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ON FEEBLY COMPACT TOPOLOGIES ON THE SEMIGROUP $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$

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We study the Gutik-Mykhalenych semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ in the case when the family $\mathscr{F}_1$ consists of the empty set and all singleton in ω. We show that $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ is isomorphic to subsemigroup $\mathscr{B}_{\omega}^{\curlyvee}(\omega_{\min})$ of the Brandt ω-extension of the semilattice $(\omega, \min)$ and describe all shift-continuous feebly compact T_1-topologies on the semigroup $\mathscr{B}_{\omega}^{\curlyvee}(\omega_{\min})$. In particular, we prove that every shift-continuous feebly compact T_1-topology τ on $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ is compact and moreover in this case the space $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_1}, \tau\right)$ is homeomorphic to the one-point Alexandroff compactification of the discrete countable space $\mathfrak{D}(\omega)$.

Key words: semitopological semigroup, feebly compact, compact, Brandt ω-extension.

We shall follow the terminology of [4, 5, 6, 7, 27]. By ω we denote the first infinite cardinal.

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents.

If S is an inverse semigroup then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the *natural partial order* on S [28].

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The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The semigroup operation on $\mathscr{C}(p,q)$ is defined as follows:

$$q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [5].

A *topological* (*semitopological*) *semigroup* is a topological space together with a continuous (separately continuous) semigroup operation. If S is a semigroup and τ is a topology on S such that (S,τ) is a topological semigroup, then we shall call τ a *semigroup topology* on S, and if τ is a topology on S such that (S,τ) is a semitopological semigroup, then we shall call τ a *shift-continuous topology* on S.

Next we shall describe the construction which is introduced by Gutik and Mykhalenych in [10].

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F\in\mathscr{P}(\omega)$ and $n,m\in\omega$ we put $n-m+F=\{n-m+k\colon k\in F\}$ if $F\neq\varnothing$ and $n-m+F=\varnothing$ otherwise. A subfamily $\mathscr{F}\subseteq\mathscr{P}(\omega)$ is called *ω-closed* if $F_1\cap(-n+F_2)\in\mathscr{F}$ for all $n\in\omega$ and $F_1,F_2\in\mathscr{F}$.

Let $\boldsymbol{B}_{\omega}$ be the bicyclic monoid and $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_{\omega}\times\mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1,j_1,F_1)\cdot(i_2,j_2,F_2)=\begin{cases} (i_1-j_1+i_2,j_2,(j_1-i_2+F_1)\cap F_2), & \text{if } j_1\leqslant i_2;\\ (i_1,j_1-i_2+j_2,F_1\cap(i_2-j_1+F_2)), & \text{if } j_1\geqslant i_2.\end{cases}$$

In [10] it is proved that if the family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ is ω-closed then $(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I}=\{(i,j,\varnothing)\colon i,j\in\omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)$. For any ω-closed family $\mathscr{F}\subseteq\mathscr{P}(\omega)$ the semigroup

$$\boldsymbol{B}_{\omega}^{\mathscr{F}}=\begin{cases} (\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot)/\boldsymbol{I}, & \text{if } \varnothing\in\mathscr{F};\\ (\boldsymbol{B}_{\omega}\times\mathscr{F},\cdot), & \text{if } \varnothing\notin\mathscr{F}\end{cases}$$

is defined in [10]. The semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in [10] that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its set of idempotents are described. The criteria of simplicity, 0-simplicity, bisimplicity, 0-bisimplicity of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and when $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ has the identity, is isomorphic to the bicyclic semigroup or the countable semigroup of matrix units are given. In particular, in [10] it is proved that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega{\times}\omega$-matrix units if and only if $\mathscr{F}$ consists of a sigleton and the empty set.

We define

$$\mathscr{F}_1=\{A\subseteq\omega\colon |A|\leqslant 1\}.$$

It is obvious that $\mathscr{F}_1$ is an ω-closed subfamily of $\mathscr{P}(\omega)$ and hence $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ is an inverse semigroup with zero. Later by $(i,j,\{k\})$ we denote a non-zero element of $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ for some $i,j,k\in\omega$ and by $\mathbf{0}$ the zero of $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$.

In this paper we study properties of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$. We show that $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ is isomorphic to the subsemigroup $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$ of the Brandt ω-extension of the semilattice $(\omega, \min)$ and describe all shift-continuous feebly compact T_1-topologies on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$. In particular, we prove that every shift-continuous feebly compact T_1-topology τ on $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$ is compact and moreover in this case the space $(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}), \tau)$ is homeomorphic to the one-point Alexandroff compactification of the discrete countable space $\mathfrak{D}(\omega)$.

Proposition 2 of [10] implies Proposition 1 which describes the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$.

Proposition 1. *Let $(i_1, j_1, \{k_1\})$ and $(i_2, j_2, \{k_2\})$ be non-zero elements of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$. Then $(i_1, j_1, \{k_1\}) \preccurlyeq (i_2, j_2, \{k_2\})$ if and only if*

$$k_2 - k_1 = i_1 - i_2 = j_1 - j_2 = p$$

for some $p \in \omega$.

Proposition 1 implies the structure of maximal chains in $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ with the respect to its natural partial order

Corollary 1. *Let i, j be arbitrary elements of ω. Then the following finite series*

$$\begin{array}{l}
\mathbf{0} \preccurlyeq (i, j, \{0\});\\
\mathbf{0} \preccurlyeq (i+1, j+1, \{0\}) \preccurlyeq (i, j, \{1\});\\
\mathbf{0} \preccurlyeq (i+2, j+2, \{0\}) \preccurlyeq (i+1, j+1, \{1\}) \preccurlyeq (i, j, \{2\});\\
\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
\mathbf{0} \preccurlyeq (i+k, j+k, \{0\}) \preccurlyeq (i+k-1, j+k-1, \{1\}) \preccurlyeq \cdots \preccurlyeq (i, j, \{k\});\\
\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\end{array}$$

describes maximal chains in the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$.

We need the following construction from [8].

Let S be a semigroup with zero and $\lambda \geqslant 1$ be a cardinal. On the set $B_\lambda(S) = (\lambda \times S \times \lambda) \sqcup \{\mathscr{O}\}$ we define a semigroup operation as follows

$$(\alpha, s, \beta) \cdot (\gamma, t, \delta) = \begin{cases} (\alpha, st, \delta), & \text{if } \beta = \gamma;\\ \mathscr{O}, & \text{if } \beta \neq \gamma \end{cases}$$

and $(\alpha, s, \beta) \cdot \mathscr{O} = \mathscr{O} \cdot (\alpha, s, \beta) = \mathscr{O} \cdot \mathscr{O} = \mathscr{O}$, for all $\alpha, \beta, \gamma, \delta \in \lambda$ and $s, t \in S$. If S is a monoid then the semigroup $\mathscr{B}_\lambda(S)$ is called the *Brandt λ-extension of the semigroup S* [8]. Algebraic properties of $\mathscr{B}_\lambda(S)$ and its generalization Brandt λ^0-extensions $\mathscr{B}_\lambda^0(S)$ of semigroups are studied in [8, 13]. The structures, topologizations of the semigroups $\mathscr{B}_\lambda(S)$ and $\mathscr{B}_\lambda^0(S)$, their algebraic, categorical properties, applications and generalizations are established in [2, 3, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 26].

By $\omega_{\min}$ we denote the set ω with the binary operation

$$xy = \min\{x, y\}, \qquad \text{for} \quad x, y \in \omega.$$

It is obvious that $\omega_{\min}$ is a semilattice.

We define a map $\mathfrak{f}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}_1} \to \mathscr{B}_\omega(\omega_{\min})$ by the formulae

$$(1) \qquad (i, j, \{k\})\mathfrak{f} = (i+k, k, j+k) \qquad \text{and} \qquad (\mathbf{0})\mathfrak{f} = \mathscr{O},$$

for $i, j, k \in \omega$.

Proposition 2. *The map* $\mathfrak{f}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}_1} \to \mathscr{B}_{\omega}(\omega_{\min})$ *is an isomorphic embedding.*

Proof. It is obvious that the map $\mathfrak{f}$ defined by formulae (1) is bijective.

Fix arbitrary $(i_1, j_1, \{k_1\}), (i_2, j_2, \{k_2\}) \in \boldsymbol{B}_{\omega}^{\mathscr{F}_1}$. Then we have that

$$((i_1, j_1, \{k_1\}) \cdot (i_2, j_2, \{k_2\}))\mathfrak{f} =$$

$$= \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + \{k_1\}) \cap \{k_2\})\mathfrak{f}, & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\ (\mathbf{0})\mathfrak{f}, & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 \neq i_2 + k_2; \\ (i_1, j_2, \{k_1\} \cap \{k_2\})\mathfrak{f}, & \text{if } j_1 = i_2 \text{ and } k_1 = k_2; \\ (\mathbf{0})\mathfrak{f}, & \text{if } j_1 = i_2 \text{ and } k_1 \neq k_2; \\ (i_1, j_1 - i_2 + j_2, \{k_1\} \cap (i_2 - j_1 + \{k_2\}))\mathfrak{f}, & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\ (\mathbf{0})\mathfrak{f}, & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 \neq i_2 + k_2 \end{cases} =$$

$$= \begin{cases} (i_1 - j_1 + i_2, j_2, \{k_2\})\mathfrak{f}, & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\ (i_1, j_2, \{k_1\})\mathfrak{f}, & \text{if } j_1 = i_2 \text{ and } k_1 = k_2; \\ (i_1, j_1 - i_2 + j_2, \{k_1\})\mathfrak{f}, & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\ (\mathbf{0})\mathfrak{f}, & \text{if } j_1 + k_1 \neq i_2 + k_2 \end{cases} =$$

$$= \begin{cases} (i_1 - j_1 + i_2 + k_2, k_2, j_2 + k_2), & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\ (i_1 + k_1, k_1, j_2 + k_1), & \text{if } j_1 = i_2 \text{ and } k_1 = k_2; \\ (i_1 + k_1, k_1, j_1 - i_2 + j_2 + k_1), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\ \mathscr{O}, & \text{if } j_1 + k_1 \neq i_2 + k_2 \end{cases} =$$

$$= \begin{cases} (i_1 + k_1, k_2, j_2 + k_2), & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\ (i_1 + k_1, k_1, j_2 + k_2), & \text{if } j_1 = i_2 \text{ and } k_1 = k_2; \\ (i_1 + k_1, k_1, j_2 + k_2), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\ \mathscr{O}, & \text{if } j_1 + k_1 \neq i_2 + k_2, \end{cases}$$

and

$$((i_1, j_1, \{k_1\})\mathfrak{f} \cdot (i_2, j_2, \{k_2\}))\mathfrak{f} = (i_1 + k_1, k_1, j_1 + k_1) \cdot (i_2 + k_2, k_2, j_2 + k_2) =$$

$$= \begin{cases} (i_1 + k_1, \min\{k_1, k_2\}, j_2 + k_2), & \text{if } j_1 + k_1 = i_2 + k_2; \\ \mathscr{O}, & \text{if } j_1 + k_1 \neq i_2 + k_2 \end{cases} =$$

$$= \begin{cases} (i_1 + k_1, k_2, j_2 + k_2), & \text{if } k_2 < k_1 \text{ and } j_1 + k_1 = i_2 + k_2; \\ (i_1 + k_1, k_1, j_2 + k_2), & \text{if } k_2 = k_1 \text{ and } k_1 = k_2; \\ (i_1 + k_1, k_1, j_2 + k_2), & \text{if } k_2 > k_1 \text{ and } j_1 + k_1 = i_2 + k_2; \\ \mathscr{O}, & \text{if } j_1 + k_1 \neq i_2 + k_2, \end{cases} =$$

$$= \begin{cases} (i_1 + k_1, k_2, j_2 + k_2), & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\ (i_1 + k_1, k_1, j_2 + k_2), & \text{if } j_1 = i_2 \text{ and } k_1 = k_2; \\ (i_1 + k_1, k_1, j_2 + k_2), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\ \mathscr{O}, & \text{if } j_1 + k_1 \neq i_2 + k_2. \end{cases}$$

Since $\mathbf{0}$ and $\mathscr{O}$ are the zeros of the semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ and $\mathscr{B}_{\omega}(\omega_{\min})$, respectively, the above equalities imply that the map $\mathfrak{f}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}_1} \to \mathscr{B}_{\omega}(\omega_{\min})$ is a homomorphism. This completes the proof of the proposition. □

Next we define

$$\mathscr{B}_{\omega}^{\Vdash}(\omega_{\min}) = \{\mathscr{O}\} \cup \{(i, k, j) \in \mathscr{B}_{\omega}(\omega_{\min}) \setminus \{\mathscr{O}\} : i, j \geqslant k\}.$$

Simple verifications show that $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$ is an inverse subsemigroup of $\mathscr{B}_{\omega}(\omega_{\min})$.

Proposition 2 implies

Theorem 1. *The semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ is isomorphic to $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$ by the map $\mathfrak{f}$.*

For any $i,j\in\omega$ we denote

$$\omega_{\min}^{(i,j)_{\Rsh}}=\left\{(i,k,j)\colon (i,k,j)\in\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})\right\}.$$

Proposition 3. *Let τ be a shift-continuous T_1-topology on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$. Then every non-zero element of $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$ is an isolated point in $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$.*

Proof. Fix arbitrary $i,j\in\omega$. Since $(i,0,i)\cdot(i,0,j)\cdot(j,0,j)=(i,0,j)$, the assumption of the proposition implies that for any open neighbourhood $W_{(i,0,j)}\not\ni\mathscr{O}$ of $(i,0,j)$ there exists its open neighbourhood $V_{(i,0,j)}$ in the topological space $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ such that $(i,0,i)\cdot V_{(i,0,j)}\cdot(j,0,j)\subseteq W_{(i,0,j)}$. The definition of the semigroup operation on $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$ implies that $V_{(i,0,j)}\subseteq\omega_{\min}^{(i,j)_{\Rsh}}$. Then the set $\omega_{\min}^{(i,j)_{\Rsh}}$ is an open subset of $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ because it is the full preimage of $V_{(i,0,j)}$ under the mapping

$$\mathfrak{h}\colon\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})\to\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\ x\mapsto(i,0,i)\cdot x\cdot(j,0,j).$$

By Corollary 1 the set $\omega_{\min}^{(i,j)_{\Rsh}}$ is finite, which implies the statement of the proposition. $\square$

Next we shall show that the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$ admits a compact shift-continuous Hausdorff topology.

Example 1. A topology τ_{Ac} on $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$ is defined as follows:

a) all nonzero elements of $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$ are isolated points in $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau_{\mathrm{Ac}}\right)$;

b) the family

$$\mathscr{B}_{\mathrm{Ac}}(\mathscr{O})=\Big\{U_{(i_1,j_1),\ldots,(i_n,j_n)}=\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})\setminus\Big(\omega_{\min}^{(i_1,j_1)_{\Rsh}}\cup\cdots\cup\omega_{\min}^{(i_n,j_n)_{\Rsh}}\Big)\colon n,i_1,j_1,\ldots,i_n,j_n\in\omega\Big\}$$

is a base of the topology τ_{Ac} at the point $\mathscr{O}\in\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$.

Corollary 1 implies that the set $\omega_{\min}^{(i,j)_{\Rsh}}$ is finite for any $i,j\in\omega$ which implies that $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau_{\mathrm{Ac}}\right)$ is the one-point Alexandroff compatification of the discrete space $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})\setminus\{\mathscr{O}\}$.

Proposition 4. *$\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau_{\mathrm{Ac}}\right)$ is a Hausdorff compact semitopological semigroup with continuous inversion.*

Proof. It is obvious that the topology τ_{Ac} is Hausdorff and compact.

Fix any $U_{(i_1,j_1),\ldots,(i_n,j_n)}\in\mathscr{B}_{\mathrm{Ac}}(\mathscr{O})$ and $(i,k,j),(l,m,p)\in\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})\setminus\{\mathscr{O}\}$. Put

$$\boldsymbol{K}=\{i,i_1,\ldots,i_n,j,j_1,\ldots,j_n\}\qquad\text{and}\qquad U_{\boldsymbol{K}}=\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})\setminus\bigcup_{x,y\in\boldsymbol{K}}\omega_{\min}^{(x,y)_{\Rsh}}.$$

Then we have that $U_{\mathbf{K}}\in\mathscr{B}_{\mathrm{Ac}}(\mathscr{O})$ and the following conditions hold

$$U_{\boldsymbol{K}}\cdot\{(i,k,j)\}\subseteq U_{(i_1,j_1),\ldots,(i_n,j_n)},$$
$$\{(i,k,j)\}\cdot U_{\boldsymbol{K}}\subseteq U_{(i_1,j_1),\ldots,(i_n,j_n)},$$

$$
\begin{gathered}
\{\mathscr{O}\}\cdot\{(i,k,j)\}=\{(i,k,j)\}\cdot\{\mathscr{O}\}=\{\mathscr{O}\}\subseteq U_{(i_1,j_1),\ldots,(i_n,j_n)},\\
\{\mathscr{O}\}\cdot U_{(i_1,j_1),\ldots,(i_n,j_n)}=U_{(i_1,j_1),\ldots,(i_n,j_n)}\cdot\{\mathscr{O}\}=\{\mathscr{O}\}\subseteq U_{(i_1,j_1),\ldots,(i_n,j_n)},\\
\{(i,k,j)\}\cdot\{(l,m,p)\}=\{\mathscr{O}\}\subseteq U_{(i_1,j_1),\ldots,(i_n,j_n)},\quad \text{if}\quad j\neq l,\\
\{(i,k,j)\}\cdot\{(l,m,p)\}=\{(i,\min\{k,m\},p)\},\quad \text{if}\quad j=l,\\
\left(U_{(j_1,i_1),\ldots,(j_n,i_n)}\right)^{-1}\subseteq U_{(i_1,j_1),\ldots,(i_n,j_n)}.
\end{gathered}
$$

Therefore, $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau_{\mathrm{Ac}}\right)$ is a semitopological inverse semigroup with continuous inversion. □

We recall that a topological space X is said to be

- *perfectly normal* if X is normal and and every closed subset of X is a G_δ-set;
- *scattered* if X does not contain a non-empty dense-in-itself subspace;
- *hereditarily disconnected* (or *totally disconnected*) if X does not contain any connected subsets of cardinality larger than one;
- *compact* if each open cover of X has a finite subcover;
- *countably compact* if each open countable cover of X has a finite subcover;
- *H-closed* if X is a closed subspace of every Hausdorff topological space in which it contained;
- *infra H-closed* provided that any continuous image of X into any first countable Hausdorff space is closed (see [24]);
- *feebly compact* (or *lightly compact*) if each locally finite open cover of X is finite [1];
- *d-feebly compact* (or *DFCC*) if every discrete family of open subsets in X is finite (see [25]);
- *pseudocompact* if X is Tychonoff and each continuous real-valued function on X is bounded;
- *Y-compact* for some topological space Y, if $f(X)$ is compact for any continuous map $f\colon X\to Y$.

The relations between above defined compact-like spaces are presented at the diagram in [22].

Lemma 1. *Every shift-continuous T_1-topology τ on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$ is regular.*

Proof. By Proposition 3 every non-zero element of the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$ is an isolated point in $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$. This implies that every open neighbourhood $V(\mathscr{O})$ of the zero $\mathscr{O}$ is a closed subset in $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$, and hence the space $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ is regular. □

Since in any countable T_1-space X every open subset of X is a F_σ-set, Theorem 1.5.17 from [7] and Lemma 1 imply the following corollary.

Corollary 2. *Let τ be a shift-continuous T_1-topology on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$. Then $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ is a perfectly normal, scattered, hereditarily disconnected space.*

By $\mathfrak{D}(\omega)$ we denote the countable discrete space and by $\mathbb{R}$ the set of all real numbers with the usual topology.

Theorem 2. *Let τ be a shift-continuous T_1-topology on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$. Then the following statements are equivalent:*

(*i*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ *is compact;*
(*ii*) $\tau=\tau_{\mathrm{Ac}}$*;*
(*iii*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ *is H-closed;*
(*iv*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ *is feebly compact;*
(*v*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ *is infra H-closed;*
(*vi*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ *is d-feebly compact;*
(*vii*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ *is pseudocompact;*
(*viii*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ *is* $\mathbb{R}$*-compact;*
(*ix*) $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ *is* $\mathfrak{D}(\omega)$*-compact.*

Proof. Implications $(ii)\Rightarrow(i)\Rightarrow(iii)\Rightarrow(iv)\Rightarrow(v)\Rightarrow(viii)\Rightarrow(ix)$ and $(i)\Rightarrow(vii)\Rightarrow(iv)\Rightarrow(vi)$ are trivial (see the diagram in [22]). Lemma 1 implies implications $(vi)\Rightarrow(iv)$ and $(iii)\Rightarrow(i)$.

$(ix)\Rightarrow(i)$ Suppose to the contrary that there exists a shift-continuous T_1-topology τ on the semigroup $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})$ such that $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ is a $\mathfrak{D}(\omega)$-compact non-compact space. Then there exists an open cover $\mathscr{U}=\{U_\alpha\}$ of $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ which has not a finite subcover. Let $U_{\alpha_0}\in\mathscr{U}$ such that $\mathscr{O}\in U_{\alpha_0}$. Since $\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)$ is not compact the set $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})\setminus U_{\alpha_0}$ is infinite. We enumerate the set $\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})\setminus U_{\alpha_0}$, i.e., put $\{x_i\colon i\in\omega\}=\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min})\setminus U_{\alpha_0}$. We identify $\mathfrak{D}(\omega)$ with ω and define a map $\mathfrak{f}\colon\left(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}),\tau\right)\to\mathfrak{D}(\omega)$ in the following way

$$(x)\mathfrak{f}=\begin{cases}0, & \text{if } x\in U_{\alpha_0};\\ i, & \text{if } x=x_i.\end{cases}$$

Proposition 3 implies that such defined map $\mathfrak{f}$ is continuous. Also, the image $(\mathscr{B}_{\omega}^{\Rsh}(\omega_{\min}))\mathfrak{f}$ is not a compact subset of $\mathfrak{D}(\omega)$, which contradicts the assumption. □

Theorem 2 implies

Corollary 3. *Every shift-continuous* T_1*-topology* $\mathfrak{D}(\omega)$*-compact* τ *on the semigroup* $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ *is compact. Moreover the semigroup* $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ *admits the unique compact shift-continuous* T_1*-topology.*

Remark 1. By Proposition 4 of [10] the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ contains an isomorphic copy of the $\omega\times\omega$-matrix units. Then Theorem 5 from [16] implies that $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ does not embed into a countably compact Hausdorff topological semigroup.

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ПРО СЛАБКО КОМПАКТНІ ТОПОЛОГІЇ НА НАПІВГРУПІ $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$

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Вивчається напівгрупа Гутіка–Михаленича $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ у випадку, коли сім'я $\mathscr{F}_1$ складається з порожньої множини та всіх одноточкових підмножин у ω. Ми доводимо, що напівгрупа $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ ізоморфна піднапівгрупі $\mathscr{B}_{\omega}^{\curvearrowright}(\omega_{\min})$ ω-розширенню Брандта напівґратки $(\omega, \min)$, описуємо всі трансляційно неперервні слабко компактні T_1-топології на напівгрупі $\mathscr{B}_{\omega}^{\curvearrowright}(\omega_{\min})$. Зокрема, доведено, що кожна трансляційно неперервна слабко компактна T_1-топологія τ на напівгрупі $\boldsymbol{B}_{\omega}^{\mathscr{F}_1}$ є компактною, ба більше, у цьому випадку простір $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_1}, \tau\right)$ гомеоморфний одноточковій компактифікації Алєксандрова дискретного зліченного простору $\mathfrak{D}(\omega)$.

Ключові слова: напівтопологічна напівгрупа, слабко компактний, компактний, ω-розширенню Брандта.