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INFINITESIMAL TRANSFORMATIONS OF A SYMMETRIC RIEMANNIAN SPACE OF THE FIRST CLASS

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The study of infinitesimal transformations in Riemannian spaces is of interest both theoretically and as an application. If a certain field is specified in the space V_n with the metric ds^2, which admits an r-parametric group of motions G_r, then this field has r conservation laws. The distribution of relativistic gas according to the Maxwell-Boltzmann law is characterized by a vector $\xi^i(x)$, which is a Killing vector (if the gas consists of particles of nonzero rest mass) or an infinitely small conformal transformation vector (if the gas consists of particles of zero rest mass). ([5], [6]) In this article, infinitesimal motions in symmetric Riemannian spaces of the first class V_n were studied. For $n = 4$ the basis of the Lie group G_8 of examined transformations is explicitly found and the structure of this group is given.

Key words: second approximation space, infinitesimal transformations, Lie group.

1. Preliminary information

P. A. Shirokov ([2, 7]) found all irreducible symmetric Riemannian spaces $V_n(x; g(x))$ of the first class. The metric tensor $g_{ij}(x)$ of such spaces in a Riemannian coordinate system with origin at a point $M_0(x^h = 0)$ has the following form:

$$g_{ij}(x) = \underset{\circ}{g}_{ij} + \frac{1}{3}\left(\underset{\circ}{h}_{i\alpha}\,\underset{\circ}{h}_{j\beta} - \underset{\circ}{h}_{ij}\,\underset{\circ}{h}_{\alpha\beta}\right) x^\alpha x^\beta, \tag{1}$$

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where

$$\left(\underset{\circ}{g}_{ij}\right) = \begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix}, \tag{2}$$

$$\left(\underset{\circ}{h}_{ij}\right) = \begin{pmatrix} e_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & e_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & e_{m-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & e_m & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}, \tag{3}$$

E_m is the unit matrix, $(e_i = \pm 1,\ i = 1, 2, ..., m)$.

In (2) and (3), $\underset{\circ}{g}_{ij}$ and $\underset{\circ}{h}_{ij}$, in the terminology of P. A. Shirokov, are values of the components of the first and second fundamental tensors at the beginning of the Riemannian coordinate system.

For an arbitrary Riemannian space $V_n(x; g(x))$ S. M. Pokas ([3, 4]) introduced the concept of a second approximation space $\widetilde{V}^2_n(y; \widetilde{g}(y))$:

$$\widetilde{g}_{ij}(y) = \underset{\circ}{g}_{ij} + \frac{1}{3} \underset{\circ}{R}_{i\alpha\beta j} y^\alpha y^\beta, \tag{4}$$

where $\underset{\circ}{g}_{ij} = g_{ij}(M_0), \underset{\circ}{R}_{i\alpha\beta j} = R_{i\alpha\beta j}(M_0)$, and $M_0 \in V_n$ is an arbitrary point of the initial space V_n.

Comparison of (1) and (4) shows that the symmetric Riemannian space of the first class V_n is isometric to the space of the second approximation $\widetilde{V}^2_n$. Therefore, the Lie group of the infinitesimal transformations $\widetilde{G}_r$ of the space $\widetilde{V}^2_n$ is isomorphic to the Lie group of the infinitesimal transformations G_r of the symmetric Riemannian space of the first class V_n.

In this article, we will use the results of the study of infinitesimal motions in Riemannian space of the second approximation $\widetilde{V}^2_n$. The following statement was proved ([3]).

Proposition 1. *Forexistence of an analytic Killing vector $\widetilde{\xi}^h(y)$ in the Riemannian space of the second approximation $\widetilde{V}^2_n(y; \widetilde{g}(y))$, in the following form:*

$$\widetilde{\xi}^h(y) = a^h_{\cdot} + a^h_{\cdot l}\, y^l + a^h_{\cdot l_1 l_2}\, y^{l_1} y^{l_2} + \ldots + a^h_{\cdot l_1 \ldots l_p}\, y^{l_1} \cdot \ldots \cdot y^{l_p} + \ldots \tag{5}$$

where $\underset{p}{a}{}^h = a^h_{\cdot l_1 \ldots l_p} y^{l_1} \ldots y^{l_p}$, $a^h_{\cdot}, a^h_{\cdot l_1}, a^h_{\cdot l_1 l_2}, \ldots, a^h_{\cdot l_1 \ldots l_k}$ are some constants, the following conditions are necessary and sufficient:

$$\underset{2p}{a}{}^h_{\cdot} = \frac{(-1)^{p+1}}{2p-1} a^\alpha t^{(p)h}_{\alpha} \tag{6}$$

(7) $$\underset{2p+1}{a}{}^{h}_{\cdot} = 0$$

(8) $$a^{\alpha}_{\cdot(i}\,\underset{\circ}{g}_{j)\alpha} = 0$$

(9) $$a^{\alpha}_{\cdot(i}\,\underset{\circ}{R}_{j)(l_1 l_2)\alpha} + a^{\alpha}{}_{\cdot(l_1}\,\underset{\circ}{R}_{l_2)(ij)\alpha} = 0$$

(10) $$\underset{2p-1}{a}{}^{\alpha}_{\cdot(i}t^{h}_{j)\alpha} + \underset{2p}{a}{}^{\alpha}_{\cdot}\mu_{\alpha ij} = 0$$

$(p = 1, 2, \ldots)$.

In the conditions (6)–(10), the following notation is introduced:

$$t^h_k = \frac{1}{3}\,\underset{\circ}{R}{}^{h}_{\cdot l_1 l_2 k}\; y^{l_1} y^{l_2}, \quad \mu_{kij} = \frac{1}{3}\,\underset{\circ}{R}_{k(ij)l}\; y^{l},$$

$$t^{(p)h}_{k} = t^{h}_{\alpha_1} t^{\alpha_1}_{\alpha_2} \cdot \ldots \cdot t^{\alpha_p}_{k}, \quad (p = 2, 3, \ldots)$$

$$\underset{2p+1}{a}{}^{h}_{\cdot i} = \frac{1}{2p+2}\,\frac{\partial}{\partial y^i}\left(\underset{2p+2}{a}{}^{h}_{\cdot}\right)$$

Considering the fact that the matrix of the tensor $\underset{\circ}{h}_{ij}$ in (3) is nilpotent, the relations (5)–(10) take the following form:

(11) $$\xi^p(x) = a^p_{\cdot} + a^p_{\cdot l}\, x^l + \frac{1}{3} a^{\alpha}_{\cdot}\left(\underset{\circ}{h}_{l_1\alpha}\,\underset{\circ}{h}{}^{p}_{l_2} - \underset{\circ}{h}_{l_1 l_2}\,\underset{\circ}{h}{}^{p}_{\alpha}\right)\; x^{l_1} x^{l_2}$$

(12) $$a^{\alpha}_{\cdot}{}_{(i}\,\underset{\circ}{g}_{j)\alpha} = 0$$

(13) $$\begin{aligned} & a^{\alpha}_{\cdot\, i}\left(\underset{\circ}{h}_{jl_1}\,\underset{\circ}{h}_{l_2\alpha} + \underset{\circ}{h}_{jl_2}\,\underset{\circ}{h}_{l_1\alpha} - 2\,\underset{\circ}{h}_{l_1 l_2}\,\underset{\circ}{h}_{j\alpha}\right) + \\ & + a^{\alpha}_{\cdot\, j}\left(\underset{\circ}{h}_{il_1}\,\underset{\circ}{h}_{l_2\alpha} + \underset{\circ}{h}_{il_2}\,\underset{\circ}{h}_{l_1\alpha} - 2\,\underset{\circ}{h}_{l_1 l_2}\,\underset{\circ}{h}_{i\alpha}\right) + \\ & + a^{\alpha}_{\cdot\, l_1}\left(\underset{\circ}{h}_{il_2}\,\underset{\circ}{h}_{j\alpha} + \underset{\circ}{h}_{jl_2}\,\underset{\circ}{h}_{i\alpha} - -2\,\underset{\circ}{h}_{ij}\,\underset{\circ}{h}_{l_2\alpha}\right) + \\ & + a^{\alpha}_{\cdot\, l_2}\left(\underset{\circ}{h}_{il_1}\,\underset{\circ}{h}_{j\alpha} + \underset{\circ}{h}_{jl_1}\,\underset{\circ}{h}_{i\alpha} - 2\,\underset{\circ}{h}_{ij}\,\underset{\circ}{h}_{l_1\alpha}\right) = 0. \end{aligned}$$

Thus the statement is proved.

Proposition 2. *An analytic Killing vector exists in a symmetric Riemannian space of the first class V_n if and only if its components have the form* (11)*, where $a^h_{\cdot}$ are arbitrary constants, and $a^h_{\cdot l}$ satisfy the algebraic equations* (12) *and* (13)*.*

Remark 1. From (11)–(13) we arrive at the well-known result ([1]) that the maximal order r of Lie groups of motions in Riemannian space V_n satisfies the inequality:

$$r \leq \frac{n(n+1)}{2}$$

2. Infinitesimal transformations in a symmetric Riemannian space of the first class V_4

Let us consider the case $n = 4$, then the matrices $\left\|\underset{\circ}{g}_{ij}\right\|$ and $\left\|\underset{\circ}{h}_{ij}\right\|$ have the form:

$$\left\|\underset{\circ}{g_{ij}}\right\| = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{14}$$
$$\left\|\underset{\circ}{h_{ij}}\right\| = \begin{pmatrix} e_1 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$(e_i = \pm 1,\ i = 1, 2)$.

Giving values from 1 to 4 to the indices i and j in the equations (12),we obtain conditions for the constants $a^i_{\cdot j}$:

$$\begin{aligned} & a^3_{\cdot 1} = a^4_{\cdot 2} = a^1_{\cdot 3} = a^2_{\cdot 4} = 0, \\ & a^4_{\cdot 1} = -a^3_{\cdot 2}, \\ & a^3_{\cdot 3} = -a^1_{\cdot 1}, \\ & a^3_{\cdot 4} = -a^2_{\cdot 1}, \\ & a^4_{\cdot 3} = -a^1_{\cdot 2}, \\ & a^4_{\cdot 4} = -a^2_{\cdot 2}, \\ & a^2_{\cdot 3} = -a^1_{\cdot 4} \end{aligned}$$

Exploring equations (13), we get the final form of the matrix $\|a^i_{\cdot j}\|$:

$$\|a^i_{\cdot j}\| = \begin{Vmatrix} a^1_{\cdot 1} & a^1_{\cdot 2} & 0 & 0 \\ a^2_{\cdot 1} & -a^1_{\cdot 1} & 0 & 0 \\ 0 & a^3_{\cdot 2} & -a^1_{\cdot 1} & -a^2_{\cdot 1} \\ -a^3_{\cdot 2} & 0 & -a^1_{\cdot 2} & a^1_{\cdot 1} \end{Vmatrix}.$$

Thus, there are exactly 4 independent arbitrary constants among $a^i_{\cdot j} : a^1_{\cdot 1}, a^1_{\cdot 2}, a^2_{\cdot 1}, a^3_{\cdot 2}$. Considering that among $a^h_{\cdot}$ there are also 4 independent constants (a^1, a^2, a^3, a^4), the following statement is true:

Proposition 3. *In the symmetric Riemannian space V_4 of the first class there exist at least 8 linearly independent Killing vectors with constant coefficients*

The vectors $\xi^h_{|p}(x)$, where $p = 1, \dots, 8$, included in the basis of the group G_r, have the following form:

(15)
$$\begin{aligned}
&\xi^h_{1|}(x) = \delta^h_1 - \frac{1}{3}\epsilon_1\epsilon_2\left[(x^2)^2\delta^h_3 - x^1x^2\delta^h_4\right],\\
&\xi^h_{2|}(x) = \delta^h_2 - \frac{1}{3}\epsilon_1\epsilon_2\left[x^1x^2\delta^h_3 - (x^1)^2\delta^h_4\right],\\
&\xi^h_{3|}(x) = \delta^h_3,\\
&\xi^h_{4|}(x) = \delta^h_4,\\
&\xi^h_{5|}(x) = x^1\delta^h_1 - x^2\delta^h_2 - x^3\delta^3_h + x^4\delta^h_4,\\
&\xi^h_{6|}(x) = x^2\delta^h_1 - x^3\delta^h_4,\\
&\xi^h_{7|}(x) = x^1\delta^h_2 - x^4\delta^h_3,\\
&\xi^h_{8|}(x) = x^2\delta^h_3 - x^1\delta^h_4.
\end{aligned}$$

Let us calculate the commutators of two operators ([8, 9]) whose components are the vectors $\xi^h_{p|}(x)$:

(16)
$$\begin{aligned}
&[X_1, X_2] = -\epsilon_1\epsilon_2\ X_8,\\
&[X_1, X_3] = 0,\\
&[X_2, X_3] = 0,\\
&[X_1, X_4] = 0,\\
&[X_2, X_4] = 0,\\
&[X_3, X_4] = 0,\\
&[X_1, X_5] = -X_1,\\
&[X_2, X_5] = X_2,\\
&[X_3, X_5] = X_5,\\
&[X_4, X_5] = -X_4,\\
&[X_1, X_6] = 0,\\
&[X_2, X_6] = -X_1,\\
&[X_3, X_6] = X_4,\\
&[X_4, X_6] = 0,\\
&[X_5, X_6] = 2X_6,\\
&[X_1, X_7] = -X_2,\\
&[X_2, X_7] = 0,\\
&[X_3, X_7] = 0,\\
&[X_4, X_7] = X_3,\\
&[X_5, X_7] = -2X_7,\\
&[X_6, X_7] = X_5,
\end{aligned}$$

$$
\begin{aligned}
&[X_1, X_8] = X_4,\\
&[X_2, X_8] = -X_3,\\
&[X_3, X_8] = 0,\\
&[X_4, X_8] = 0,\\
&[X_5, X_8] = -2X_8,\\
&[X_6, X_8] = 0,\\
&[X_7, X_8] = 0.
\end{aligned}
$$

Since commutators of any two operators, whose components are vectors $\xi^h_{p|}(x)$, are linearly expressed through the same operators, we come to the theorem:

Proposition 4. *The symmetric Riemannian space of the first class V_4 admits a Lie group of motions G_8 with basis* (15) *and structure* (16).

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НЕСКІНЧЕННО МАЛІ ПЕРЕТВОРЕННЯ СИМЕТРИЧНОГО РІМАНОВОГО ПРОСТОРУ ПЕРШОГО КЛАСУ

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Дослідження нескінченно малих перетворень у ріманових просторах становлять теоретичний і практичний інтерес. Якщо вказано певне поле у просторі V_n з метрикою ds^2 допускає r-параметричну групу рухів G_r, то це поле має r законів збереження. Поширення релятивістського газу за законом Максвелла-Больцмана характеризується вектором ξ і (x), що є вектором Кіллінга (якщо газ складається з частинок з ненульовою масою спокою) або нескінченно малим вектором конформного перетворення (якщо газ складається з частинок з нульовою масою спокою) ([5], [6]). Досліджено нескінченно малі рухи в симетричних ріманових просторах першого класу V_n. Для $n = 4$ в явному вигляді знайдено базис групи Лі G_8 розглянутих перетворень і наведена структура цієї групи.

Ключові слова: простір другого наближення, нескінченно малі перетворення, група Лі.