\varepsilon = \frac{1}{2^i}$ we get a desired sequence (φ_i). Then

$$\mu(\varphi) = \lim_{i\to\infty} \mu(\varphi_i) = \lim_{i\to\infty} \nu(\varphi_i) = \nu(\varphi).$$

The symmetry of the functions $\tilde{d}$ obviously follows from the definition: $\tilde{d}(\mu,\nu) = \tilde{d}(\nu,\mu)$ for all $\mu,\nu \in M^*(X)$.

Now we need to prove the strong triangle inequality. Let $\mu,\nu,\tau \in M^*(X)$ and $\tilde{d}(\mu,\nu) = a, \tilde{d}(\nu,\tau) = b$. Without loss of generality, we may assume that $a \leqslant b$. Then, for every $\varepsilon > 0$ and every $\varphi \in \mathcal{F}_{a+\varepsilon}(X)$, $\psi \in \mathcal{F}_{b+\varepsilon}(X)$, we have $\mu(\varphi) = \nu(\varphi)$ and $\nu(\psi) = \tau(\psi)$. And then, for every $\varphi \in \mathcal{F}_{b+\varepsilon}(X)$, we have $\mu(\varphi) = \nu(\varphi) = \tau(\varphi)$. Hence, $\tilde{d}(\mu,\tau) \leqslant b + \varepsilon$ and letting $\varepsilon \to 0$, we see that $\tilde{d}(\mu,\tau) \leqslant b$.

We denote by **Ultr** the category of ultrametric spaces and non-expanding maps. Let $(X,d),(Y,d)$ be ultrametric spaces. Let $f\colon X \to Y$ be a non-expanding map.

Define $M^*(f)\colon M^*(X) \to M^*(Y)$ by the formula:

$$M^*(f)(\mu)(\varphi) = \mu(\varphi f),$$

$\mu \in M^*(X), \varphi \in C(X,\mathbb{I})$.

Proposition 1. *The map $M^*(f)$ is non-expanding.*

Proof. Let $\mu,\nu \in M^*(X)$ and $\tilde{d}(\mu,\nu) < r$.

Note that, since f is non-expanding, given $\varphi \in \mathcal{F}_r(Y)$, one has $\varphi f \in \mathcal{F}_r(X)$.

Then

$$M^*(f)(\mu)(\varphi) = \mu(\varphi f) = \nu(\varphi f) = M^*(f)(\nu)(\varphi).$$

Therefore, $\tilde{\rho}(M^*(f)(\mu), M^*(f)(\nu)) < r$ and we see that the map $M^*(f)$ is non-expanding.

Actually, we obtain a functor in the category **Ultr**. We keep the notation M^* for this functor.

A functor F in the category **Ultr** is called locally non-expanding, if

$$\tilde{\rho}(F(f), F(g)) = \rho(f, g)$$

(see [15]).

Proposition 2. *The functor M^* is locally non-expanding.*

Proof. Let $r > 0$ and $\rho(f, g) < r$.

Then for all $x \in X$, $\rho(f(x), g(x)) < r$ and then $g(x) \in B_r(f(x))$.

Let $\mu \in M^*(X)$. We need to show that $\tilde{\rho}(M^*(f)(\mu), M^*(g)(\mu)) < r$.

Let $\varphi \in B_r(Y)$, that is we need to check the equality

$$M^*(f)(\mu)(\varphi) = \mu(\varphi f) = \mu(\varphi g) = M^*(g)(\mu)(\varphi).$$

Let $x \in X$, then $\rho(f(x), g(x)) < r$ and since φ is a constant function on the balls of radius r it follows that $g(x) \in B_r(f(x))$. This means that $\varphi f(x) = \varphi g(x)$ and therefore $\mu(\varphi f) = \mu(\varphi g)$.

On the other hand, let $x \in X$ and $\delta_x \in M^*(X)$. We have that

$$\tilde{\rho}(M^*(f)(\delta_x), M^*(g)(\delta_x)) = \tilde{\rho}(M^*(f)(\varphi)(x), M^*(g)(\varphi)(x)) \geqslant \rho(f(x), g(x)).$$

And this proves the fact that the functor M^* is locally non-expanding.

Recall that the hyperspace $\exp X$ of a metric space X is the set of all nonempty compact subsets of X endowed with the Hausdorff metric

$$d_H(A, C) = \inf\{r > 0 \mid A \subset B_r(C),\ C \subset B_r(A)\},\ A, C \in \exp X.$$

For any $\mu \in M^*X$ its support is a nonempty compact subset of X, i.e., an element of the hyperspace $\exp X$.

It is well-known that the Hausdorff metric on the hyperspace of an ultrametric space is also an ultrametric space. Moreover, exp is a functor on the category **Ultr**.

Proposition 3. *The support map $s = s_X : M^*(X) \to \exp X$ is non-expanding.*

Proof. Let $\mu, \nu \in M^*(X)$ and $\tilde{d}(\mu, \nu) < r$.

Suppose that $d_H(s(\mu), s(\nu)) > r$, then $M^*(f)(\mu) = M^*(f)(\nu)$ and $f(s(\mu)) \neq f(s(\nu))$, where $f : X \to f(X)$ is the quotient map with respect to the decomposition of X into disjoint balls of radius r.

Without loss of generality one may assume that $f(s(\mu)) \setminus f(s(\nu)) \neq \emptyset$.

By the definition of support, there exist $\varphi, \psi \in C(f(X), \mathbb{I})$ such that $\varphi|f(s(\mu)) = \psi|f(s(\nu))$ and $M^*(f)(\mu)(\varphi) \neq M^*(f)(\nu)(\psi)$. This clearly contradicts to the choice of r.

Note that $s = (s_X)$ is a natural transformation of the functor M^* to the hyperspace functor exp.

Proposition 4. *Let (X, d) be an ultrametric space. Then the map $\delta : X \to M^*(X)$, $\delta(x) = \delta_x$, is an isometric embedding.*

Proof. We need to show that the equality $d(x, y) = \tilde{d}(\delta_x, \delta_y)$ is satisfied for all $x, y \in X$.

By the definition,

$$\tilde{d}(\delta_x, \delta_y) = \inf\{r > 0 \mid \delta_x(\varphi) = \delta_y(\varphi), \forall \varphi \in \mathcal{F}_r\}$$

and $\delta_x(\varphi) = \varphi(x), \delta_y(\varphi) = \varphi(y)$.

Let $d(x,y) < r$, then $x, y \in B_r(X)$ and $\varphi(x) = \varphi(y)$ for every $\varphi \in \mathcal{F}_r$. Therefore $\tilde{d}(\delta_x, \delta_y) \leqslant d(x,y)$.

Now let $\tilde{d}(\delta_x, \delta_y) < d(x,y)$. Then there exists $r > 0$ such that $d(x,y) > r$ and $\varphi(x) = \varphi(y)$ for every $\varphi \in \mathcal{F}_r$.

Since $d(x,y) > r$, we see that $B_r(x) \cap B_r(y) = \varnothing$. We take $\varphi \in \mathcal{F}_r$ that $\varphi|B_r(x) = 0$ and $\varphi|X \setminus B_r(x) = 1$. From that $\varphi \in \mathcal{F}_r$ it follows that $\varphi(x) = \varphi(y)$. And we got to a contradiction.

Note that $\delta = (\delta_X)$ is a natural transformation of the identity functor into the functor M^*.

We denote by $M^*_\omega(X)$ the subset of $M^*(X)$ consisting of $*$-measures of finite support, i.e., $*$-measures of the form $\mu = \bigvee_{i=1}^{n} \lambda_i * \delta_{x_i}$.

Proposition 5. *The set $M^*_\omega(X)$ is dense in $M^*(X)$.*

Proof. Let $\mu \in M^*(X)$ and let $r > 0$. Let $\{B_r(x_i) \mid i = 1, \dots, n\}$ be a finite disjoint cover of the set $\mathrm{supp}(\mu)$ by balls of radius r. By φ_i we denote the characteristic function of the set $B_r(x_i)$. Now let $\varphi \in \mathcal{F}_r(X)$. Without loss of generality one may assume that $\varphi \equiv 0$ on the set $X \setminus \bigcup_{i=1}^{n} B_r(x_i)$. Then $\varphi = \bigvee_{i=1}^{n} \varphi(x_i) * \varphi_i$.

Let $\nu = \bigvee_{i=1}^{n} \mu(\varphi_i) * \delta_{x_i}$. Note that

$$1 = \mu(1_X) = \mu\left(\bigvee_{i=1}^{n} \varphi_i\right) = \bigvee_{i=1}^{n} \mu(\varphi_i),$$

therefore $\nu \in M^*(X)$.

Now, given $\psi \in \mathcal{F}_r(X)$, one can write $\psi = \bigvee_{i=1}^{n} \psi(x_i) * \varphi$. Then

$$\nu(\psi) = \nu\left(\bigvee_{i=1}^{n} \psi(x_i) * \varphi\right) = \bigvee_{i=1}^{n} \psi(x_i) * \nu(\varphi_i) = \bigvee_{i=1}^{n} \psi(x_i) * \nu(\varphi_i) = \mu(\psi).$$

Therefore, $\tilde{d}(\mu, \nu) < r$.

In the sequel, we endow the set $C(X, \mathbb{I})$ with the sup-metric.

Lemma 1. *Let X be a compact ultrametric space. The set $\mathcal{F}(X) = \bigcup_{r>0} \mathcal{F}_r(X)$ is dense in $C(X, \mathbb{I})$.*

Theorem 2. *Suppose that (X, d) is a complete ultrametric space. Then the space $(M^*(X), \tilde{d})$ is also complete.*

Proof. Let (μ_i) be a Cauchy sequence in $M^*(X)$. From Proposition 3 it easily follows that the set $Y = \overline{\bigcup_{i=1}^{\infty} \mathrm{supp}(\mu_i)}$ is compact. Without loss of generality, one may assume that $Y = X$.

Let $\varphi \in \mathcal{F}(Y)$. There exists $r > 0$ such that $\varphi \in \mathcal{F}_r(Y)$. There exists $N \in \mathbb{N}$ such that, for any $m, n \geq N$, $\tilde{d}(\mu_m, \mu_n) < r$. Therefore $\mu_m(\varphi) = \mu_n(\varphi)$ for any $m, n \geq N$. We let $\mu(\varphi) = \lim_{i\to\infty} \mu_i(\varphi) = \mu_N(\varphi)$.

Thus, we have defined a map $\mu\colon \mathcal{F}(Y) \to \mathbb{I}$. It is straightforward to verify that μ satisfies the conditions from Definition 1 if we replace $C(X, \mathbb{I})$ by $\mathcal{F}(Y)$.

Now we are going to extend μ over the set $C(Y, \mathbb{I})$.

Claim. The map $\mu\colon \mathcal{F}(Y) \to \mathbb{I}$ is uniformly continuous.

Let $\varepsilon > 0$. Since the map $*\colon \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ is uniformly continuous, there exists $\delta > 0$ such that $|a - a'| < \delta$ and $|b - b'| < \delta$ together imply $|a * b - a' * b'| < \varepsilon$.

Let $\varphi', \varphi'' \in \mathcal{F}(Y)$. One may assume that $\varphi', \varphi'' \in \mathcal{F}_r(Y)$, for some $r > 0$. Let $\{B_r(x_i) \mid i = 1, \ldots, n\}$ be the disjoint cover of Y by balls. Let χ_i denote the characteristic function of the ball $B_r(x_i)$, $= 1, \ldots, n$. Then one can write

$$\varphi' = \bigvee_{i=1}^{n} \alpha'_i * \chi_i, \;\; \varphi'' = \bigvee_{i=1}^{n} \alpha''_i * \chi_i,$$

for some $\alpha'_i, \alpha''_i \in \mathbb{I}$.

If $\|\varphi' - \varphi''\| < \delta$, then $\bigvee_{i=1}^{n} |\alpha'_i - \alpha''_i| < \delta$ and we obtain

$$\begin{aligned}
|\mu(\varphi') - \mu(\varphi'')| &= \left|\mu\left(\bigvee_{i=1}^{n} \alpha'_i * \chi_i\right) - \mu\left(\bigvee_{i=1}^{n} \alpha''_i * \chi_i\right)\right| \\
&= \left|\bigvee_{i=1}^{n} \alpha'_i * \mu(\chi_i) - \bigvee_{i=1}^{n} \alpha''_i * \mu(\chi_i)\right| \\
&\leq \bigvee_{i=1}^{n} |\alpha'_i * \mu(\chi_i) - \alpha''_i * \mu(\chi_i)| \leq \varepsilon.
\end{aligned}$$

Let us return to the proof of the theorem. The map admits a unique continuous extension over the set $C(Y, \mathbb{I})$ (we keep the notation μ for this extension). Clearly, $\mu \in M^*(X)$ and $\mu = \lim_{i\to\infty} \mu_i$.

3. Remarks

Some of the results concerning fuzzy ultrametrization of functorial constructions are considered in numerous publications. Recall that fuzzy ultrametric spaces were introduced in [7, 11].

Fuzzy ultrametrization of the sets of probability measures is considered in [12]. The case of idempotent measures is treated in [5]. We formulate the general problem of fuzzy ultrametrization of the sets of ∗-measures of compact support defined on fuzzy ultrametric spaces.

Note that the space of probability measures of compact support on a complete ultrametric space is complete as well [15].

An ultrametric space (X, d) is said to be spherically complete if every descending collection of closed balls in X has nonempty intersection.

It is not known whether the space of $*$-measures of compact support on a spherically complete ultrametric space is also spherically complete.

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ПРО ПРОСТОРИ $*$-МІР НА УЛЬТРАМЕТРИЧНИХ ПРОСТОРАХ

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Поняття $*$-міри на компактному гаусдорфовому просторі запроваджено і досліджено першим автором. Ми розглядаємо множину всіх $*$-мір з компактними носіями на ультраметричному просторі. Наведено ультраметризацію цієї множини, яка визначає функтор на категорії ультраметричних просторів і нерозтягуючих відображень. Доведено, що цей функтор локально нерозтягуючий і зберігає клас повних ультраметричних просторів.

Ключові слова: ультраметричний простір, нерозтягуюче відображення, $*$-міра.